A self-dual modality for non-commutative contraction and duplication in the category of coherence spaces

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We define a modality "flag" in the category of coherence spaces (or hypercoherences) with two inverse linear (iso)morphisms: "duplication" from (flag A) to ((flag A) < (flag A)) and "contraction" in the opposite direction — where < is the self dual and non commutative connective of pomset logic or of BV. The coherence space A is a retract of its modal image (flag A). This semantic construction is likely to help the design of proof rules for a modality enjoying those properties.

1 Presentation

Given the aftermath [9, 21, 12, 20] of pomset logic [14, 17] and of the BV calculus of structures [7, 10, 11] we decided to update and submit this ancient work [16], which answers a question of Jean-Yves Girard in [5]. This semantic work might help to find the proper rules for a self dual modality as recently sketched by Alessio Guglielmi in [9].

The structural rules of classical logic are responsible for the non-determinism of classical logic, and linear logic which carefully handles these rules is especially adequate for a constructive treatment of classical logic, as the afore mentioned paper shows. Linear logic handles structural rules by the modalities (a.k.a. exponentials) "?" and "!". The modality "?" allows contraction and weakening in positive position, and the modality "!" in negative position. The formula $!A$ linearly implies A while the formula $?A$ is linearly implied by A In semantical words this means we have the linear morphisms:

$$\begin{array}{ccc} !A & \multimap & (!A \otimes !A) \\ \downarrow & \searrow & \\ 1 & & A \end{array} \qquad\qquad \begin{array}{ccc} ?A & \multimapinv & ?A \parr ?A \\ \uparrow & \nwarrow & \\ 1 & & A \end{array}$$

The major difficulty when dealing with classical logic are the cross-cuts, appearing in the cut elimination theorem of Gentzen as a rule called MIX [2] . This rule is a generalised cut between several occurrences of A and several occurrences of $\neg A$, i.e. a cut between two formulas both coming from contractions. This is a major cause of non-determinism: an example can be found in [5, Appendix B, Example 2, p 294].

In linear logic, such a cut may not happen, since contraction only applies to $?A$ formulas, while their negation is $!A^{\perp}$ which can not come from a contraction — hence no such CUT is possible in linear logic.

Let us now quote the precise paragraph of [5, p. 257] which initially motivated this note:

The obvious candidate for a classical semantics was of course coherence spaces which had already given birth to linear logic; the main reason for choosing them was the presence of the involutive linear negation. However the difficulty with classical logic is to accommodate structural rules (weakening and contraction); in linear logic, this is possible by considering coherent spaces $?X$. But since classical logic allows contraction and weakening both on a formula and its negation, the solution seemed to require the linear negation of $?X$ to be

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of the form $?Y$, which is a nonsense (the negation of $?X$ is $!X^{\perp}$ which is by no means of isomorphic to a space$?Y$). Attempts to find a self-dual variant $\S Y$ of $?Y$ (enjoying $(\S Y)^{\perp} = \S Y^{\perp}$) systematically failed. The semantical study of classical logic stumbled on this problem of self-duality for years.

Here too we focus on coherence spaces because of their tight relation to linear logic [4, 22, 15, 18, 6]. Once the modality is found in the category of coherence spaces, we briefly show that this modality also exists in the category of hypercoherences of Thomas Ehrhard [1].

In our previous work on pomset logic [14, 17, 20], we studied a self-dual connective before, together with partially ordered multisets of formulae. This lead us to the modality $\urcorner$ (flag) to be described in this paper. Flag is a functor, it is self-dual, and it enjoys both *left and right contraction linear isomorphism* with respect to "before" $\urcorner A \multimap\!\!\circ (\urcorner A < \urcorner A)$ and A is a retract of $\urcorner A$. Fortunately, there is no weakening, which would yield unwanted morphisms like $1 \multimap A$ and $A \multimap 1$ for all A...

