On Randomization in (Higher-Order) Programming

Part I

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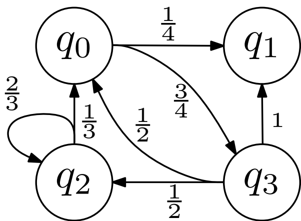

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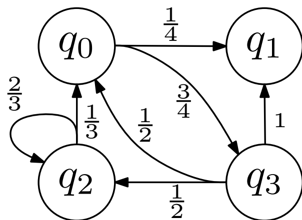
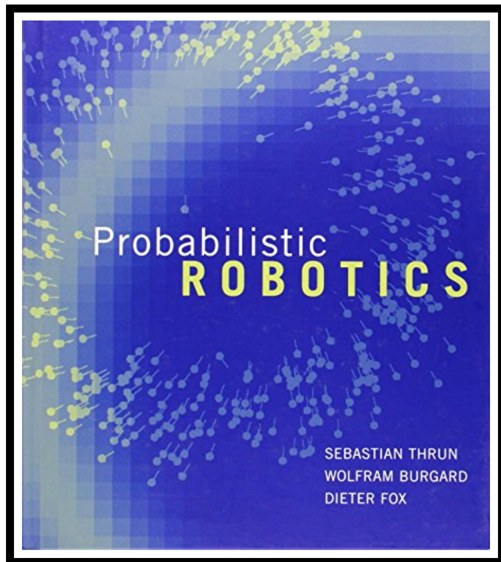


- **Abstractions:**
  - (Labelled) Markov Chains.

Probabilistic Models



ROBOTICS

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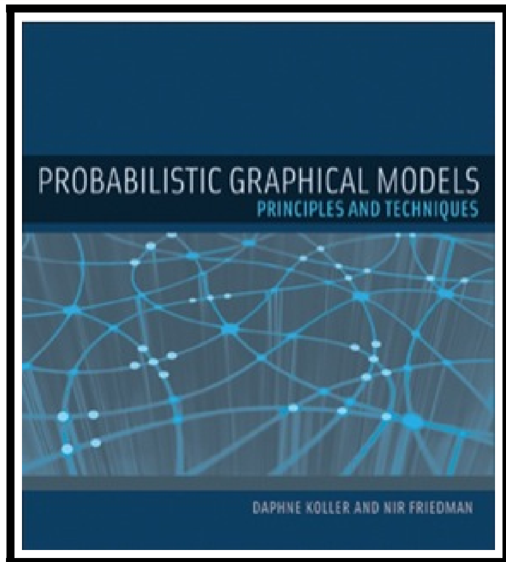
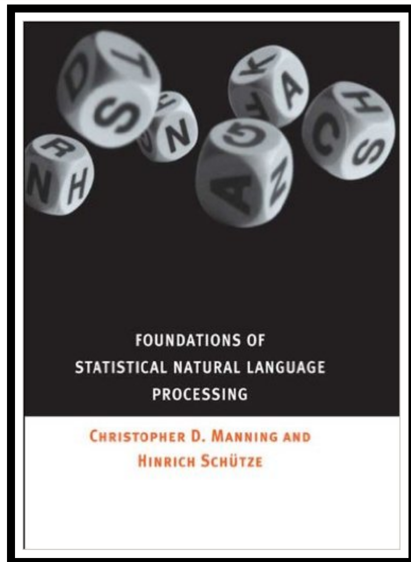


ARTIFICIAL INTELLIGENCE

Probabilistic Models



NATURAL LANGUAGE
PROCESSING

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end /* Done. $n-1 = 2^s t$. */
Choose $x \in \{1, 2, \ldots, n-1\}$ uniformly at random.
Compute each of the numbers $x^t, x^{2t}, x^{4t}, \ldots, x^{2^s t} = x^{n-1} \bmod n$.
If $x^{n-1} \not\equiv 1 \pmod{n}$, return **composite.**
for $i = 1, 2, \ldots, s$
  If $x^{2^i t} \equiv 1 \pmod{n}$ and $x^{2^{i-1} t} \not\equiv \pm 1 \pmod{n}$, return **composite.**
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- **Abstractions:**
  - Randomized algorithms;
  - Probabilistic Turing machines.
  - Labelled Markov chains.

