Polarities in topological vector spaces

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Linear Logic (LL) is the result of a decomposition of Intuitionistic Logic via an involutive *linear* negation. This linear negation takes its root in semantics: it is interpreted as the *dual* of a vector space While LL's primary intuitions lie in algebra, the study of vectorial models [5, 6] of it led to the introduction of Differential Linear Logic [7] (DiLL). This new proof system introduces the possibility to differentiate proofs and led to advances in the semantics of probabilistic and differentiable programming. Obtaining models of DiLL using traditionnal smooth objects of analysis is a non-trivial task. Indeed, while one would like to interpret differentiation on proofs as differentiation of smooth functions, interpreting traditional computational equations in a smooth setting such as cartesian closedness, monodal associativity or involutive linear negations reveals challenging. This abstract focuses on monoïdality and involutivity, that is obtaining a model of MLL in a smooth setting. We thus provide new polarized models of MLL which would results in models of DiLL, although the interpretation of exponential connectives is not detailled here.

Infinite dimensional vector spaces are necessary to interpret all proofs of DiLL. However these spaces seldom are good interpretation for classical MLL. Indeed, the class of all *reflexive* topological vector spaces, that is of spaces invariant via double-dual, enjoys poor stability properties. More crucially, duality in topological vector spaces does not define a closure operator: simply considering E'' does not produce a reflexive space. Thus historical models of Linear Logic traditionally interpret formulas via very specific vector spaces: vector spaces of sequences [5], vector spaces over discrete field [6]. How close is the differentitation at stakes in DiLL from the one of real analysis ? Denotational models of DiLL in real-analysis either don't interpret the involutivity of linear negation [2] or imply a certain discretisation for the interpretation of non-linear proofs [12, 4].

Polarization is a syntactical refinement of Linear logic arising for matters of proof-search [1, 10]. By making vary the topology on the dual, this paper unveils polarized models behind preexisting models of DiLL and construct new ones. Meanwhile, it attaches topological notions to the concept of polarity in proof theory.

We revisit the poor stability properties of reflexive spaces by decomposing it in a polarized version model of MLL. We show that reflexive spaces in their canonical definition — that is topological spaces invariant by double strong dual — are best defined in a polarized context. In particular, we show that barrelled spaces - introduced Bourbaki to generalize Banach-Steinhauss theorem— are the interpretation for positive formulas, while Banach-Steinhass is nothing but monodal closedness in a polarized setting. The theory of topological vector spaces is known for its dense network of interdependent definitions. What we suggest here is that structures inherited from proof theory provide a framework for these definitions.

We also revisit the notion of *bornological* spaces persistent in DiLL's denotational semantics [17, 2] as an interpretation for positives. Models of DiLL are in particular cartesian closed category, that is higher-order models of functional programming. An elegant solution for constructing instances of these with smooth functions is to work primarily with bounded subsets instead of open subsets [2]. However, this setting does not accommodate involutive duals: we show that chiralities are the way to obtain that, by working with two different interpretations of negation.

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1 Smooth and polarized differential linear logic

Categorically, the exponential ! is interpreted as a co-monad on $\mathscr{L}$, arising from a strong monoïdal adjunction: $! := \mathscr{E}' \circ U$ and $\mathscr{E}' : \mathbb{C} \to \mathscr{L} \vdash U : \mathscr{L} \to \mathbb{C}$. On top of that, interpreting DiLL necessitates an additive categorical structure on $\mathscr{L}$ and a natural transformation $\bar{d} : ! \to Id$ enabling the linearization of proof (hence their differentiation) [8].

Topological vector spaces are a generalization of normed or metric spaces necessary to higher-order functions. Smooth functions between topological vector spaces are those functions which can be infintely or everywhere differentiated. To handle composition or differentiation of smooth functions, the topology of their codomain must verify some completeness property[1]. However, this requirement for completeness mixes badly with reflexive spaces (those interpreting an involutive linear negation) and create an obstacle in the construction of smooth models of DiLL.

