A TERM IN $PCF_{\oplus}$ FOR THE RANDOM WALK

$$RW = \left(\text{fix } f.\ \lambda x.\ \text{if } x \text{ then } * \text{ else } \begin{array}{l} (\text{let } pred(x) = y \text{ in } f y) \oplus \\ (\text{let } succ(x) = y \text{ in } f y) \end{array}\right) 1$$

$$RW' = (\text{fix } f.\ \lambda x.\ \text{if } x \text{ then } * \text{ else let } pred(x) \oplus succ(x) = y \text{ in } f y)\ 1$$

$$enc(M\,N) = \text{let } M = x \text{ in let } N = y \text{ in } x y$$
$$enc'(M\,N) = \text{let } N = x \text{ in let } M = y \text{ in } y x$$

A TERM FOR MINIMIZATION IN PCF

$$f : \mathbb{N}^{k+1} \to \mathbb{N} \qquad g : \mathbb{N}^k \rightharpoonup \mathbb{N}$$

$$g(\bar{x}) = \mu y.\ (f(y, \bar{x}) = 0) \qquad M_f$$

$$M_g = \left(\text{fix } g.\ \lambda x_0.\lambda x_1 \ldots \lambda x_k.\ \begin{array}{l} \text{let } M_f\, x_0\, x_1 \ldots x_k = y \text{ in} \\ \quad \text{if } y \text{ then } x_0 \text{ else} \\ \quad\quad \text{let } succ(x_0) = z \text{ in} \\ \quad\quad\quad g\, z\, x_1, \ldots x_k \end{array}\right) 0$$

$$enc(\text{if } L \text{ then } M \text{ else } N) = \text{let } L = x \text{ in if } x \text{ then } M \text{ else } N$$

ON THE FAILURE OF CONFLUENCE IN THE PROBABILISTIC SETTING

$$M = (\lambda x.\lambda y.x)(II)$$

$$(\lambda x.\lambda y.x)(II) \xrightarrow{\text{value}} (\lambda x.\lambda y.x)\, I \xrightarrow{\text{value}} \lambda y. I$$

$$(\lambda x.\lambda y.x)(II) \xrightarrow{\text{name}} \lambda y.(II)$$

FOR TERMS OF GROUND TYPE IN PCF, WE CAN PROVE UNICITY OF NORMAL FORMS:

IF $\vdash M : NUM$ AND

$M \Rightarrow_n V$ IN CBV, WHILE

$M \Rightarrow_m W$ IN CBN, THEN

$V = W$

· IN A RANDOMIZED SETTING, EVEN THIS VERY RESTRICTED NOTION OF UNF FAILS:

$$\overbrace{(\lambda x.\, sum\_mod\_2\,(x,x))}^{N}(0 \oplus 1)$$

$$\xrightarrow{\text{value}} \{N0 : \tfrac{1}{2},\ N1 : \tfrac{1}{2}\}$$

$N0 \longrightarrow sum\_mod\_2(0,0) \longrightarrow 0$

$N1 \longrightarrow sum\_mod\_2(1,1) \longrightarrow 0$

$$\longrightarrow \{sum\_mod\_2\,(0\oplus 1, 0\oplus 1) : 1\}$$

$$\longrightarrow \{sum\_mod\_2\,(0,0) : 1/4,\ sum\_mod\_2\,(0,1) : 1/4,\ sum\_mod\_2\,(1,0) : 1/4,\ sum\_mod\_2\,(1,1) : 1/4\}$$

$$\longrightarrow \{0 : 1/2,\ 1 : 1/2\}$$

EVERY POSITIVELY ALMOST-SURELY TERMINATING TERM IS ALMOST-SURELY TERMINATING.

$$\text{Explength}(M) = \sum_{m=0}^{\infty} \left(1 - \sum_{n=0}^{m} \sum \langle M \rangle_n\right)$$