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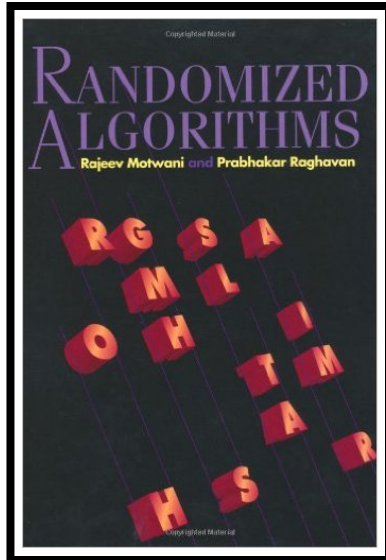

ALGORITHMICS

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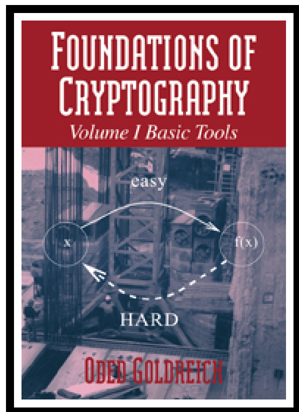
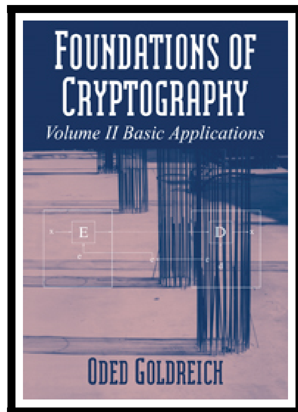
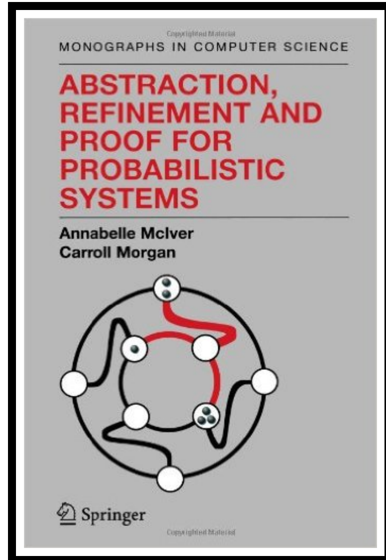




CRYPTOGRAPHY

Randomized Computation



PROGRAM VERIFICATION

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- **Models**:
  - λ-calculus

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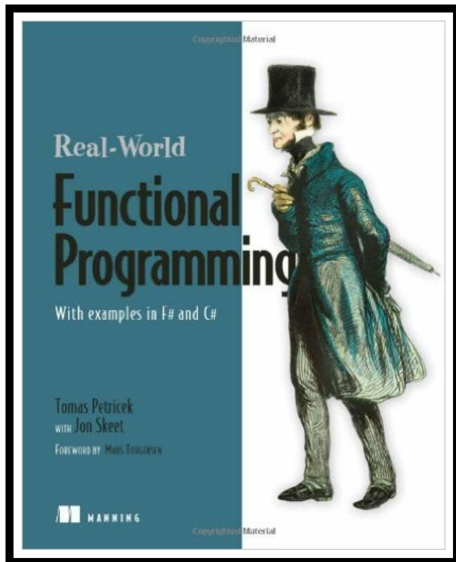
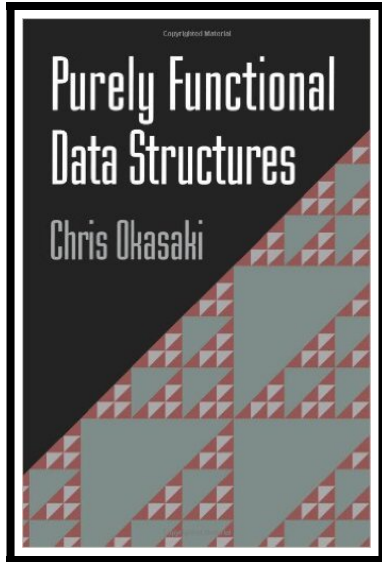


