

A TERM IN PCF₀ FOR THE RANDOM WALK

$$RW = \left(\text{fix } f. \lambda x. \text{if } x \text{ then } * \text{ else } \left(\begin{array}{l} \text{let pred}(x)=y \text{ in } f y \\ \text{let succ}(x)=y \text{ in } f y \end{array} \right) \right) 1$$

$$RW' = \left(\text{fix } f. \lambda x. \text{if } x \text{ then } * \text{ else } \text{let pred}(x) \oplus \text{succ}(x)=y \text{ in } f y \right) 1$$

$$\text{enc}(MN) = \text{let } M=x \text{ in let } N=y \text{ in } x y$$

$$\text{enc}'(MN) = \text{let } N=x \text{ in let } M=y \text{ in } (y x)$$

A TERM FOR MINIMIZATION IN PCF

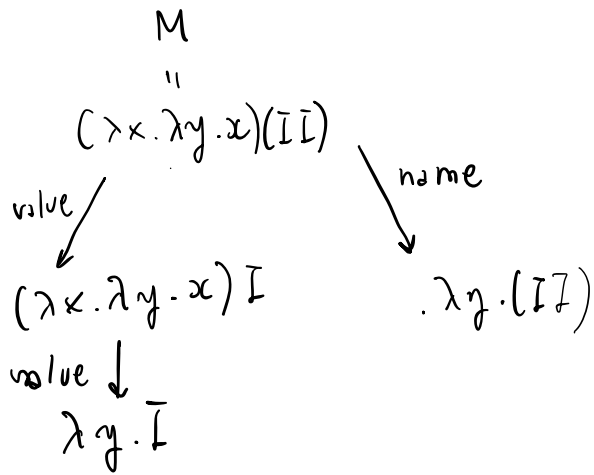
$$f: N^{k+1} \rightarrow N \quad g: N^k \rightarrow N$$

$$f(\bar{x}) = \mu y. (f(y, \bar{x}) = 0) \quad M_f$$

$$M_g = \left(\text{fix } g. \lambda x_0. \lambda x_1. \dots \lambda x_k. \left(\begin{array}{l} \text{let } M_f x_0 x_1 \dots x_k = y \text{ in} \\ \text{if } y \text{ then } x_0 \text{ else} \\ \text{let succ}(x_0) = z \text{ in} \\ g z x_1 \dots x_k \end{array} \right) 0 \right)$$

$$\text{enc}(\text{if } L \text{ then } M \text{ else } N) = \text{let } L=x \text{ in if } x \text{ then } M \text{ else } N$$

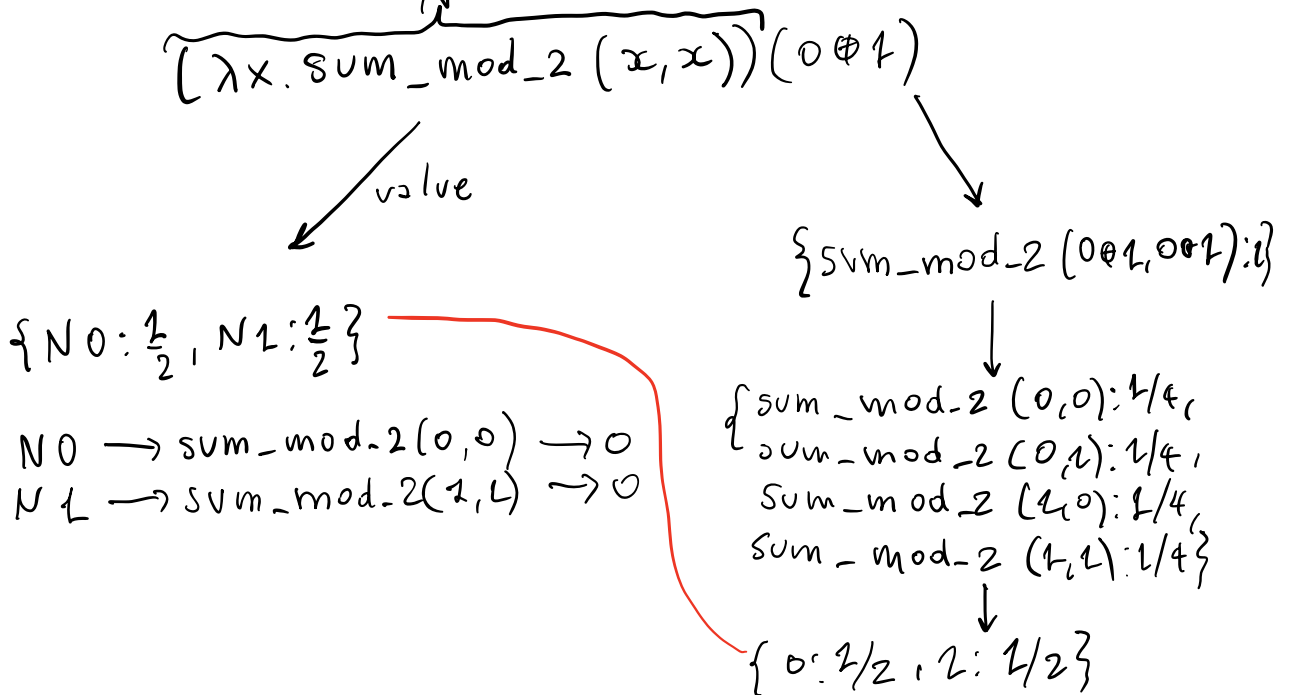
ON THE FAILURE OF CONFLUENCE IN THE PROBABILISTIC SETTING



FOR TERMS OF
GROUND TYPE IN PCF, WE
CAN PROVE UNICITY
OF NORMAL FORMS:

