On Randomization in (Higher-Order) Programming

Part IV

Ugo Dal Lago

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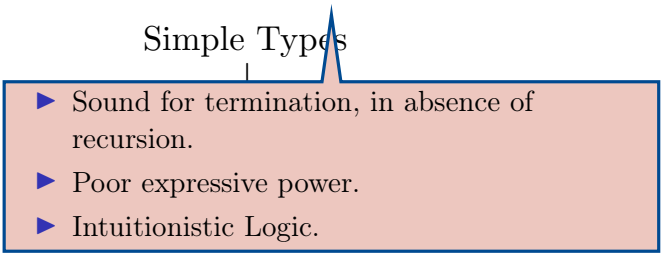
The Landscape: *Type* Theory

Simple Types

$$\tau ::= \iota \quad \Big| \quad \tau \to \tau$$

The Landscape: *Type* Theory



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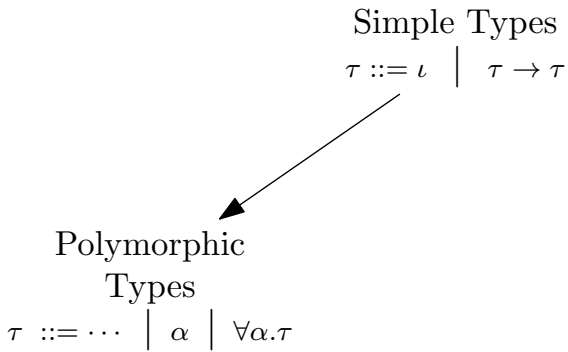


The Landscape: *Type* Theory

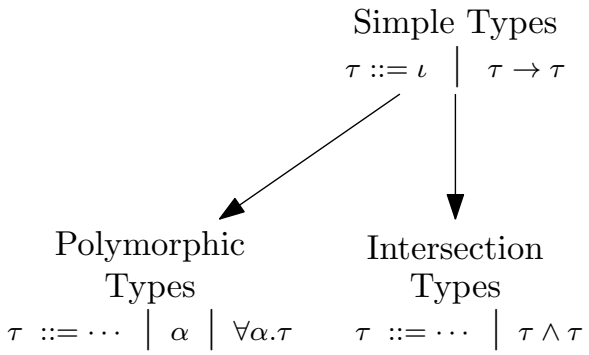
- Second-order Intuistionistic Logic.
- Very expressive, extensionally.
- Still poor, intensionally.

$\tau ::= \iota \mid \tau \to \tau$

Polymorphic Types

$\tau ::= \cdots \mid \alpha \mid \forall \alpha.\tau$

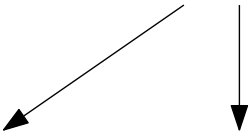
The Landscape: *Type* Theory



The Landscape: *Type* Theory

- Motivated by Semantics.
- *Complete* for termination.
- Type inference is undecidable.

$\tau ::= \iota \mid \tau \to$



Polymorphic Types

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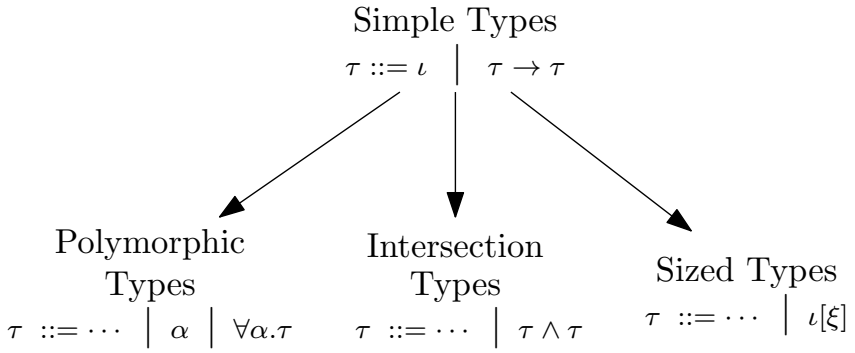
Intersection Types

$\tau ::= \cdots \mid \tau \wedge \tau$

The Landscape: *Type* Theory



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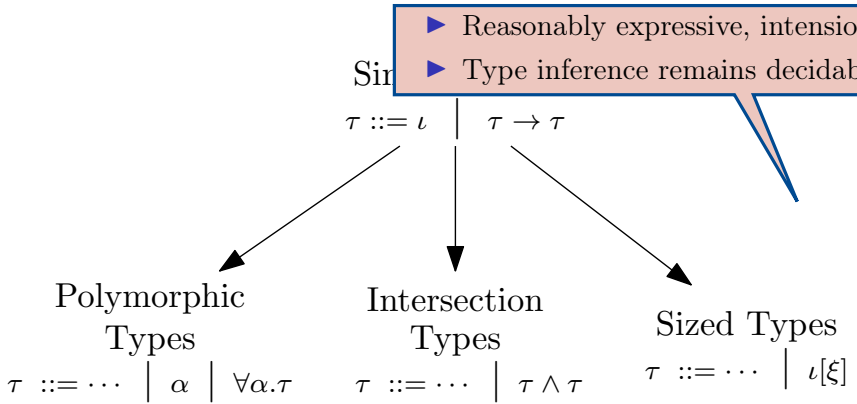

Determinism

$M\overline{s} \to^* N_s$

The Landscape: *Recursion* Theory

Determinism	**Probabilism**
$M\overline{s} \to^* N_s$	$\langle M\overline{s} \rangle = \mathcal{D}_s$

The Landscape: *Recursion* Theory

$\sum \mathcal{D}_s$ can be smaller than 1.