There is not yet any syntax extending pomset logic to this modality, however Alession Guglielmi sketched a possible syntax in the calculus of structures in [9]. We firstly need to study the basic steps of cut-elimination, in particular the contraction/contraction case and the commutative diagrams it requires, as Myriam Quatrini did in [13] for the logical calculus LC of [5]

2 Preliminary remarks

2.1 Before a.k.a. "sequential" connective

We refer the reader to [3, 6] for the definition of coherence spaces. Let us simply recall the multiplicative connective before studied in [14, 17, 20], written $A < B$:

Definition 1. *Given two coherence spaces A and B, the coherence space $A < B$ is defined by: web $|A < B| = |A| \times |B|$ coherence $(a,b) \frown (a',b')\,[A < B]$ whenever $(a \frown a'[A]$ and $b = b')$ or $b \frown b'[B]$*

From [14, 17] we know that the following easy proposition holds:

Proposition 1. *This connective is: non-commutative, $A < B \neq B < A$ self-dual,$(A < B)^{\perp} \equiv A^{\perp} < B^{\perp}$ associative $A < (B < C) \equiv (A < B) < C$ admits 1 as a unit, $A < 1 \equiv A \equiv 1 < A$ in between* p *and* $\otimes$ *:for all formulae A and B, we have $A \otimes B \multimap A < B$ and $A < B \multimap A \parr B$.*

2.2 Some simple remarks on the Cantor space and finite trees

We write $\mathbb{2}$ for $\{0,1\}$ and $\mathbb{2}^*$ for the set of finite words on $\mathbb{2}$, including the empty word, $\mathbb{2}^{\omega}$ for the set of infinite words on $\mathbb{2}$. Letters like w, v, u range over $\mathbb{2}^{\omega}$, while m range over $\mathbb{2}^*$. We use the standard notation $w = m(m')^*$ for $w = mm'm'm'm'm'\cdots$

The set $\mathbb{2}^{\omega}$ of infinite words on $\mathbb{2}$ is assumed to be endowed with:

- the usual total lexicographical order on defined by:

$$w_1 < w_2 \text{ iff } \exists m \in \mathbb{2}^* \ \exists w_1', w_2' \in \mathbb{2}^{\omega} \ w_1 = m0w_1' \textit{ and } w_2 = m1w_2'$$

- the usual product topology generated by the basis of clopen sets $(U_m)_{m \in \mathbb{2}^*}$ with

$$U_m = \{w \in \mathbb{2}^{\omega} | \exists w' \in \mathbb{2}^{\omega} \ w = mw'\}$$

Proposition 2. $\mathbb{2}\,(\mathrm{gt}_M$ *generic trees on* $M)$ *The set* gt_M *of continuous functions from* $\mathbb{2}^\infty$ *to a set* M *(discrete topology) is in a one-to-one correspondence with the set of finite binary trees on M such that any two sister leaves have distinct labels.*

Thus, an element f *of* gt_M *may either be described:*

1. *As a finite set* $\{(m_1,a_1),\ldots,(m_k,a_k)\} \subset \mathcal{P}_{fin}(\mathbb{2}^* \times M)$ *satisfying:*
   (a) $\forall w \in \mathbb{2}^\omega\ \exists! i \leq k\ \exists w' \in \mathbb{2}^\omega \quad w = m_i w'$
   (b) $\forall i,j \leq k\ \left[\exists m \in \mathbb{2}^*\ m_i = m0 \text{ and } m_j = m1\right] \Rightarrow a_i \neq a_j$

   In this formalism $f(w)$ *is computed as follows: applying* (a), *there exists a unique* i *such that there exists* w' *with* $w = m_i w'$, *let* $f(w) = f(m_i(0)^*) = a_i$

2. *As the normal form of a term of the grammar:*

$$\mathcal{T}_M :: \underline{M} | \langle \mathcal{T}_M\, \mathcal{T}_M \rangle$$

   where the reduction is $\forall x \in M\ t[\langle \underline{x}\underline{x} \rangle] \longrightarrow t[\underline{x}]$ *where* $t[u], u \in \mathcal{T}_M$ *means a term of* $\mathcal{T}_M$ *having an occurrence of the subterm* $u \in \mathcal{T}_M$. *In this formalism* $f(w)$ *is computed as follows:*

$$\begin{array}{rclcrcl} f & = & \underline{a} & : & f(w) & = & a \\ f & = & \langle t_0 t_1 \rangle & : & f(0w) & = & t_0(w) \\ & & & & f(1w) & = & t_1(w) \end{array}$$

Example: Let $M = \{a,b,c\}$ Here are the three description of the same element of gt_M :

0. As a function

$$\begin{array}{rclcrcl} f(000w) & = & a & \quad & f(100w) & = & a \\ f(001w) & = & a & & f(101w) & = & b \\ f(010w) & = & a & & f(110w) & = & a \\ f(011w) & = & a & & f(111w) & = & b \end{array}$$