Beyond the distinction between linear and non-linear proofs, *polarization* in LL [15] distinguishes between *positive* and *negative* formulas.

$$\text{Negative Formulas: } N, M := a \mid ?P \mid \uparrow P \mid N \parr M \mid \bot \mid N \& M \mid \top.$$
$$\text{Positive Formulas: } P, Q := a^{\perp} \mid !N \mid \downarrow N \mid P \otimes Q \mid 0 \mid P \oplus Q \mid 1.$$

Semantically, polarization splits $\mathscr{L}$ in two categories $\mathscr{P}$ and $\mathscr{N}$ This was formalized categorically by Mellies: mixed chiralities [16][2] are a decomposition of $*$-autonomous categories in two adjunctions. A strong monoïdal adjunction $(-)^{\perp_L} : \mathscr{P} \to \mathscr{N}^{op} \vdash (-)^{\perp_N} : \mathscr{N}^{op} \to \mathscr{P}$ interprets negations, accompanied with an adjunction interpreting shifts $\uparrow : \mathscr{P} \to \mathscr{N} \vdash \downarrow : \mathscr{N} \to \mathscr{P}$. To interpret soundly proofs of MLL, one asks for two additional equations, respectively accounting for polarized closedness and the unicity of a saturation procedure on positive.

$$\chi_{p,n,m} : \mathscr{N}(\uparrow p, n \parr m) \sim \mathscr{N}(\uparrow(p \otimes n^{\perp_N}), m) \qquad (\textit{curryfication})$$

$$clos_p : \downarrow(p^{\perp_P}) \simeq (\uparrow p)^{\perp_N}. \qquad (\textit{closure})$$

This semantics enables an internal interpretation of polarized connectives - thus refining traditional interpretation in terms of dual pairs.

Definition 1. *A **dialogue chirality** is a mixed chirality in which the monoidal adjunction is an equivalence. A **negative chirality** is a mixed chirality in which the monoidal adjunction is reflective. A **positive chirality** is a mixed chirality in which the monoidal adjunction is co-reflective.*

Definition 2. *A **topological chirality** takes place between two adjoint categories $\mathscr{T}$ and $\mathscr{C}$. It adds to this first adjunction a strong monoidal contravariant adjunction between a full subcategory of $\mathscr{T}$ and a full subcategory of $\mathbb{C}$,*

$$\begin{tikzcd} (\mathscr{P}, \otimes, 1) \arrow[r, bend left, "(-)^{\perp_P}"] \arrow[r, phantom, "\perp"] & (\mathscr{N}^{op}, \parr, \bot) \arrow[l, bend left, "(-)^{\perp_N}"] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{T} \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend left, "\downarrow"] \end{tikzcd} \tag{1}$$

and such that curryfication and closure are still validated in $\mathscr{T}$ and $\mathscr{C}$ respectively. The other commutative diagrams required from chiralities are also required here.

[1] As an example, Mackey-Completeness is a minimal completeness condition used by Frölicher, Kriegl and Michor [9, 14] to develop a theory of higher-order smooth functions

[2] We warn the reader that chiralities have no obvious link with the orientation-related chiralities in physics

2 Weak spaces as negatives, Mackey spaces as positive.

Definition 3. *A* ***Hausdorff and locally convex topological vector space*** *(lcs)is a vector space endowed with a Haussdorff topology making scalar multiplication and addition continuous, and such that every point has a basis of convex* 0*-neighbourhoods.*

We denote by TOPVEC the category of real lcs and linear continuous maps between them.

Definition 4. *The space of all linear continuous functions between lcs* E *and* F *is denoted* $\mathscr{L}(E,F)$*. The dual of a lcs* E *is denoted* $E' := \mathscr{L}(E,\mathbb{R})$.

