On Randomization in (Higher-Order) Programming

Part III

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Equivalence and Distance Checking

- How could we check two higher-order probabilistic programs to be observationally equivalent?

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- How could we check two higher-order probabilistic programs to be observationally equivalent?
- How about their distance?
- Contextual equivalence and contextual distance are good answers, *definitionally*.
  - They are the coarsest compatible and adequate relation and metric between programs.
  - There is however an **explicit universal quantification** over all contexts, which make arguments inherently complicated.

Examples from $\Lambda_{\oplus}$

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Examples from $\Lambda_\oplus$

$\lambda x.x$

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Examples from $\Lambda_{\oplus}$

$\Delta\Delta = (\lambda x.xx)(\lambda x.xx)$

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Examples from $\Lambda_{\oplus}$

Not Context Equivalent: $C = [\cdot]$.

Context Distance? Consider $C_n = (\lambda x.\ \underbrace{x \ldots x}_{n \text{ times}})[\cdot]$.

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Examples from $\Lambda_\oplus$

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

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Examples from $\Lambda_{\oplus}$

Not Context Equivalent: $C = [\cdot]$.
Context Distance? Cannot Easily Amplify.

$I \oplus \Omega$ vs. I

$I \oplus \Omega$ vs. Ω

Examples from $\Lambda_\oplus$

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$$(\lambda x.I) \oplus (\lambda x.\Omega) \qquad \text{vs.} \qquad \lambda x.I \oplus \Omega$$

Examples from $\Lambda_\oplus$

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Not Context Equivalent: $C = (\lambda x.x(xI))[\cdot]$
Apparently Context Equivalent in $\Lambda_\oplus^{\mathsf{name}}$.

$$I \oplus \Omega \qquad \text{vs.} \qquad \Omega$$

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$$Y_1 \qquad \text{vs.} \qquad Y_2$$

Examples from $\Lambda_\oplus$

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

$$Y_1 M \to^* M(Y_2 M) \oplus M(Y_3 M)$$
$$Y_2 M \to^* M(Y_1 M) \oplus M(Y_3 M)$$
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Equivalence Through Bisimulation

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- A **Labelled Transition System** is a triple $(\mathcal{S}, \mathcal{L}, \rightarrow)$ such that:
  - $\mathcal{S}$ is a countable set of *states*;
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  - $\rightarrow$ is a subset of $\mathcal{S} \times \mathcal{L} \times \mathcal{S}$, namely a nondeterministic transition function.

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- A relation $\mathcal{R} \subseteq \mathcal{S} \times \mathcal{S}$ is said to be a *bisimulation* iff, whenver $s \mathrel{\mathcal{R}} t$, it holds that:
  1. If $s \stackrel{a}{\rightarrow} q$, then $t \stackrel{a}{\rightarrow} r$, where $q \mathrel{\mathcal{R}} r$;
  2. If $t \stackrel{a}{\rightarrow} r$, then $s \stackrel{a}{\rightarrow} q$, where $q \mathrel{\mathcal{R}} r$.

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  2. If $t \xrightarrow{a} r$, then $s \xrightarrow{a} q$, where $q \,\mathcal{R}\, r$.
- If we drop clause 2. above, we get a *simulation*.

Probabilistic Bisimulation in the Abstract [LS1992]

- **Labelled Markov Chain** (LMC): a triple $\mathcal{M} = (\mathcal{S}, \mathcal{L}, \mathcal{P})$, where
  - $\mathcal{S}$ is a countable set of *states*;
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  - $\mathcal{P}$ is a *transition probability matrix*, i.e., a function $\mathcal{P} : \mathcal{S} \times \mathcal{L} \times \mathcal{S} \to \mathbb{R}$ such that for every state t and for every label ℓ, $\mathcal{P}(t, \ell, \mathcal{S}) = \sum_{s \in \mathcal{S}} \mathcal{P}(t, \ell, s) \leq 1$;

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- Variation: **Simulation**, which is required to be a preorder.
- **Bisimilarity** and **Similarity**, namely the greatest bisimulation and simulation relations, can always be formed.
- The challenge now is to find **a good enough LMC**, namely one which models randomized calculi faithfully.
  - We will work with $\Lambda_\oplus$.