1. as a finite set of pairs $\{(m_i,a_i)/m_i \in \mathbb{2}^* \text{ and } a_i \in M\}$:

$$f = \{(0,a),(100,a),(101,b),(110,a),(111,b)\}$$

2. as a normal term of $\mathcal{T}_M : f = \langle \underline{a} \langle \langle \underline{a}\,\underline{b} \rangle \langle \underline{a}\,\underline{b} \rangle \rangle \rangle$ such a term is the normal form of, e.g. $\langle \langle \underline{a}\,\underline{a} \rangle \langle \langle \underline{ab} \rangle \langle \langle \underline{aa} \rangle \langle \underline{b}\,\underline{b} \rangle \rangle \rangle \rangle$

Here is one more easy remark:

Proposition 3. *Let* $f,g \in \mathrm{gt}_M$. *If* $f \neq g$, *then there exists* $w \in \mathbb{2}^\omega$ *such that*

$$f(w) \neq g(w) \text{ and } \forall w' > w \quad f(w') = g(w')$$

Proof. Now, let us see that whenever two continuous functions from $\mathbb{2}^\omega$ to a set M (discrete topology) differ, there exists w such that $f(w) \neq g(w)$ and $\forall v > w\ f(v) = g(v)$ The product of the discrete topological space M by itself is the discrete topological space $M \times M$ Hence the function Δ from $M \times M$ to $\mathbb{2}$ defined by $\Delta(x,y) = 1$ iff $x = y$ is continuous. The function (f,g) from $\mathbb{2}^\omega$ to $M \times M$ defined by $(f,g)(w) = (f(w),g(w))$ is continuous, because we use the product topology on $M \times M$. Therefore $(\Delta \circ (f,g))^{-1}(0)$ is a clopen set, which has a greatest element w (of the shape $m(1)^*$). Thus this w enjoys $f(w) \neq g(w)$ and $f(v) = g(v)$ whenever $v > w$. □

3 The modality and its properties

3.1 The modality

The first attempt to find such a modality, inspired by the product of Q copy of A lead me to consider sequences of finitely many tokens in A, indexed by the rational numbers of $[0,1[$. It works but it has $2^{\aleph_0}$ contraction isomorphisms — hence too many of them. Achim Jung told me binary trees should work and avoid this drawback, and thanks to his suggestion, I arrived to the following:

Definition 2. *Let A be a coherence space. We define $\urcorner A$ as follows:*

web $|\urcorner A| = gt_{|A|}$ *the set of continuous functions from 2^ω to $|A|$ (discrete topology).*

coherence *Two functions f and g of* $\mathrm{gt}_{|\mathrm{A}|} = |\urcorner A|$ *are said to be strictly coherent whenever*

$$\exists w \in 2^\omega f(w) \frown g(w)[A] \text{ and } \forall w' > w f\left(w'\right) = g\left(w'\right)$$

3.2 Properties

The following is clear from proposition 2:

Proposition 4. *(denumerable web) If $|A|$ is denumerable, so is $|\urcorner A|$.*

And next come a key property:

Proposition 5. *(self-duality) The modality $\urcorner$ is self-dual, i.e. $(\urcorner A)^\perp \equiv \urcorner(A^\perp)$*

Proof. Those two coherence spaces obviously have the same web. Hence it is equivalent to show that, given two distinct tokens f,g in $|\urcorner A|$ either $f \frown g[\urcorner A]$ or $f \frown g[\urcorner(A^\perp)]$ holds. If $f \neq g$, then, because of the previous proposition 3, there is an infinite word of 2^ω such that $f(w) \neq g(w)$ and $\forall w' > w.\ f(w') = g(w')$. Therefore, either $[f(w) \frown g(w)[A]$ and $\forall w' > w.\ f(w') = g(w')]$ or $[f(w) \frown g(w)[A^\perp]$ and $\forall w' > w.\ f(w') = g(w')]$ holds. This is component-wise equivalent to the expected exclusive disjunction. □

Proposition 6. *(contraction isomorphism) There is a canonical linear isomorphism $\urcorner A \multimapboth \urcorner A < \urcorner A$*

Proof. Consider the following subset of $|\urcorner A| \times |\urcorner A < \urcorner A|$:

$$\mathcal{C} = \{(h,(h_0,h_1)) \mid \forall w \in 2^\omega\ h(0w) = h_0(w) \text{ and } h(1w) = h_1(w)\}$$

