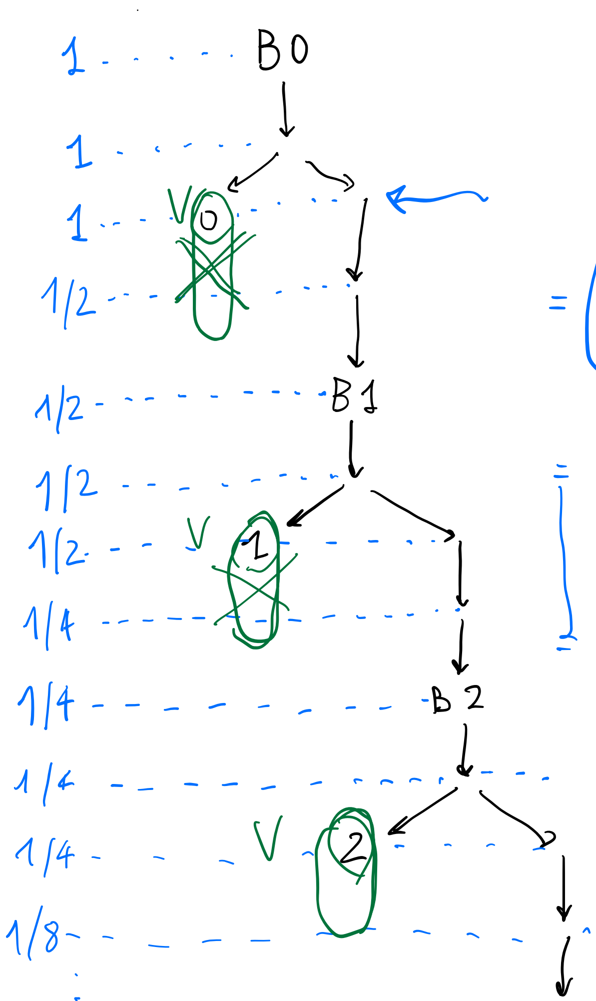
$$\text{GEO} = \overbrace{\left(\text{fix } f.\lambda x.\, x \oplus (\text{let } succ(x)=y \text{ in } f\,y)\right)}^{B} 0$$



$$3\cdot 1 + 4\cdot\frac{1}{2} + 4\cdot\frac{1}{4} + 4\cdot\frac{1}{8} + \dots =$$

$$= \left(4\sum_{n=0}^{\infty}\frac{1}{2^n}\right) - 1 =$$

$$= 4\cdot\frac{1}{1-\frac{1}{2}} - 1 =$$

$$= 4\cdot\frac{1}{1/2} - 1 = 4\cdot 2 - 1 = 7$$

$V : \text{NUM} \to \text{NUM}$

$V\,r \Rightarrow 2^r$.

$V = \text{EXP} = \text{fix } f.\, \lambda x.\ \text{if } x \text{ then } 1 \text{ else}$
$\quad \text{let } pred(x) = y \text{ in}$
$\quad \text{let } f\,y = z \text{ in}$

DOUBLE z

$$
\begin{aligned}
\text{DOUBLE} = \text{fix } f.\ \lambda x.\ \text{if } x \text{ then } 0 \text{ else } & \\
& \text{let } pred(x) = y \text{ in} \\
& \text{let } f\ y = z \text{ in} \\
& \text{let } succ(z) = w \text{ in} \\
& \text{let } succ(w) = p \text{ in} \\
& p
\end{aligned}
$$

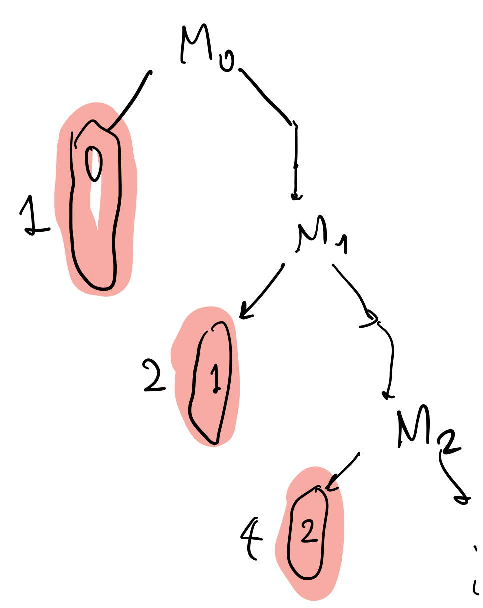
ONE CAN PROVE THAT $\forall n.\ \exists m \geq 2^n.$

$$\text{EXP}\ n \Rightarrow_m V$$

. LET US CONSIDER WHAT HAPPENS WHEN WE REDUCE

$$\text{let GEO} = y \text{ in } (\text{EXP}\ y)$$

WE CAN BUILD A TREE VERY MUCH LIKE THE ONE FOR GEO IN WHICH HOWEVER, THE LEAVES $0, 1, 2, \ldots$ ARE REPLACED BY TRACES OF LENGTH AT LEAST $1, 2, 4, 8, \ldots$



$$\mathrm{ExpLength}(B0) \geq \sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1 = +\infty$$

$\boxed{\text{LOGICAL RELATIONS}}$

ST

~~ST⊕~~

$\mathbb{RCT}_\tau \qquad \mathbb{RCV}_\tau$

$$\mathbb{RCV}_{UNIT} = \{(\star,\star)\}$$
$$\mathbb{RCV}_{NUM} = \{(n,n)\}$$
$$\mathbb{RCT}_\tau = \left\{ (M,N) \;\middle|\; \exists n,m.\ V,W \quad M \Rightarrow_m V \quad N \Rightarrow_n W \quad (V,W) \in \mathbb{RCV}_\tau \right\}$$

FUNDAMENTAL LEMMA

$(M,M) \in \mathbb{RCT}_\tau$ FOR EVERY $M \in \mathbb{CT}_\tau$

PROOF.