A Labelled Markov Chain for $\Lambda_\oplus$

Terms

A Labelled Markov Chain for $\Lambda_{\oplus}$

Terms **Values**

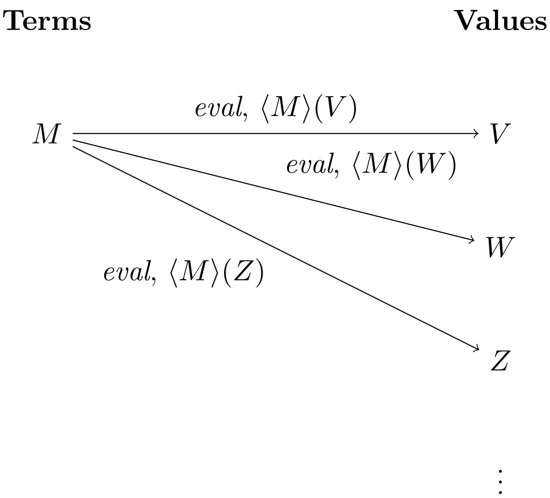
A Labelled Markov Chain for $\Lambda_{\oplus}$

Terms

Values

M

A Labelled Markov Chain for $\Lambda_{\oplus}$



A Labelled Markov Chain for $\Lambda_{\oplus}$

Terms	**Values**
	$\lambda x.N$

A Labelled Markov Chain for $\Lambda_{\oplus}$

Terms		**Values**
$N[W/x]$	$\xleftarrow{W,\ 1}$	$\lambda x.N$

Probabilistic Applicative Bisimulation

$$\lambda x.M \;\; \mathcal{R} \;\; \lambda x.N$$

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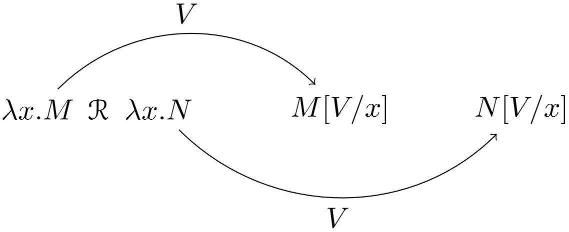
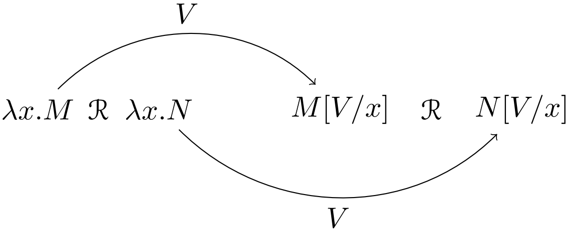
$$\lambda x.M \;\; \mathcal{R} \;\; \lambda x.N$$

$$\lambda x.M \xrightarrow{V} M[V/x]$$

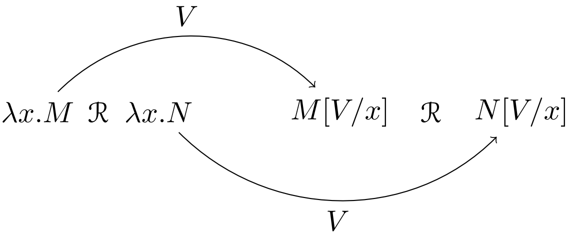
Probabilistic Applicative Bisimulation



Probabilistic Applicative Bisimulation

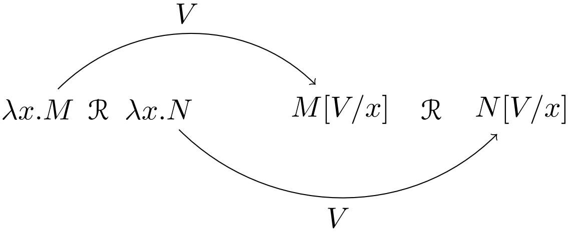


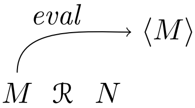
Probabilistic Applicative Bisimulation



$$M \ \mathcal{R} \ N$$

Probabilistic Applicative Bisimulation





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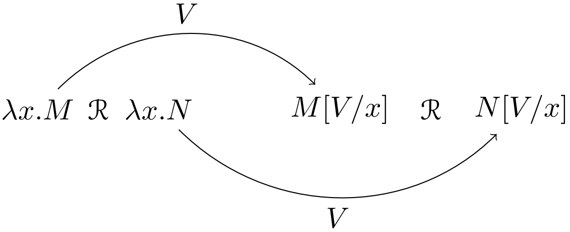

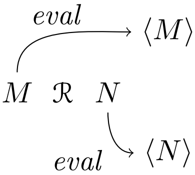

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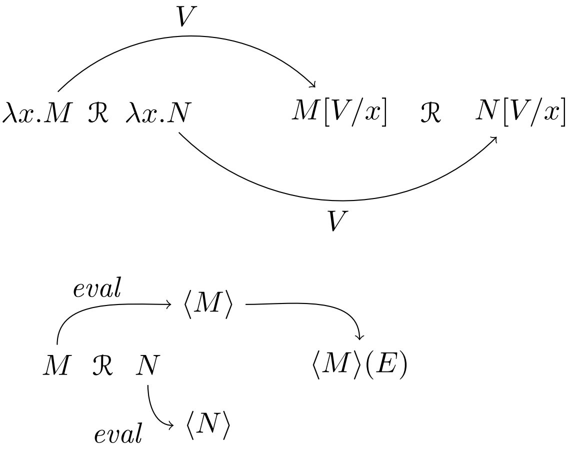