The weak topology on a topological vector space E is the *coarsest topology* preserving the dual of E' of E, while the Mackey topology is the *finest topology* preserving E'. We denote respectively by E_μ and E_σ the lcs E endowed with the Mackey and the weak topology described above.

In earlier work [12], the author built a model of DiLL in which formulas were interpreted by weak spaces. We argued that the fact that spaces of linear maps endowed with the pointwise convergence topology preserve weak spaces gave this model a polarized flavour. The space $E \parr_\sigma F := \mathscr{L}_\sigma(E'_w, F)$ is always endowed with its weak topology ([11, 15.4.7]) and the MLL model described in [12] easily refines in a chirality:

Proposition 1. *The following adjunctions define a negative chirality:*

$$\begin{tikzcd}(\textsc{Topvec}, \otimes_w, \mathbb{R}) \arrow[r, bend left, "(-)'_\sigma"] \arrow[r, phantom, "\perp"] & (\textsc{Weak}^{op}, \parr_\sigma, \mathbb{R}) \arrow[l, bend left, "(-)'_\sigma"]\end{tikzcd} \qquad \begin{tikzcd}\textsc{Topvec} \arrow[r, bend left, "(-)_\sigma"] \arrow[r, phantom, "\perp"] & \textsc{Weak} \arrow[l, bend left, "\iota"]\end{tikzcd}$$

in which ι *denotes the inclusion functor.*

We show that Mackey spaces leave stable the positive constructions, and in particular a certain topological tensor product. It is however not enough to construct a positive chirality between Mackey spaces and general lcs, due to a lack of associative interpreation for the $\parr$ on general topological vector spaces.

Thus we investigate the interpretation of positive formulas MLL in Mackey spaces. . The goal now is to handle as negatives spaces with *some completeness*, in order to work with smooth functions and differentiability.

3 Decomposing reflexivity via polarities

The *strong* topology on E' is defined as the topology of uniform convergence of bounded subsets of E. The strong dual of E (denoted E'_β) is the traditional interpretation the LL negation. As such, reflexive spaces, that is spaces in which strong duality is involutive is the traditional denotational interpretation for formulas of classical LL.

Definition 5. *E is said to be* ***semi-reflexive*** *when E and* $(E'_\beta)'$ *are isomorphic as vector spaces, and* ***reflexive*** *when E* $(E'_\beta)'_\beta$ *are linearly homeomorphic – that is they have the same algebraic and topological structure.*

Reflexive spaces are stable by product or direct sums, but are not in general stable under inductive limits or tensor product. Thus using the strong dual as interpretation for the negation of linear logic gives us very little chance to construct a model of DiLL without strongly restricting the kind of vector spaces one handles.

A lcs E is called **barrelled** when it is endowed with the topology induced by $(E'_{\beta(E)})'_{\beta(E')}$. Barrelled spaces are in particular Mackey [11, 11.1]. Thus a lcs is reflexive *iff* is it semi-reflexive and barrelled.

We show that this decomposition of reflexivity corresponds in fact to a *polarized decomposition*. We show that barrelled spaces are stable under the Mackey tensor product, and that the Banach-Steinhauss theorem - which motivates the introduction of barrelled spaces by Bourbarki [3] - corresponds to curryfication in chiralities. We denote by BARR the full subcategory of barrelled lcs, and by SREFL the full subcategory of semi-reflexive spaces.