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The Landscape: *Recursion* Theory

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Termination	$\exists N_s \in NF$	

The Landscape: *Recursion* Theory

Undecidable;
Σ_1^0-complete.

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The Landscape: *Recursion* Theory

Almost-Sure Termination
Π^0_2-complete.

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Π^0_3-complete.

Section 1

Sized Types

Deterministic Sized Types

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  - But useless as a programming language.

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- For every type τ, define a set of reducible terms $\mathbb{RCT}_\tau$.
- Prove that all reducible terms are normalizing...
- ...and that all typable terms are reducible.

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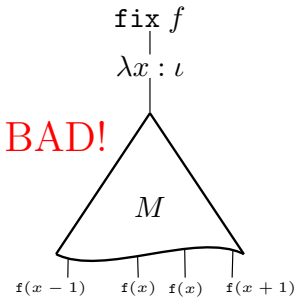

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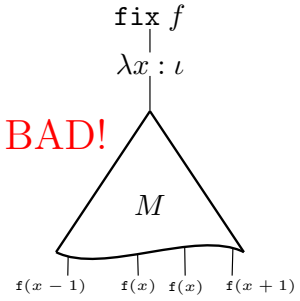

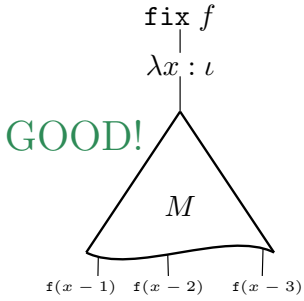

Deterministic Sized Types, Technically

- **Types**.

$$\xi ::= a \mid \omega \mid \xi + 1; \qquad \tau ::= \iota[\xi] \mid \tau \to \tau.$$

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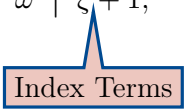

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$$\frac{\Gamma, x : \iota[a] \to \tau \vdash V : \iota[a+1] \to \tau}{\Gamma \vdash \texttt{fix}\ x.V : \iota[\xi] \to \tau}$$

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- **Typing Fixpoints**.

- Reducibility sets are of the form $\mathbb{RCT}^{\theta}_{\tau}$.
- θ is an environment for index variables.
- Proof of reducibility for $\texttt{fix}\, x.V$ is rather delicate.

- Can type many forms of structural recursion.

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- **Quite Powerful**.
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- **Termination**.
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- **Type Inference**.
  - It is indeed *decidable*.
  - But *nontrivial*.

Almost-Sure Termination

- **Examples**:

  $\texttt{fix}\ f.\lambda x.\texttt{if}\ x > 0\ \texttt{then}\ (\texttt{if}\ \mathit{FairCoin}\ \texttt{then}\ f(x-1)\ \texttt{else}\ f(x+1))\ \texttt{else}\ \star;$

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Unbiased Random Walk

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  $\texttt{fix}\ f.\lambda x.$ Unbiased Random Walk, with **two** upward calls. $\texttt{else}\ \star;$

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Unbiased Random Walk, with **two** upward calls.

Biased Random Walk, the "wrong" way.

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- Probabilistic termination **is** thus:
  - Sensitive to *the actual distribution* from which we sample.
  - Sensitive to *how many recursive calls* we perform.

One-Counter Blind Markov Chains

- They are automata of the form (Q, δ) where
  - Q is a finite set of *states*.
  - $\delta : Q \to \mathbf{D}(Q \times \{-1, 0, 1\})$.
- They are a very special form of One-Counter Markov Decision Processeses [BBEK2011].
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- They are a very special form of One-Counter Markov Decision Processeses [BBEK2011].
  - Everything is purely deterministic.
  - The counter value is ignored.
- The probability of reaching a configuration where the counter is 0 can be approximated arbitrarily well *in polynomial time*.

Probabilistic Sized Types [DLGrellois2017]

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Probabilistic Sized Types [DLGrellois2017]

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  recursive structure by a OCBMC.
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A **distribution type**, i.e., a finite distribution of types.

$$\Gamma \mid \Delta \vdash M : \mu$$

Every higher-order variable occurs **at most once**.