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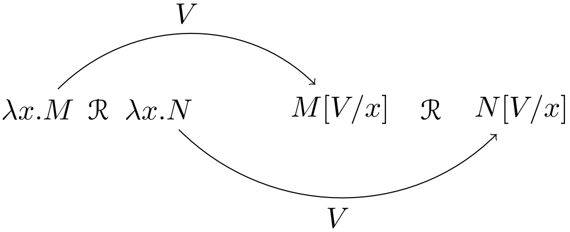

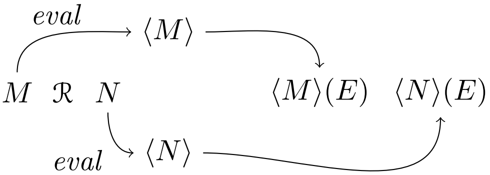

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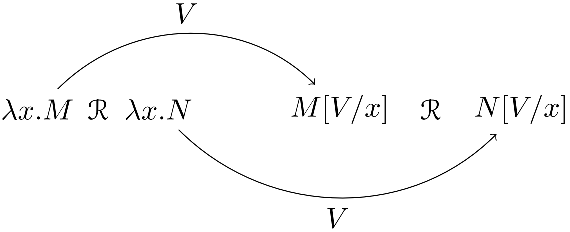

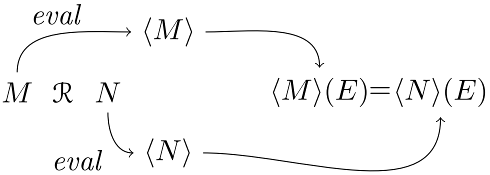

Applicative Bisimilarity vs. Context Equivalence

- **Bisimilarity**: the union $\sim$ of all bisimulation relations.
- **Similarity**: the union $\lesssim$ of all simulation relations.
- Is it that $\sim$ is included in $\equiv$? How to prove it?
- Natural strategy: is $\sim$ a congruence?
  - If this is the case:

$$M \sim N \implies C[M] \sim C[N] \implies \sum \langle C[M] \rangle = \sum \langle C[N] \rangle$$
$$\implies M \equiv N.$$

  - This is a necessary sanity check anyway.
- The naïve proof by induction **fails**, due to application: from $M \sim N$, one cannot directly conclude that $LM \sim LN$.

Howe's Technique

$\mathcal{R}$ $\mathcal{R}^H$

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$$\mathcal{R} \xrightarrow{\subseteq} \mathcal{R}^{H}$$

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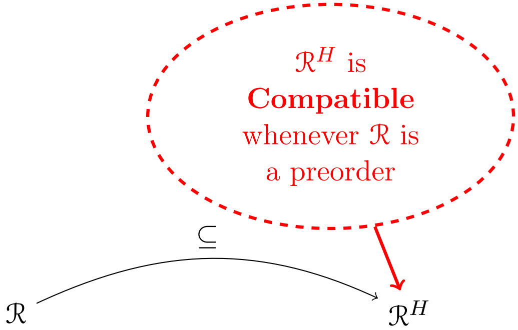

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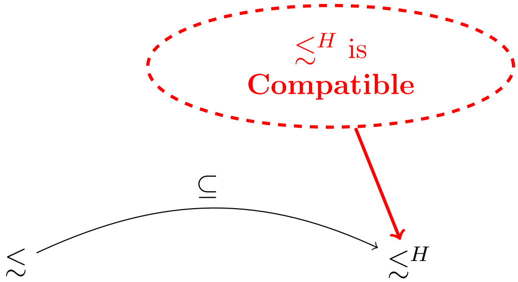

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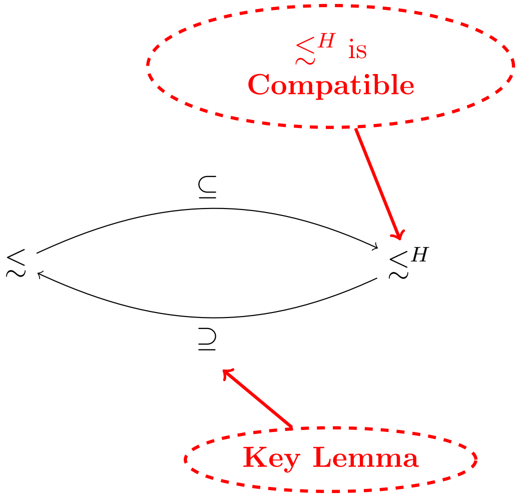

