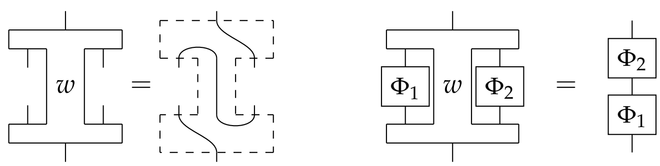
Work in Progress:
MLL+MIX as a logic of influence and causation

Aleks Kissinger, *University of Oxford*

May 4, 2020

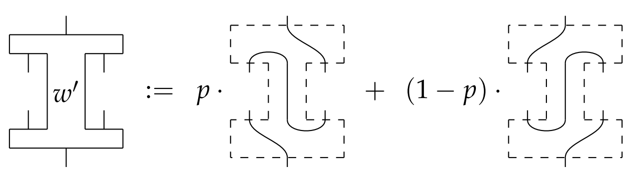
The framework of *higher-order causal categories* [10] gives a means of reasoning about influences between the inputs and outputs of networks of 'black box' processes using multiplicative linear logic (MLL). Notably, the higher-order structure is able to put not only the processes themselves, but also their causal ordering (i.e. how they are wired together), into a black box. For example, we may picture a process that connects two processes in sequence as a second-order process of type $w : \big[(A \multimap B) \otimes (B \multimap C)\big] \multimap (A \multimap C)$, and picturially as:



(1)

Such higher-order constructions are very familiar in theoretical computer science and functional programming, where processes are interpreted as functions, lambda terms, etc.

Recently, higher-order processes have become of particular interest in theoretical physics as well, particularly when they act on stochastic processes (a.k.a. conditional probability distributions) or quantum channels [2, 1]. This stems from an interesting feature of 'black-boxing' causal ordering: it enables us to capture situations where the causal ordering is not fixed in advance. For example, by interpreting processes probabilistically, we can consider situations where the causal order is not fully known as probabilistic mixtures of multiple different 'wirings':



(2)

There are even higher-order processes which admit no such decomposition, which are known as *indefinite causal orders*. Theories admitting such processes have been shown to be consistent with quantum theory [11] while modelling geniunely new phenomena such as time-delocalised processes [12] and superpositions of causal orders due to quantum gravitational effects [14]. Processes exhibiting indefinite causal structure may also lead to solving new (quantum) computational tasks [3, 7, 6], and have already formed the basis of new experiments in quantum optics [13].

The key notion to making higher-order processes physically or computationally meaningful is a notion of *consistency*, which in the case of stochastic (and quantum) processes amounts to the fact that closed diagrams of processes, i.e. processes with no inputs or outputs, yield well-defined probabilities between 0 and 1. In particular, these conditions forbid the existence of time-loops in higher-order processes.

For example, the higher-order process w depicted in (1) corresponds to a bilinear functional $w(P,Q)(y|a) := \sum_x Q(y|x)P(x|a)$, capturing the usual way in which two stochastic processes are composed in sequence. In particular, it has the property that if P and Q are properly normalised stochastic processes, so too is $w(P,Q)$. Hence, any closed diagram involving P, Q, or $w(P,Q)$ will indeed yield well-defined probabilities. The same holds for w' depicted in equation (2). On the other hand, a linear functional *loop* : $(A \multimap A) \multimap \mathbb{R}$ that

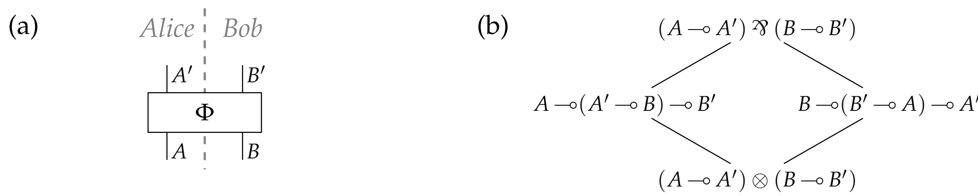


Figure 1: (a) A 'black box' process shared by Alice and Bob, (b) a hierarchy of possible *signalling conditions* for this process, represented as MLL types: non-signalling (bottom), Alice can only signal Bob (left), Bob can only signal Alice (right), and no restrictions (top). Entailments upwards are all provable in MLL+MIX.

'plugs' the output of a stochastic process into its input $loop(P) := \sum_i P(i|i)$ can produce values ranging from 0 to $|A|$. If such a higher-order process were physically meaningful, it would entail that even deterministic processes could 'happen' with probabilities different from 1, leading to logical inconsistences such as the Grandfather Paradox [15].

It was shown in [10] that this notion of consistency can be captured in a general, categorical framework by considering a compact closed category $\mathcal{C}$ of 'raw', unnormalised processes (e.g. $\mathbb{R}_{\geq 0}$-valued matrices) and building a $*$-autonomous category of consistent higher-order processes $\text{Caus}[\mathcal{C}]$ (e.g. higher-order stochastic processes) using a variation of the double-gluing construction described in [8]. From the $*$-autonomous structure of $\text{Caus}[\mathcal{C}]$, it follows that the logic dictating which processes can be consistently composed contains MLL, and since the two multiplicative units coincide in $\text{Caus}[\mathcal{C}]$, it also contains the (n-ary) MIX rule.