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- **Typing Fixpoints**.

$$\frac{\Gamma \mid x : \sigma \vdash V : \iota[a+1] \to \tau \quad OCBMC(\sigma) \text{ terminates.}}{\Gamma \mid \Theta \vdash \texttt{fix}\ x.V : \iota[\xi] \to \tau}$$

Probabilistic Sized Types [DLGrellois2017]

- **Basic Idea**: craft a sized-ty
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- **Judgments**.

Form σ, one can build a OCBMC:

- σ is a distribution type.
- It keeps track of the probability of each recursive call.

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This is sufficient for typing:

- Unbiased random walks;
- Biased random walks.

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- **Typing Probabilistic Choice**

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- **Termination**.
  - By a quantitative nontrivial refinement of reducibility.

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- Reducibility sets are now on the form $\mathbb{RCT}_\tau^{\theta,p}$
- p stands for the *probability* of being reducible.
- Reducibility sets are continuous:

$$\mathbb{RCT}_\tau^{\theta,p} = \bigcap_{q<p} \mathbb{RCT}_\tau^{\theta,q}$$

$$\frac{\Gamma \mid \Delta \vdash M : \tau \qquad \Gamma \mid \Omega \vdash N : \rho}{\Gamma \mid \frac{1}{2}\Delta + \frac{1}{2}\Omega \vdash M \oplus N : \frac{1}{2}\tau + \frac{1}{2}\rho}$$

- **Termination**.
  - By a quantitative nontrivial refinement of reducibility.

Section 2

Intersection Types

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- **Completeness**
  - By *subject expansion*, the dual of subject reduction.

Oracle Intersection Types [BreuvartDL2018]

- Probabilistic choice can be seen as a form of read operation:

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This is **unavoidable**, due to recursion theory.

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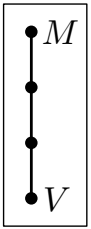
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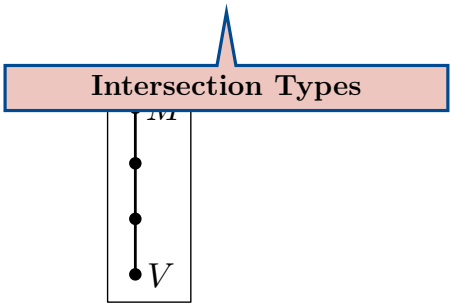
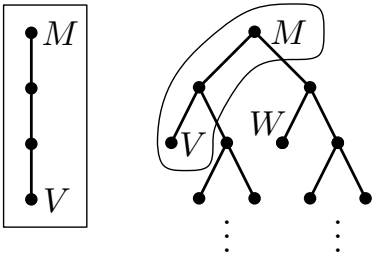
Intersection Types and Computations

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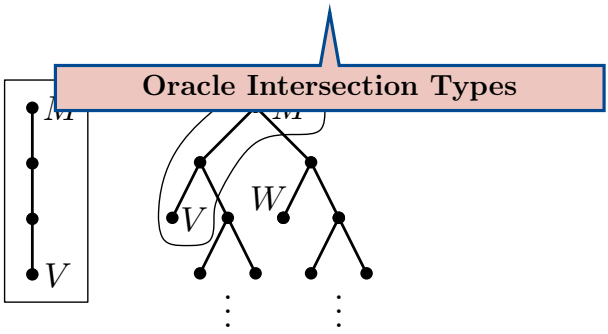
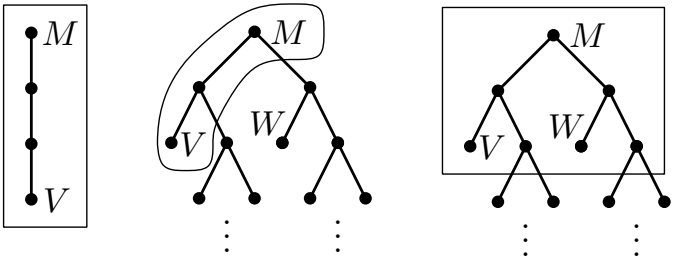
Intersection Types and Computations




M
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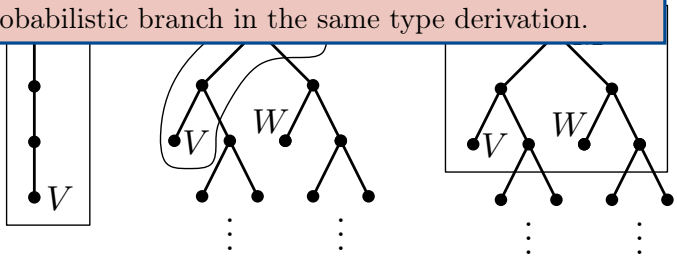
Intersecti

Monadic Intersection Types [BDL2018,DLFR2021]

- They are a combination of oracle and sized types.
- Intersections are needed for preciseness.
- Distributions of types allow to analyse more than one probabilistic branch in the same type derivation.



Thank You!

Questions?