Let us see that it is the trace of a linear isomorphism between $\urcorner A$ and $\urcorner A < \urcorner A$.
Firstly, $\mathcal{C}$ clearly defines a bijection, between the webs $|\urcorner A|$ and $|\urcorner A < \urcorner A| = |\urcorner A| \times |\urcorner A|$.
Secondly, let us see that, given $(h,(h_0,h_1))$ and $(g,(g_0,g_1))$ in $\mathcal{C}$ we have

$$(1): \quad h \frown h'[\urcorner A] \quad \Longleftrightarrow \quad (h_0,h_1) \frown \left(h'_0,h'_1\right)[\urcorner A < \urcorner A] \quad :(2)$$

$(1) \Longrightarrow (2)$ We assume that $h \frown h'[\urcorner A]$, i.e. $\exists w \in 2^\omega\ h(w) \frown g(w)$ and $\forall v > w \quad h(v) = g(v)$. Two cases may occur: either $w = 0w'$ or $w = 1w'$, and in both cases we show that $(h_0,h_1) \frown (g_0,g_1)[\urcorner A < \urcorner A]$

0. If $w = 0w'$ we have $h_0 \frown g_0[\urcorner A]$ and $h_1 = g_1$, and thus $(h_0,h_1) \frown (g_0,g_1)[\urcorner A < \urcorner A]$. Let us check that $h_0 \frown g_0[\urcorner A]$ and that $h_1 = g_1$.
   - $h_0 \frown g_0[\urcorner A]$
     - We have $h_0(w') \frown g_0(w')[A]$ because $h_0(w') = h(0w') = h(w)$ and $g_0(w') = g(0w') = g(w)$ and $h(w) \frown g(w)[A]$

   - We also have $\forall v' > w' h_0(v') = g_0(v')$ because $\forall v' > w'\ 0v' > 0w' = w$ and therefore $h_0(v') = h(0v') = g(0v') = g_0(v')$
 - $h_1 = g_1$ because for all u, we have $1u > 0w' = w$ and therefore $h_1(u) = h(1u) = g(1u) = g_1(u)$.

1. If $w = 1w'$, then we have $h_1 \frown g_1[\urcorner A]$ and thus $(h_0,h_1) \frown (g_0,g_1)[\urcorner A < \urcorner A]$.
   - $h_1(w') \frown g_1(w')[A]$ because $h_1(w') = h(1w') = h(w)$ and $g_1(w') = g(1w') = g(w)$ while we know that $h(w) \frown g(w)[A]$.
   - $\forall v' > w'\ h_1(v') = g_1(v')$ because $\forall v' > w'\ 1v' > 1w' = w$ hence $h_1(v') = h(1v') = g(1v') = g_1(v')$.

(2) $\Longrightarrow$ (1) We assume that $(h_0,h_1) \frown (g_0,g_1)[\urcorner A < \urcorner A]$ i.e. that either ($h_0 \frown g_0[\urcorner A]$ and $h_1 = g_1$) or $h_1 \frown g_1[\urcorner A]$. We show that in both cases we have $h \frown g[\urcorner A]$

0. If $h_0 \frown g_0[\urcorner A]$ and $h_1 = g_1$ then there exists w' such that $h_0(w') \frown g_0(w')[A]$ and $h_0(v') = g_0(v')$ for all $v' > w'$. Let $w = 0w'$
   - $h(w) \frown g(w)[A]$ Indeed we have $h(w) = h(0w') = h_0(w')$ and $g(w) = g(0w') = g_0(w')$ and we know that $h_0(w') \frown g_0(w')[A]$
   - $\forall v > w h(v) = g(w)$ Let $v > w$.
     - If $v = 0v'$ then $v' > w'$ and therefore $h(v) = h(0v') = h_0(v') = g_0(v') = g(0v') = g(v)$
     - If $v = 1v'$ then $h(v) = h(1v') = h_1(v') = g_1(v') = g(1v') = g(v)$
1. If $h_1 \frown g_1[\urcorner A]$ then there exists w' such that $h_1(w')' g_1(w')[A]$ and $h_1(v') = g_1(v')$ for all $v' > w'$. Let $w = 1w'$
   - $h(w) \frown g(w)[A]$ because $h(w) = h(1w') = h_1(w')$ $g(w) = g(1w') = g_1(w')$ and $h_1(w') \frown g_1(w')[A]$.
   - $\forall v > w\ h(v) = g(v)$ indeed, when $v > w$ one has $v = 1v'$ with $v' > w'$ and therefore $h(v) = h(1v') = h_1(v') = g_1(v') = g(1v') = g(v)$.