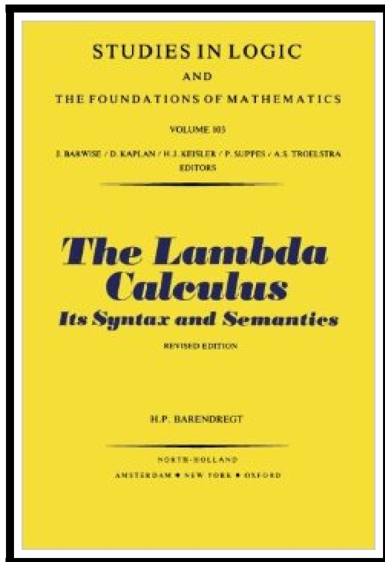
FUNCTIONAL PROGRAMMING

Higher-Order Computation



FUNCTIONAL
DATA STRUCTURES

Higher-Order Computation



λ-CALCULUS

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Interesting Research Problems?

This Course

- Five lectures:
  1. **Randomized Computation, Complexity, and Programming**.
  2. **Calculi for Randomized Higher-Order Computation**.
  3. **Relational Reasoning**.
  4. **Termination and Complexity Analysis**.
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- The course material will be made available on the course's webpage:

`http://www.cs.unibo.it/~dallago/ECI2023/`

The Lectures' Structure

- Each lecture will be divided into *three parts*:
  1. An **overview**, based on some slides.
     - Around *90 minutes* long.
     - Slides are progressively made available on the course webpage.
  2. An **in-depth analysis** of some of the presented concepts, using an online whiteboard.
     - Around *60 minutes* long.
     - The transcript will be made available, again at the course webpage.
  3. **A short exercise session** about the topics of the days.
     - Around *30 minutes*.
     - I will ask you to solve some simple exercises about the topics of the day. Please try to have with you pen and paper, and a laptop if possible.
- There will be **a break** between the first and second part of the lecutre, around *10 minutes*.
- I would be glad to answer **your questions**, either during the lecture, or at the end of it.

The Exam

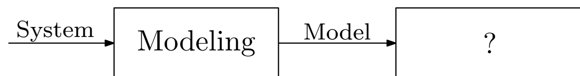
- It will consist of an **handout exam**, which will be made public after the last lecture.
- You have to work **individually**.
- You are required to complete as many exercises you can in at most ten days, namely by **August 8th**.
- Your solutions to the exercises have to be typeset, preferably in LaTeX, and **sent to me** via email, the subject beginning with `[ECI2023]`.

Part I

The Basics of Randomized Computation

Incepting Probability into Algorithms

- Probability theory provides effective tools to facilitate **modeling** of existing systems, e.g.



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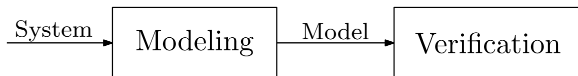

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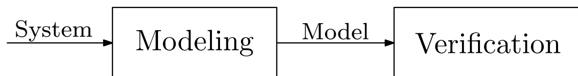

- Probability, however, can also be seen as a very powerful way to **compute**.
  - Algorithms can "flip a coin";
  - This is somehow a paradigm shift: probabilism is not there for modeling uncertainty, but rather for the purpose of **relaxing determinism** as an algorithm design principle.
  - This "new" computation paradigm is called **randomized computation**.
  - Your algorithms (then your programs) can be *seen* and *treated* as probabilistic systems.

But Why?

- **Achieving Better Performances**
  - Allowing computation to proceed probabilistically *somehow* allows to exploit different computation paths **at the same time**.
  - Of course, each computation path comes with its own probability.
  - ... but all this can (sometime) be turned into a way to alleviate exponential blowups.

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- **Allowing a Form of "Confusion" in Computation**
  - Deterministic computation has its own shortcomings.
  - In particular, in some interactive scenarios probabilistic computation is a way to make computation unpredictable.

Example: Primality Testing

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If x^{n−1} ≢ 1 (mod n), return composite.
for i = 1, 2, ..., s
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- Similarly when Π is **private-key** and we want to prove it to be secure against *chosen-plaintext* attacks.
  - Saying it differently, the so called **one-time-pad** is simply not secure against active attacks.

What do we Assume?