A NECESSARY CONDITION FOR Explength TO BE FINITE

$$\lim_{m\to\infty} \sum_{n=0}^{m} \sum \langle M \rangle_n = 1$$

$$\|$$

$$\lim_{m\to\infty} \sum \sum_{n=0}^{m} \langle M \rangle_n = 1$$

$$\downarrow$$

$$\sum \sum_{n=0}^{\infty} \langle M \rangle_n = 1 \Rightarrow \sum \langle M \rangle = 1$$

$$\text{AST} \iff \sum \langle M \rangle = 1$$

$$\text{PAST} \Rightarrow \lim_{m\to\infty} \sum \sum_{n=0}^{m} \langle M \rangle_n = 1$$

ST IS TERMINATING

- WE GIVE A PROOF OF TERMINATION FOR ST BY WAY OF THE REDUCIBILITY TECHNIQUE

$$M \in \mathbb{CT}_\tau \Rightarrow \exists n.\ M \Rightarrow_n V$$

WE DEFINE, FIRST OF ALL, SUBSETS OF TERMS AND VALUES, INDEXED BY TYPES:

$$\mathbb{RCV}_\tau \subseteq \mathbb{CV}_\tau \qquad \mathbb{RCT}_\tau \subseteq \mathbb{CT}_\tau$$

$$\mathbb{RCV}_{\text{UNIT}} = \{*\} \qquad \mathbb{RCV}_{\text{NUM}} = \{r \mid r \in \mathbb{N}\}$$

$$\mathbb{RCV}_{\tau \to \rho} = \{\lambda x.M \in \mathbb{CV}_{\tau \to \rho} \mid M[V/x] \in \mathbb{RCT}_\rho \ \forall V \in \mathbb{RCV}_\tau\}$$

$$\mathbb{RCT}_\tau = \{M \in \mathbb{CT}_\tau \mid \exists n.\ M \Rightarrow_n V \text{ AND } V \in \mathbb{RCV}_\tau\}$$

<u>LEMMA</u>

IF $M \in \mathbb{RCT}_\tau$ THEN M IS TERMINATING.

<u>LEMMA</u>

$$\mathbb{CT}_\tau \subseteq \mathbb{RCT}_\tau$$

PROOF.

- YOU CANNOT HOPE TO PROVE THE LEMMA DIRECTLY, AND YOU HAVE TO STRENGTHEN THE STATEMENT AS FOLLOWS:

  FOR EVERY $x_1 : \tau_1, \ldots x_n : \tau_n \vdash M : \rho$ IT HOLDS THAT IF $V_i \in \mathbb{RCV}_{\tau_i}$ THEN $M[V_1/x_1, \ldots V_n/x_n] \in \mathbb{RCT}_\rho$

- THE STATEMENT ABOVE IS PROVED BY INDUCTION ON THE STRUCTURE OF THE PROOF THAT $x_1:\tau_1, \dots, x_n:\tau_n \vdash M:\rho$. SOME CASES
  - IF $M = x_i$, THEN

$$M[V_1/x_1, \dots, V_n/x_n] = V_i \in \mathbb{RCV}_{\tau_i} \subseteq \mathbb{RCT}_{\tau_i}$$

  - IF $M = WY$ THEN

$$\frac{\Gamma \vdash W : \theta \to \rho \qquad \Gamma \vdash Y : \theta}{\Gamma \vdash WY : \rho}$$

$$WY[V_1/x_1, \dots, V_n/x_n] = \underbrace{W[V_1/x_1, \dots, V_n/x_n]}_{\mathbb{RCV}_{\theta\to\rho}} \underbrace{Y[V_1/x_1, \dots, V_n/x_n]}_{\mathbb{RCV}_{\theta}}$$

SUPPOSE THAT $W = \lambda x.N$ (OTHERWISE, EXERCISE). YOU CAN SAY THAM

$$(N[Y_1/x_1, \dots, Y_n/x_n])[Y[V_1/x_1, \dots, V_n/x_n]/x] \in \mathbb{RCT}_\rho^{\,n}$$

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$$M[Y_1/x_1, \dots Y_n/x_n] \in \mathbb{RCT}_\rho^{\,n+1}$$

- PROVE THAT:
  - GEO IS PAST

- THERE EXISTS A TERM $V : \mathrm{NUM} \to \mathrm{NUM}$ SUCH THAT $V \in \mathrm{PCF}$ AND $V r$ I TERMINATING FOR EVERY r AND $V(\mathrm{GEO})$ IS <u>NOT</u> PAST