□

Proposition 7 (A retract of $\urcorner A$)**.** *Any coherence space A is a linear retract of* $\urcorner A$*:*

$$A \quad f_A :\!\circ\!\!-\!\!\!-\!\!\circ\!: t_A \quad \urcorner A$$

Proof. Consider $Tr(f_A) = \{(a,\underline{a})/a \in |A|\} \subset |A| \times |\urcorner A|$, where $\underline{a} \in |\urcorner A|$ stands for the constant function mapping any element of 2^ω to a. Let us call $Tr(t_A)$ the symmetric of $Tr(f_A)$, that is $Tr(t_A) = \{(\underline{a},a)/a \in |A|\} \subset |\urcorner A| \times |A|$. The trace $Tr(f_A)$ is the linear trace of f_A from A to $\urcorner A$ while $Tr(t_A)$ is the linear trace of t_A from $\urcorner A$ to A. The compound $f_A \circ t_A$ is $\{(\underline{a},\underline{a})/a \in |A|\}$ which is a strict subset of $Id_{\urcorner A}$, while the compound $\mathrm{t}_A \circ \mathrm{f}_A$ is exactly Id_A □

Proposition 8 ($\urcorner$ is a functor)**.** *Given* $\ell : A \to B$ *defines* $\urcorner\ell : \urcorner A \to \urcorner B$ *by the following trace:*

$$\urcorner\ell = \{(f,g)/\forall w \in 2^\omega (f(w),g(w)) \in \ell\}$$

This makes $\urcorner$ *an endo-functor.*

Proof. **Firstly, let us show that $\lhd\!|\ell$ defines a linear map from $\lhd\!|A$ to $\lhd\!|B$.** Let $(f,g),(f',g')$ be in $\lhd\!|\ell$.

- Assume that $f' \frown f'[\lhd\!|A]$. Thus there exists w such that $f(w) \frown f'(w)[A]$ and $f(v) = f'(v)$ for all $v > w$. since we know that $(f(w),g(w))$ and $(f'(w),g'(w))$ are in ℓ which is linear, we have $g(w) \mathrel{\overset{\frown}{\smile}} g'(w)[B]$ Now let $v > w$. We have $f(v) = a = f'(v)$ and since both $(a,g(v))$ and $(a,g'(v))$ are in ℓ which is linear we have $g(v) \mathrel{\overset{\frown}{\smile}} g'(v)[B]$. Applying proposition 3, there exists an u such that $g(u) \neq g'(u)$ and $g(t) = g'(t)$ for all $t > u$. We necessarily have $u \geqslant w$ and therefore $g(u) \frown g'(u)[B]$ Hence $g \frown g'[\lhd\!|B]$
- Assume $f = f'$. For all w, both $(f(w),g(w))$ and $(f(w),g'(w))$ are in ℓ which is linear. Therefore, for all w, one has $g(w) \mathrel{\overset{\frown}{\smile}} g'(w)[B]$. Applying proposition 3 either $g = g'$ or there exists an u such that $g(u) \neq g'(u)$ and $g(v) = g'(v)$ for all $v > u$. In the second case we have $g(u) \frown g'(u)[B]$ since $g(w) \mathrel{\overset{\frown}{\smile}} g'(w)[B]$ for all w. In both case we have $g \mathrel{\overset{\frown}{\smile}} g'[\lhd\!|B]$

It is easily seen that $\lhd\!|Id_A = Id_{\lhd\!|A}$
Let us finally show that $\lhd\!|$ commutes with linear composition. Let $\ell : A \multimap B$ and $\ell' : B \multimap C$, be two linear morphisms.