- Familiarity with **Turing machines and automata**.
  - We briefly review the rudiments of computability and complexity theory.
- The basics of **elementary probability theory**.
- Some knowledge about the **theory of programming languages**.
  - We will review almost all what is needed to understand randomized programming languages.

Probabilistic Turing Machines

- Generalizing ordinary Turing Machines to the probabilistic setting is a relatively easy task.
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  - In presence of **determinism**,

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- Every time it makes a step, a PTM *flips a coin*, without reusing the results of previous coin flips.

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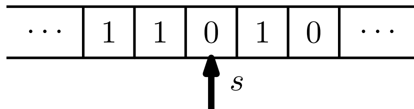

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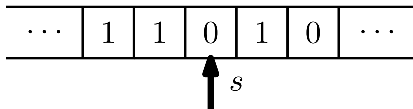

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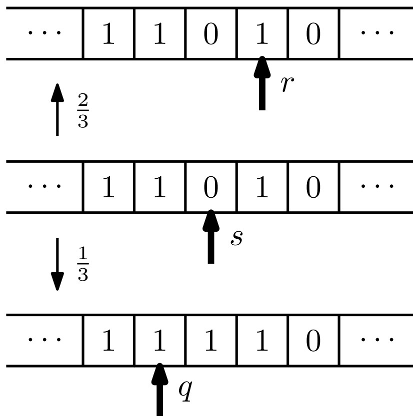

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- The **probabilistic space** we work with has the form

$$(\mathbb{F}^{\mathbb{N}}, \mathcal{F}, \mu)$$

  where $\mathbb{F} = \{0, 1\}$, $\mathcal{F}$ is any σ-algebra on $\mathbb{F}^{\mathbb{N}}$ including all the cylinders, and μ assigns probability $\frac{1}{2}$ to each cylinder $C_{b,i}$.
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  - $C_{b,i}$ is the subset of $\mathbb{F}^{\mathbb{N}}$ consisting of all functions f such that $f(i) = b$.
- Any property of interest (e.g. the outcome of the computation process or the number of computation steps) can be seen as a **random variable**.
  - Very often, we do not care of mentioning the random variable itself, and concentrate on its distribution.

PTMs, Languages and Functions

- In PTM, **termination** and **acceptance of a string** become probabilistic events.
- A PTM can terminate with probability 1 even if there is the *possibility* of nontermination.
- A given string $s \in \Sigma^*$ is accepted *with a probability* between 0 and 1, included.
- How can we define language recognition?
  - One can consider different, alternative, constraints on the **error** probability:
    - It must be 0.
    - It must be smaller than $\frac{1}{2}$;
    - It must be smaller than a given fixed constant ε, itself smaller than $\frac{1}{2}$.
  - One can stipulate that the given result is the one with **highest probability**.

What Algorithms Compute

- **Deterministic Computation**
  - For every input x, there is *at most* one output y any algorithm $\mathcal{A}$ produces when fed with x.
  - As a consequence:

$$\mathcal{A} \qquad \rightsquigarrow \qquad [\![\mathcal{A}]\!] : \mathbb{F}^* \rightharpoonup \mathbb{F}^*.$$

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$$\mathcal{A} \qquad \rightsquigarrow \qquad [\![\mathcal{A}]\!] : \mathbb{F}^* \rightarrow \mathbf{D}(\mathbb{F}^*).$$

  - The distribution $[\![\mathcal{A}]\!](n)$ sums to anything between 0 and 1, thus accounting for the probability of divergence.

Quick Recap of Computability Theory

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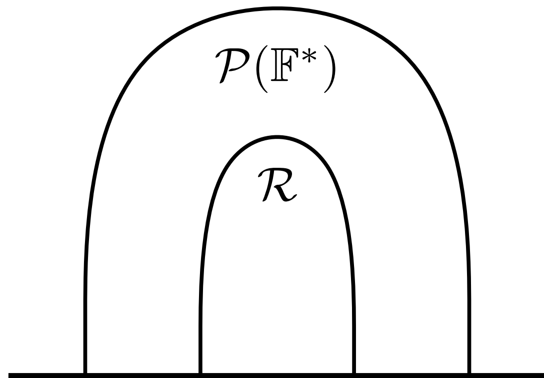
- What could we say about **computational problems**, i.e., about any $L \subseteq \mathbb{F}^*$?
  - L is said to be **recursive** iff the function f_L defined as follows is computable:

$$f_L(x) = \begin{cases} 0 & \text{if } x \in L \\ 1 & \text{otherwise} \end{cases}$$

  - $\mathfrak{R}$ is the corresponding class of functions.