Our Neighborhood

- Λ, where we observe **convergence**

	$\sim \subseteq \equiv$	$\equiv \subseteq \sim$
CBN	✓	✓
CBV	✓	✓

[Abramsky1990, Howe1993]

- $\Lambda_{\oplus}$ with nondeterministic semantics, where we observe **convergence**, in its **may** or **must** flavors.

	$\sim \subseteq \equiv$	$\equiv \subseteq \sim$
CBN	✓	×
CBV	✓	×

[Ong1993, Lassen1998]

The Probabilistic Case

- $\Lambda_{\oplus}$ with probabilistic semantics.

	$\sim \subseteq \equiv$	$\equiv \subseteq \sim$
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- Counterexample for CBN: $(\lambda x.I) \oplus (\lambda x.\Omega) \not\sim \lambda x.I \oplus \Omega$
- **Where** these discrepancies come from?

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- From **testing**!

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- **Where** these discrepancies come from?
- From **testing**!
- Bisimulation can be characterized by testing equivalence as follows:

Calculus	**Testing**
Λ	$T ::= \omega \mid a \cdot T$
probabilistic $\Lambda_{\oplus}$	$T ::= \omega \mid a \cdot T \mid \langle T, T \rangle$
nondeterministic $\Lambda_{\oplus}$	$T ::= \omega \mid a \cdot T \mid \wedge_{i \in I} T_i \mid \ldots$

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	$\precsim \subseteq \leq$	$\leq \subseteq \precsim$
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- Full abstraction can be recovered if endowing $\Lambda_\oplus$ with parallel disjunction [CDLSV2015].

	$\precsim \subseteq \leq$	$\leq \subseteq \precsim$
CBN	✓	×
CBV	✓	✓

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Context Distance: the Affine Case [CDL2015]

$$\frac{}{\Gamma, x \vdash x} \qquad \frac{x, \Gamma \vdash M}{\Gamma \vdash \lambda x.M} \qquad \frac{\Gamma \vdash M \qquad \Delta \vdash N}{\Gamma, \Delta \vdash MN} \qquad \frac{\Gamma \vdash M \qquad \Gamma \vdash N}{\Gamma \vdash M \oplus N}$$

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  - The maximum distance induced by traces, i.e., sequences of actions: $\delta^t(M, N) = \sup_{\mathsf{T}} |Pr(M, \mathsf{T}) - Pr(N, \mathsf{T})|$.

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- **Soundness and Completeness Results:**

$\delta^b \leq \delta^c$	$\delta^c \leq \delta^b$	$\delta^t \leq \delta^c$	$\delta^c \leq \delta^t$
✓	✗	✓	✓

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✓	×	✓	✓

- **Example**: $\delta^t(I, I \oplus \Omega) = \delta^t(I \oplus \Omega, \Omega) = \frac{1}{2}$.

Logical Relations for $\mathsf{ST}_{\oplus}$

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For example,

$$\mathbb{RCV}_{\tau \to \rho} = \{(\lambda x.M, \lambda x.N) \mid (V, W) \in \mathbb{RCV}_{\tau} \Longrightarrow (M[V/x], N[W/x]) \in \mathbb{RCT}_{\rho}\}$$

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$$\mathbb{RCT}_\tau \subseteq \mathbb{CT}_\tau \times \mathbb{CT}_\tau \qquad \mathbb{RCV}_\tau \subseteq \mathbb{CV}_\tau \times \mathbb{CV}_\tau$$

  For example,

$$\mathbb{RCV}_{\tau\to\rho} = \{(\lambda x.M, \lambda x.N) \mid (V,W) \in \mathbb{RCV}_\tau \implies (M[V/x], N[W/x]) \in \mathbb{RCT}_\rho\}$$

  But how about $\mathbb{RCT}_\tau$? This requires a little bit of ingenuity to be defined.
- The key step towards proving compatibility is the following:

Theorem (Fundamental Lemma)

For every τ, the relation $\mathbb{RCT}_\tau$ is reflexive.

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  - Inherently difficult endeavour, because the category of measurable spaces is not cartesian closed.

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- **Game and GoI Models** [DanosHarmer2002, DLFVY2017].
  - Higher-order programs are interpreted as strategies or automata.
  - Game models are fully abstract, in presence of states.

Thank You!

Questions?