- Assuming that (f,h) is in $(\lhd\!|\ell') \circ (\lhd\!|\ell)$ it is easily seen that (f,h) is in $\lhd\!|(\ell' \circ \ell)$ Indeed there exists a g in $|\lhd\!|B|$ such that (f,g) is in $\lhd\!|\ell$ and (g,h) is in $\lhd\!|\ell'$. Thus, for all w the pair $(f(w),g(w))$ is in ℓ and the pair $(g(w),h(w))$ is in ℓ', so that $(f(w),h(w))$ is in $\ell' \circ \ell$ for all w, i.e. (f,h) is in $\lhd\!|(\ell' \circ \ell)$
- We now assume that (f,h) is in $\lhd\!|(\ell' \circ \ell)$ and show that it is in $(\lhd\!|\ell') \circ (\lhd\!|\ell)$ too. If (f,h) is in $\lhd\!|(\ell' \circ \ell)$ then $(f(w),h(w))$ is in $\ell' \circ \ell$ for all w, i.e. for all w there exists some $g(w)$ such that $(f(w),g(w))$ is in ℓ and $((g(w),h(w))$ is in ℓ'. The point is to show that among the functions from 2^ω to $|B|$ that this existential quantifier may define, one of them actually is an element of $|\lhd\!|B|$. Consider the function (f,h) from 2^ω to $|A| \times |C|$ with the discrete topology on $|A| \times |C|$. It is continuous, because the product topology of a finite product of discrete spaces is the discrete topology on the (finite) product of the involved sets. Therefore it may be described as a finite binary tree, with leaves in $|A| \times |C|$. We write it as a finite set $\{(m_i,(a_i,c_i))\}$ with the properties (1a) and (1b) of proposition 2.
  For all i, there exists w such that $f(w) = a_i$ and $h(w) = c_i$ take e.g. $w = m_i(0)^*$. Therefore for all i there exists b_i in $|B|$ such that (a_i,b_i) is in ℓ and (b_i,c_i) is in ℓ', and for each i we choose one (there are finitely many i). Now, we can define $g(w)$ with the help of 1a. For each w there exists a unique i such that $w = m_i w'$, and we define $g(w)$ to be b_i, and thus g is clearly a continuous function from 2^ω to $|B|$. Notice that the generic tree of g is not necessarily $\{(m_i,b_i)\}$ but *its normal form* according to 2 of proposition 2: the property 1b may fail since there possibly exist i,j and m such that $m_i = m0, m_j = m1$ while $b_i = b_j$. Now it is easily seen that (f,g) is in $\lhd\!|\ell$ and (g,h) in $\lhd\!|\ell'$. Indeed for all w there exists a unique i such that $w = m_i w'$ and we then have $f(w) = a_i, g(w) = b_i$ $h(w) = c_i$ and thus $(f(w),g(w)) = (a_i,b_i)$ is in ℓ and $(g(w),h(w)) = (b_i,c_i)$ is in ℓ'

□

4 The modality in the category of hypercoherences

The construction is very similar, we shall be brief, and closely refer to [1]:

Definition 3. *Let X and Y be hypercoherences. The hypercoherence $X < Y$ is the hypercoherence whose web is $|X| \times |Y|$ and whose strict atomic coherence is defined by:*

$$w \in \Gamma^*(X).\ \mathit{iff}\ .((\pi_1(w) \in \Gamma^*(X)\ \mathit{and}\ \#\pi_2(x) = 1)\ \mathit{or}\ \pi_2(w) \in \Gamma^*(X))$$

It is easily seen that this connective is associative, self-dual, non-commutative and in between the tensor product and the par – just like in the category of coherence spaces.

Now, the self-dual modality enjoying the wanted properties is defined in the category of hypercoherences by:

Definition 4. *Let X be an hypercoherence. The hypercoherence* $\urcorner X$ *is the hypercoherence whose web is:*

$$|\urcorner X| = \left\{ f \in \mathrm{C}\left(2^{\omega}, |X|\right) | f\left(2^{\omega}\right) \in \mathcal{P}_{fin}(|X|) \right\}$$

and whose strict atomic coherence is defined by:

$$\{f_1, \ldots, f_k\} \in \Gamma^*(X).\ \mathit{iff}\ .\exists m \in 2^{\omega} \begin{cases} \{f_1(m), \ldots, f_k(m)\} \in \Gamma^*(X) \\ \quad \mathit{and} \\ \forall m' > m \quad \#\{f_1(m'), \ldots, f_k(m')\} = 1 \end{cases}$$

The proofs that the hypercoherence version of $\urcorner$ enjoys the same properties as the coherence version studied in the previous section are the same *mutatis mutandis*.

5 Conclusion

A challenging issue is to define a deductive system enjoying cut elimination including a syntactical match of the self dual modality presented in this paper, that is a calculus for non commutative contraction and duplication. This logical calculus could be defined as an extension of the calculus of structures with deep inference (roughly speaking, internal rewriting) [7, 10, 8, 11] or with pomset proof nets with or without links [17, 19, 20], or with one of the sequent calculi introduced for pomset logic [14, 17, 20] not to forget the proposal by Sergey Slavnov [21] which is complete w.r.t. pomset proof nets.

The ongoing syntactic work by Alessio Guglielmi with a self dual modality [9] looks quite appealing.

We think that this ancient work may give guidelines fro defining such a deductive system as coherence semantics often did for linear logic. Such a logic would be a multiplicative exponential non commutative linear logic close to classical logic, as MELL is to intuitionnistic logic.

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