$\mathcal{P}(\mathbb{F}^*)$
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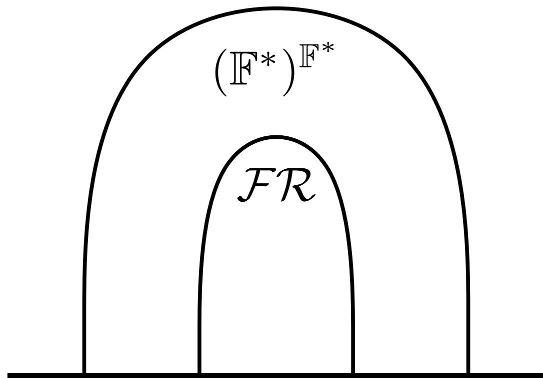

$(\mathbb{F}^*)^{\mathbb{F}^*}$
$\mathcal{FR}$

Computing with Limited Resources

- Some computable functions seem to require too much **time** (and/or **space**) to be computed.
- Let us consider the function $A : \mathbb{N}^2 \to \mathbb{N}$ defined as follows:

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0; \\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0; \\ A(m-1, A(m,n-1)) & \text{if } m > 0 \text{ and } n > 0. \end{cases}$$

- The value $A(m,n)$ increases **too fast** when m and n grow, e.g.,

$$A(4,3) \approx 10^{6031 \cdot 10^{19727}}.$$

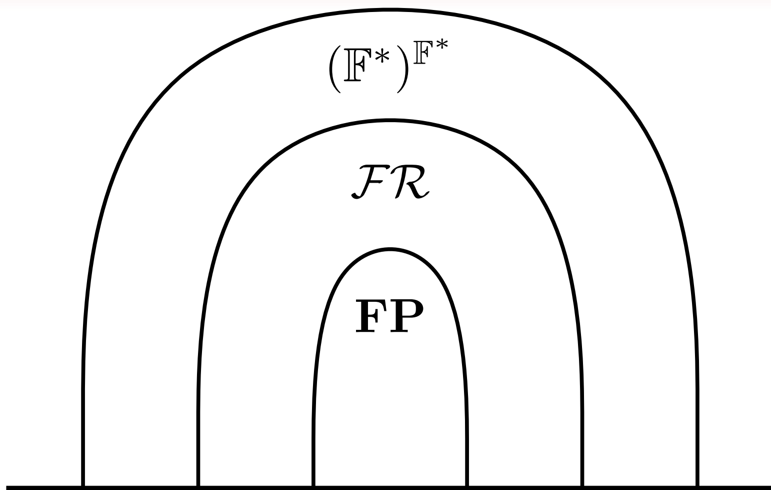
- Ideally, we would like the time necessary to compute a function on $s \in \mathbb{F}^*$ to be bounded by a **fixed** polynomial on $|s|$.

Polynomial Time

- A TM M **works** in time $g : \mathbb{N} \to \mathbb{N}$ iff for every $s \in \mathbb{F}^*$, the computation of M on input s takes at most $g(s)$ steps. In this case, we will write that $M \in \mathit{Time}(g)$.
- The complexity class **FP** is the collection of all functions from $\mathbb{F}^*$ in $\mathbb{F}^*$ which are computable by a TM M in