A pleasant feature of these categorical models is they give a clear, intuitive interpretation for the logical connectives of MLL in terms of *signalling conditions*, which have been studied extensively in the foundations of physics due to their relevance in studying the interplay of quantum theory and special relativity. Supposing a pair of agents Alice and Bob each interact with a 'black box' by giving it an input and receiving an output, in Figure 1(a), one could ask whether either party is able to send a message, i.e. *signal* the other, by using their input to affect the other's output. There are four possibilities. The most restrictive possibility is that Φ is a *non-signalling* process, which corresponds to a tensor of Alice and Bob's local input/output types, whereas the least restrictive possibility is that Alice and Bob can use the process for arbitrary 2-way communication, which corresponds to the par. Then, there are two possibilities in-between where only one agent can communicate to the other. These four possibilities are depicted in Figure 1.

As $\text{Caus}[\mathcal{C}]$ always forms an (ISOMIX [4]) $*$-autonomous category, an entailment of types in MLL+MIX gives sufficient conditions for two processes to be composable without inconsistences. However, this is not the full story. In this work, I will discuss progress toward necessary conditions in order to completely characterise which processes can be composed. In the concrete instances of classical probabilistic and quantum maps, necessary and sufficient conditions are known, but they involve solving systems of linear equations on vector spaces of exponentially large dimension. For example, the simplest non-trivial space of N-party quantum processes that allow for indefinite causal structure is 13^N-dimensional. Hence, a logical characterisation could not only shed light on these somewhat mysterious conditions, but could possibly lead to more practical scaling than the concrete approach.

Toward finding necessary logical conditions, I will discuss several related avenues being explored. The first is the special role played by the 'atoms' in this logic, i.e. the state spaces. These satisfy interesting properties which do not pass to higher-order, such as the fact that, when A and B at atomic, the usual linear distributivity mapping $\delta : A \otimes (X \parr B) \to (A \otimes X) \parr B$, which always exists in a $*$-autonomous category, becomes invertible for any X. In particular, this implies that tensor and par coincide at first order. The second avenue involves looking directly at relaxations of proof net correctness criteria, such as the Danos-Regnier conditions [5] and variations thereof. These already encode some aspect of the notion of 'no time loops' and several versions of these criteria can easily be adapted or weakened to incorporate the MIX rule. I will show that some further modifications to the correctness criteria can be incorporated to admit strictly more consistent compositions than MLL+MIX alone. I will also briefly discuss some experiments with these new criteria based on an open-source graphical theorem prover for MLL+MIX called PyPN [9].

References

[1] Alessandro Bisio and Paolo Perinotti. Theoretical framework for higher-order quantum theory. *Proceedings of the Royal Society A*, 475(2225):20180706, 2019.

[2] G. Chiribella, G. M. D'Ariano, and P. Perinotti. Quantum circuit architecture. *Physical Review Letters*, 101:060401, 2008.

[3] G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron. Quantum computations without definite causal structure. *Physical Review A*, 88:022318, August 2013.

[4] J.R.B. Cockett and R.A.G. Seely. Proof theory for full intuitionistic linear logic, bilinear logic, and mix categories. *Theory and Applications of Categories*, 1991.

[5] Vincent Danos and Laurent Regnier. The structure of multiplicatives. *Archive for Mathematical logic*, 28(3):181–203, 1989.

[6] Daniel Ebler, Sina Salek, and Giulio Chiribella. Enhanced communication with the assistance of indefinite causal order. *Physical review letters*, 120(12):120502, 2018.

[7] Philippe Allard Guérin, Adrien Feix, Mateus Araújo, and Časlav Brukner. Exponential communication complexity advantage from quantum superposition of the direction of communication. *Phys. Rev. Lett.*, 117:100502, Sep 2016.

[8] M. Hyland and A. Schalk. Glueing and orthogonality for models of linear logic. *Theoretical Computer Science*, 294(1–2):183 – 231, 2003.

[9] Aleks Kissinger. PyPN: A graphical MLL+MIX theorem prover and proof net library. `https://github.com/akissinger/pypn`.

[10] Aleks Kissinger and Sander Uijlen. A categorical semantics for causal structure. *Logical Methods in Computer Science*, Volume 15, Issue 3, August 2019.

[11] O. Oreshkov, F. Costa, and C. Brukner. Quantum correlations with no causal order. *Nature Communication*, 1092(3).

[12] Ognyan Oreshkov. Time-delocalized quantum subsystems and operations: on the existence of processes with indefinite causal structure in quantum mechanics. *Quantum*, 3:206, 2019.

[13] Lorenzo M Procopio, Amir Moqanaki, Mateus Araújo, Fabio Costa, Irati Alonso Calafell, Emma G Dowd, Deny R Hamel, Lee A Rozema, Časlav Brukner, and Philip Walther. Experimental superposition of orders of quantum gates. *Nature communications*, 6:7913, 2015.

[14] Magdalena Zych, Fabio Costa, Igor Pikovski, and Časlav Brukner. Bell's theorem for temporal order. *Nature communications*, 10(1):1–10, 2019.

[15] Ämin Baumeler, Fabio Costa, Timothy C Ralph, Stefan Wolf, and Magdalena Zych. Reversible time travel with freedom of choice. *Classical and Quantum Gravity*, 36(22):224002, oct 2019.