Theorem 1. *Barrelled spaces and semi-reflexive spaces organise in the following topological dialogue chirality:*

$$\begin{tikzcd}(\textsc{Barr}, \otimes_\beta, \mathbb{R}) \arrow[r, bend left, "(-)'_\sigma"] \arrow[r, phantom, "\perp"] & (\textsc{SRefl}^{op}, \parr_w, \mathbb{R}) \arrow[l, bend left, "(-)'_\mu"]\end{tikzcd} \qquad \begin{tikzcd}\textsc{Mackey} \arrow[r, bend left, "(-)_\sigma"] \arrow[r, phantom, "\perp"] & \textsc{Weak} \arrow[l, bend left, "(-)_\mu"]\end{tikzcd}$$

4 Bornological spaces as positives

Bornological spaces were at the heart of the duality in vectorial models of LL [17, III.5], and in the first smooth intuitionistic model of DiLL [2]. However, it was shown that in the context of intuitionistic smooth models, bornological topologies were unecessary, and the first model made of bornological and Mackey-complete spaces was refined into a model made only of Mackey-complete space [13]. We show that *bornologicality is in fact the key to make smooth models classical*, through polarization.

Definition 6. *Consider E a lcs. A (strongly) bounded subspace of E is a* ***vector bornology*** *if it is* absorbed *by any neighborhood of* 0*. A bounded map is a map for which the image of a bounded set is bounded. We denote by* $\mathbf{L}(E,F)$ *the vector space of all bounded linear maps between* $E,F \in$ TOPVEC*.*

While the converse is not true, a linear continuous map is always bounded. *Bornological lcs* are those lcs on which these a linear bounded map is always continuous. We denote by bTOPVEC the category of bornological lcs and continuous (iff bounded) linear maps between them.

Proposition 2. *As bornological lcs are in particular Mackey, we have a contravariant adjunction and a coreflection:*

$$\begin{tikzcd}(\textsc{bTopVec}, \otimes_\beta, \mathbb{R}) \arrow[r, bend left, "(-)'_\mu"] \arrow[r, phantom, "\perp"] & (\textsc{Mackey}^{op}) \arrow[l, bend left, "\mathsf{Top}\circ\mathsf{Born}((-)'_\mu)"]\end{tikzcd} \qquad \begin{tikzcd}\textsc{bTopVec} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp"] & \textsc{Mackey} \arrow[l, bend left, "\mathsf{Top}\circ\mathsf{Born}"]\end{tikzcd}$$

This however is not enough to have a chirality: *we do not have a suitable interpretation for the dual of* $\otimes_\beta$ *which would be associative on all Mackey spaces*, and not just on duals of bornological spaces. More generally, bornological spaces do not verify a duality theorem with some kind of complete spaces, or at least not some kind involving duals which preserves reflexivity [11, 13.2.4]. One solution detailed is to add a suitable notion of completeness. Indeed, to the notion of bornology corresponds a good notion of completeness, enforcing the convergence of Cauchy sequences with respect the norms generated by bounded subsets. We denote the full subcategory of TOPVEC made of Mackey-complete lcs by MCO. Mackey-complete spaces are the heart of several smooth models of DiLL [2, 4, 13].

In particular, in work by Blute Ehrhard and Tasson [2] formulas were interpreted by **convenient spaces**, that is bornological lcs which are also Mackey-complete[3]. We denote by CONV the full subcategory of bornological and Mackey-complete lcs, endowed with linear bounded (iff continuous) maps.

Theorem 2. *Convenient spaces and Mackey-Complete spaces organise in the following topological positive chirality:*

$$\begin{tikzcd} (\mathrm{CONV}, \hat{\otimes}^M_\beta, \mathbb{R}) \arrow[r, bend left, "(-)'_\mu"] \arrow[r, phantom, "\perp"] & (\mathrm{MCO}^{op}, \parr_b, \mathbb{R}) \arrow[l, bend left, "(\mathsf{Born}((-)'_\mu))^{conv}"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{bTopVec} \arrow[r, bend left, "\mathsf{Top}"] \arrow[r, phantom, "\perp"] & \mathrm{TOPVEC} \arrow[l, bend left, "\mathsf{Born}"] \end{tikzcd}$$

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[3] Mackey-completeness in fact is what makes bornological lcs ultrabornological, and in particular barrelled. Theorem 2 can be seen as an adaptation of convenient spaces to the chirality of barrelled spaces