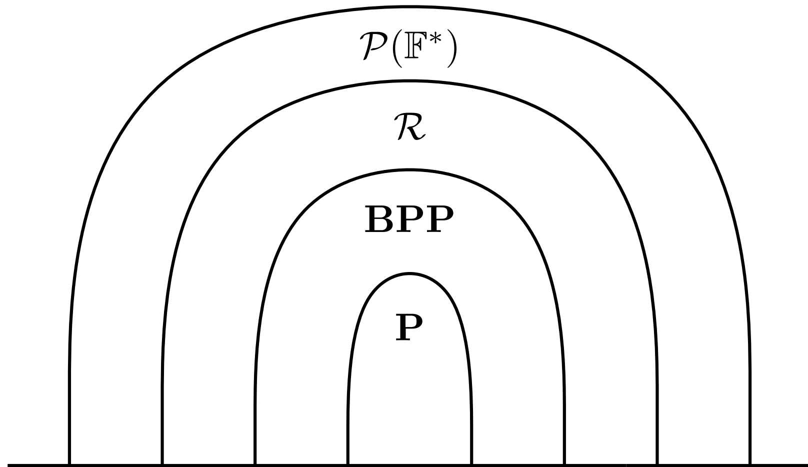
$$\bigcup_{g \in \mathsf{POLY}} \mathit{Time}(g).$$

  where POLY is the space of all polynomials from $\mathbb{N}$ to $\mathbb{N}$.
- Knowing that a function is **not** in **FP** can sometime help a lot. . . .

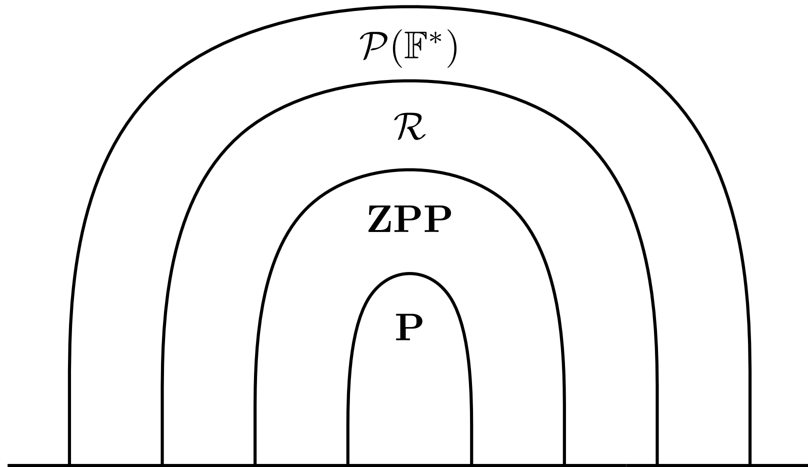

$(\mathbb{F}^*)^{\mathbb{F}^*}$
$\mathcal{FR}$
$\mathbf{FP}$

Probabilistic Polynomial Time

- How should we capture the concept of feasibility in a probabilistic setting?
- There is a tradeoff between bounding running times and bounding the error...
  - We can insist on having **no** error, but allow computation to take polynomial time only on the average, obtaining the class **ZPP**.
  - We can allow the error to be smaller than a constant $\varepsilon < \frac{1}{2}$, but insist on polynomial time bounds to hold independently on the random choices, getting the class **BPP**.
- There are many other possibilities, but these are somehow less interesting, giving rise to classes which are too large.



$\mathcal{P}(\mathbb{F}^*)$
$\mathcal{R}$
BPP
P


$\mathcal{P}(\mathbb{F}^*)$
$\mathcal{R}$
ZPP
P

$$\mathrm{BPP} \stackrel{?}{=} \mathrm{P}$$

Part II

Why Higher-Order (Randomized) Programming?

MergeSort (1)

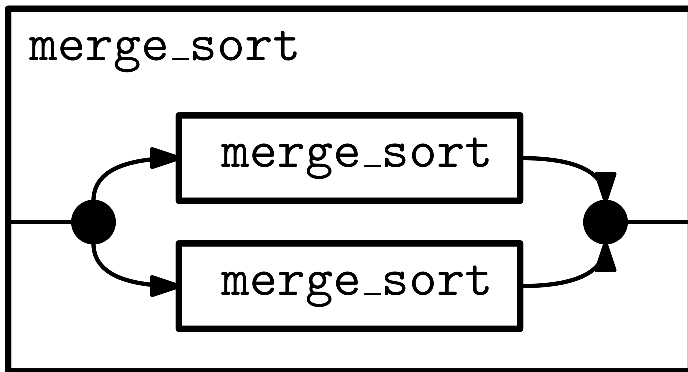
```
let rec merge = function
      | list, []
      | [], list -> list
      | h1::t1, h2::t2 ->
          if h1 <= h2 then
            h1 :: merge (t1, h2::t2)
          else
            h2 :: merge (h1::t1, t2);;

let rec halve = function
      | []
      | [_] as t1 -> t1, []
      | h::t ->
          let t1, t2 = halve t in
            h::t2, t1;;
```