WE FIRST HAVE TO GENERALIZE THE RELATIONS $\mathbb{RCT}_\tau$ AND $\mathbb{RCV}_\tau$ TO THE FOLLOWING ONES

$$\Gamma \vdash M,N : \tau$$

$(M,N) \in \mathbb{RCT}_\tau^\Gamma$ IFF

$$(M[\vec{V}/\vec{x}],\ N[\vec{W}/\vec{x}]) \in \mathbb{RCT}_\tau$$

WHERE

$\Gamma = x_1 : \tau_1, \dots x_n : \tau_n$

$\vdash V_i, W_i : \tau_i \qquad (V_i, W_i) \in RCV_{\tau_i}$

WE WILL NOW PROVE THE FOLLOWING BY INDUCTION:

$$\Gamma \vdash M : \tau \implies (M, M) \in RCT_{\tau}^{\Gamma}$$

LET'S CONSIDER A COUPLE OF CASES:

- IF $M = x$, THEN $\Gamma = x : \tau, \Delta$. SUPPOSE NOW THAT $\vec{V}, \vec{W}$ ARE TYPED ACCORDING TO Γ AND THAT $(V_i, W_i) \in RCV_{\tau_i}$

  NOW

$$x[\vec{V}/\bar{x}] \qquad x[\vec{W}/\bar{x}]$$

$$(V_i \qquad , \qquad W_i) \in RCV_{\tau_i} \subseteq RCT_{\tau_i}$$

- IF $M = VW$, THEN... EXERCISE

PROOF OF CONGRUENCE FOR APPLICATIVE BISIMILARITY

- WE WILL DO THE PROOF JUST LOOKING AT THE DEFINITION OF $\mathcal{R}^H$ OUT OF ANY RELATION $\mathcal{R}$ ON TERMS. WE LEAVE THE KEY

LEMMA OUT.

- WE HAVE TO WORK WITH LAMBDA RELATIONS:
  - $\Lambda_\oplus$ IS THE SET OF ALL TERMS
  - $\Lambda_\oplus(\overline{x})$ IS THE SUBSET OF $\Lambda_\oplus$ OF ALL TERMS WHOSE FREE VARIABLES ARE INCLUDED IN $\overline{x}$
  - A LAMBDA RELATION $\mathcal{R}$ IS A FAMILY OF RELATIONS $\mathcal{R}_{\overline{x}}$ WHERE
    $\mathcal{R}_{\overline{x}} \subseteq \Lambda_\oplus(\overline{x}) \times \Lambda_\oplus(\overline{x})$
  - IF $\mathcal{R}$ IS A LAMBDA-RELATION AND $(M,N) \in \mathcal{R}_{\overline{x}}$, WE WRITE

$$\overline{x} \vdash M \,\mathcal{R}\, N$$

  - <u>HOWE'S EXTENSION</u>

$$\frac{\overline{x} \vdash x \,\mathcal{R}\, M \quad \leftarrow}{\overline{x} \vdash x \,\mathcal{R}^H M}$$

$$\frac{\overline{x} \cup \{x\} \vdash N \,\mathcal{R}^H L \qquad \overline{x} \vdash \lambda x.L \,\mathcal{R}\, M \qquad x \notin \overline{x}}{\overline{x} \vdash \lambda x.N \,\mathcal{R}^H M}$$

$$\frac{\overline{x} \vdash M \,\mathcal{R}^H P \qquad \overline{x} \vdash N \,\mathcal{R}^H Q \qquad \overline{x} \vdash PQ \,\mathcal{R}\, L}{\overline{x} \vdash MN \,\mathcal{R}^H L}$$

$$\frac{\overline{x} \vdash M \,\mathcal{R}^H P \qquad \overline{x} \vdash N \,\mathcal{R}^H Q \qquad \overline{x} \vdash P \oplus Q \,\mathcal{R}\, L}{\overline{x} \vdash M \oplus N \,\mathcal{R}^H L}$$

LEMMA

(i) IF $\mathcal{R}$ IS REFLEXIVE, THEN $\mathcal{R}^H$ IS COMPATIBLE

(ii) IF $\mathcal{R}$ IS REFLEXIVE AND TRANSITIVE, THEN $\overline{x} \vdash M \mathcal{R} N$ IMPLIES $\overline{x} \vdash M \mathcal{R}^H N$.

PROOF.

LET'S CONSIDER POINT (ii) AND LET US PROCEED BY INDUCTION ON THE STRUCTURE OF M:

- THE CASE OF VARIABLES IS TRIVIAL
- LET'S CONSIDER THE CASE OF APPLICATIONS:

$$\overline{x} \vdash ST \, \mathcal{R} \, N$$

$\mathcal{R}$ IS REFLEXIVE, THEN $\overline{x} \vdash S \mathcal{R} S$ AND $\overline{x} \vdash T \mathcal{R} T$, AND BY INDUCTIVE HYPOTHESIS:

$$\frac{\overline{x} \vdash S \, \mathcal{R}^H S \qquad \overline{x} \vdash T \, \mathcal{R}^H T \qquad \overline{x} \vdash ST \, \mathcal{R} \, N}{\overline{x} \vdash ST \, \mathcal{R}^H N}$$

POINT (i) IS LEFT AS AN EXERCISE.