IF $\vdash M; \text{NUM}$ AND
 $M \Rightarrow_n V$ IN CBV, WHERE
 $M \Rightarrow_m W$ IN CBN, THEN
 $V = W$

IN A RANDOMIZED SETTING, EVEN THIS VERY
RESTRICTED NOTION OF UNF FAILS:



EVERY POSITIVELY ALMOST-SURELY
TERMINATING TERM IS ALMOST-SURELY
TERMINATING.

$$\text{ExpLength}(M) = \sum_{m=0}^{\infty} \left(1 - \sum_{n=0}^m \langle M \rangle_n \right)$$

A NECESSARY CONDITION FOR ExpLength TO
BE FINITE

$$\lim_{m \rightarrow \infty} \sum_{n=0}^m \langle M \rangle_n = 1$$

$$\lim_{m \rightarrow \infty} \sum_{n=0}^m \langle M \rangle_n = 1$$

$$\downarrow$$

$$\sum_{n=0}^{\infty} \langle M \rangle_n = 1 \Rightarrow \langle M \rangle = 1$$

$$\text{AST} \Leftrightarrow \langle M \rangle = 1$$

$$\text{PAST} \Rightarrow \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle M \rangle_n = 1$$

ST IS TERMINATING

WE GIVE A PROOF OF TERMINATION FOR ST
BY WAY OF THE REDUCIBILITY TECHNIQUE

$$M \in \mathcal{C}\Pi_T \Rightarrow \exists v. M \Rightarrow_v V$$

WE DEFINE, FIRST OF ALL, SUBSETS OF TERMS AND VALUES, INDEXED BY TYPES:

$$\mathcal{R}\mathcal{C}\mathcal{V}_T \subseteq \mathcal{C}\mathcal{V}_T \quad \mathcal{R}\mathcal{C}\Pi_T \subseteq \mathcal{C}\Pi_T$$

$$\mathcal{R}\mathcal{C}\mathcal{V}_{\text{UNIT}} = \{\star\} \quad \mathcal{R}\mathcal{C}\mathcal{V}_{\text{NUM}} = \{v \mid v \in \mathbb{N}\}$$

$$\mathcal{R}\mathcal{C}\mathcal{V}_{T \rightarrow P} = \left\{ \lambda x. M \in \mathcal{C}\mathcal{V}_{T \rightarrow P} \mid \begin{array}{l} M[V/x] \in \mathcal{R}\mathcal{C}\Pi_P \\ \forall V \in \mathcal{R}\mathcal{C}\mathcal{V}_T \end{array} \right\}$$

$$\mathcal{R}\mathcal{C}\Pi_T = \left\{ M \in \mathcal{C}\Pi_T \mid \exists v. M \Rightarrow_v V \text{ AND } v \in \mathcal{R}\mathcal{C}\mathcal{V}_T \right\}$$

LEMMA

IF $M \in \mathcal{R}\mathcal{C}\Pi_T$ THEN M IS TERMINATING.

LEMMA

$$\mathcal{C}\Pi_T \subseteq \mathcal{R}\mathcal{C}\Pi_T$$

PROOF.

• YOU CANNOT HOPE TO PROVE THE LEMMA DIRECTLY, AND YOU HAVE TO STRENGTHEN THE STATEMENT AS FOLLOWS:

FOR EVERY $x_1:T_1, \dots, x_n:T_n \vdash M:P$ IT HOLDS THAT IF $v_i \in \mathcal{R}\mathcal{C}\mathcal{V}_{T_i}$ THEN $M[v_1/x_1, \dots, v_n/x_n] \in \mathcal{R}\mathcal{C}\Pi_P$

THE STATEMENT ABOVE IS PROVED BY INDUCTION ON THE STRUCTURE OF THE PROOF THAT $x_1: \tau_1, \dots, x_n: \tau_n \vdash M: P$.
SOME CASES

IF $M = x_i$, THEN

$$M[V_1/x_1, \dots, V_n/x_n] = V_i \in \text{RCV}_{\tau_i} \subseteq \text{RCT}_{\tau_i}$$

IF $M = WY$ THEN

$$\frac{\Gamma \vdash W: \theta \rightarrow P \quad \Gamma \vdash Y: \theta}{\Gamma \vdash WY: P}$$

$$WY[V_1/x_1, \dots, V_n/x_n] =$$

$$W[V_1/x_1, \dots, V_n/x_n] Y[V_1/x_1, \dots, V_n/x_n]$$

$$\uparrow \text{RCV}_{\theta \rightarrow P}$$

$$\uparrow \text{RCV}_{\theta}$$

SUPPOSE THAT $W = \lambda x. N$ (OTHERWISE, EXERCISE) YOU CAN SAY THAT

$$(N[Y_1/x_1, \dots, Y_n/x_n])[Y[V_1/x_1, \dots, V_n/x_n]/x] \in \text{RCT}_{\tau_P}$$

$$M[V_1/x_1, \dots, V_n/x_n] \in \text{RCT}_{\tau_P}$$

PROVE THAT:

GEO IS PAST

· THERE EXISTS A TERM $V: \text{NUM} \rightarrow \text{NUM}$
SUCH THAT $\forall e \text{ PCF}$ AND $\forall r \mid$
TERMINATING FOR EVERY r AND
 $V(\text{GEO})$ IS NOT PAST