MergeSort (2)

```
let rec merge_sort = function
      | []
      | [_] as list -> list
      | list ->
          let l1, l2 = halve list in
            merge (merge_sort l1, merge_sort l2);;
```

The Structure of MergeSort



MergeSort, HO

```
let rec merge = function
      | (list, []), _ -> list
      | ([], list), _ -> list
      | (h1::t1, h2::t2),el ->
        if h1 <= h2 then
          h1 :: merge ((t1, h2::t2),el)
        else
          h2 :: merge ((h1::t1, t2),el);;

let rec halve = function
      | []
      | [_] as t1 -> (t1, []),()
      | h::t ->
        let (t1, t2),el = halve t in
          (h::t2, t1),el;;

let rec dac divide conquer = function
      | []
      | [_] as list -> list
      | list ->
        let (l1, l2),el = divide list in
          conquer ((dac divide conquer l1, dac divide conquer l2),el);;

let rec merge_sort = dac halve merge;;
```

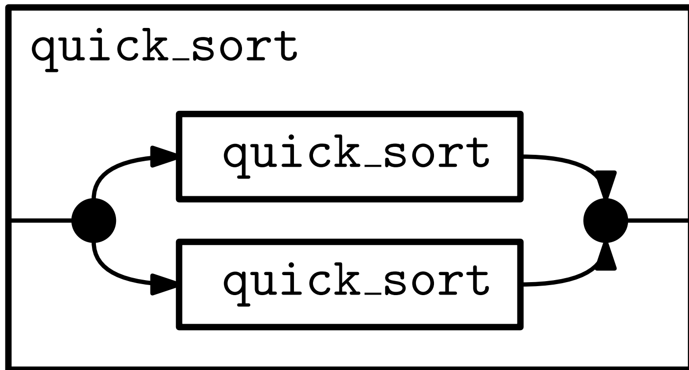
QuickSort

```
let app x y z = x @ (z::y);;

let partition = function
    | pivot :: rest -> (List.partition (( > ) pivot) (rest)),pivot;;

let rec quick_sort = function
    | []
    | [_] as list -> list
    | list ->
      let (l1, l2), el = partition list in
        app (quick_sort l1) (quick_sort l2) el;;
```

The Structure of MergeSort



QuickSort, HO

```
let app = function
    | (x,y),z -> x @ (z::y);;

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    | pivot :: rest -> (List.partition (( > ) pivot) (rest)),pivot;;

let rec dac divide conquer = function
    | []
    | [_] as list -> list
    | list ->
        let (l1, l2),el = divide list in
          conquer ((dac divide conquer l1, dac divide conquer l2),el);;

let quick_sort = dac partition app;;
```
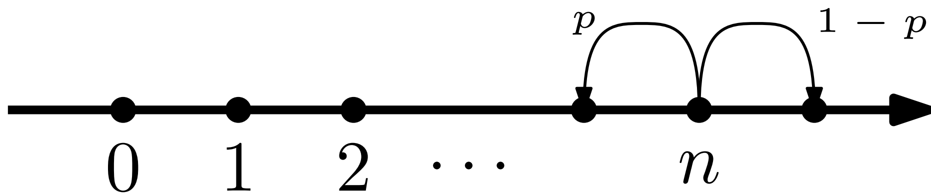

Randomized QuickSort (1)

```
let app = function
    | (x,y),z -> x @ (z::y);;

let rec extract = function
  | [],_ -> ([],0)
  | hd::tl,n ->
     if n==0 then
       (tl,hd)
     else
       let (l,el) = extract(tl,n-1) in
         (hd::l,el);;

let partition list =
  let (rest,pivot) = extract (list,(Random.int (List.length list))) in
    (List.partition (( > ) pivot) (rest)),pivot;;
```

Randomized QuickSort (2)

```
let rec dac divide conquer = function
      | []
      | [_] as list -> list
      | list ->
          let (l1, l2),el = divide list in
            conquer ((dac divide conquer l1, dac divide conquer l2),el);;

let rand_quick_sort = dac partition app;;
```

Random Walk



Two Kinds of Random Walks

```
let rec iter f g n = if n==0 then g else let m=pred(n) in f m (iter f g m);;

let mult m n = succ(m)*n;;

let fact = iter mult 1;;

let rec param_iter f g step n =
    if n==0 then g else let m=step(n) in f m (param_iter f g step m);;

let succ_2 m n = n+1;;

let updown_fair x = x+(2*Random.int(2)-1);;

let fair_random_walk = param_iter succ_2 0 updown_fair;;

let updown_biased x = if Random.int(3)==0 then x+1 else x-1;;

let biased_random_walk = param_iter succ_2 0 updown_biased;;
```

The Coupon Collector

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- **Example**: if $n = 5$, you could get the following coupons:

$$3, 1, 5, 2, 3, 1, 5, 2, 3, 1, 5, 2, \ldots$$

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- You win a prize when you collect a set of n coupons, each with a distinct label.
- **Example**: if $n = 5$, you could get the following coupons:

$$3, 1, 5, 2, 3, 1, 5, 2, 3, 1, 5, 2, \ldots$$

- Are you guaranteed to win the prize with probability 1? After how many days, on the average?

The Coupon Collector

```
let rec base_param_iter f g step base e =
  if base(e) then g else let d=step(e) in f d (base_param_iter f g step base d);;

let second_zero = function
  | (_,0) -> true
  | _ -> false;;

let succ_2 m n = n+1;;

let step_2 = function
  | (n,m) -> if Random.int(n)<=m then (n,m-1) else (n,m);;

let coupon_collector x = base_param_iter succ_2 0 step_2 second_zero (x,x);;
```

Part III

Randomized Programming in Other Paradigms

Imperative Programming

- Programming randomized algorithms is possible in most modern imperative programming languages, e.g., `Python` or `Rust`.

Imperative Programming

- Programming randomized algorithms is possible in most modern imperative programming languages, e.g., `Python` or `Rust`.
- A `Python` program implementing the **fair random walk** would, e.g., look like the following:

```
import random

def fair_random_walk(x):
        if x==0:
            return True
        else:
            b=random.choice([-1,1])
            return fair_random_walk(x+b)
```

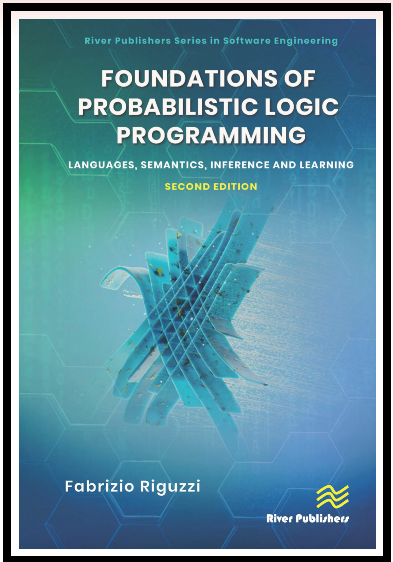
- Many of the verification techniques which are well-known in the deterministic setting (e.g. Floyd-Hoare logics, symbolic execution, or abstract interpretation) can be generalized to the presence of probabilistic choice.
  - This is highly nontrivial, we will see something about that, if time permits.

Logic Programming

- In logic programming, computation is seen as deduction: programs are sets of **clauses** and inputs to them are **facts**, e.g.

$$edge(a,b) \quad edge(b,c) \quad edge(a,c) \quad edge(b,d)$$
$$reachable(x,x) \leftarrow$$
$$reachable(x,y) \leftarrow edge(x,z) \wedge reachable(z,y)$$

- Computation consists in trying deducing the truth of a **query**, e.g. $reachable(a,d)$, from the facts and clauses.
- Various ways of generalizing logic programming to accommodate for randomization have been introduced, e.g.:
  - Facts can be hold **with a certain probability**, but clauses remain deterministic.
  - Whenever the variables in a clause are **substituted**, randomization enters into the playground.



LOGIC PROGRAMMING