ECI 2023
On Randomization in (Higher-order) Programming
Final Exam
July 28th, 2023

Please notice that:

- Exercises are meant to be solved *individually*.
- Solutions should be typeset in LaTeX, and sent, in pdf format, to `ugo.dallago@unibo.it`. Students are encouraged to use the template `template.tex`, which can be retrieved from the course's webpage.
- The deadline for sending the solutions is August 8th, 2023, at midnight AoE.
- Please start the subject of your email with `[ECI2023]`.

Exercise 1.
Please consider the following variation on the random walk on $\mathbb{N}$: rather than going up by one or down by one, each with probability $\frac{1}{2}$, you go down with probability $\frac{3}{4}$, or up (but *by two*) with probability $\frac{1}{4}$. Determine whether this form of random walk is (positive) almost-surely terminating.

Exercise 2.
Consider the operational semantics of the language $\mathsf{PCF}_{\oplus}$. In principle, nothing guarantees that $\langle M\rangle$, defined as $\sum_{n\in\mathbb{N}}\langle M\rangle_n$ is a distribution, because nothing in principle prevents $\sum\langle M\rangle$ to be strictly greater than 1. Prove that $\sum\langle M\rangle$ is indeed less or equal to 1. Do that by considering the partial sums $\sum_{n=0}^{m}\langle M\rangle_n$ and prove, by induction on m, that all of them are distributions. Be careful about the precise form of the statement to which you apply the induction principle.

Exercise 3.

1. We have not said much about the termination behavior of $\mathsf{ST}_{\oplus}$ terms. Can we say that they are all almost-surely terminating? Can we go beyond and say that they are *positively* almost-surely terminating? Prove your answer.
2. Consider the calculus $\mathsf{ST}_{\oplus}$, but endowed with a operator `iter` for iteration and subject to the typing rule
$$\frac{\Gamma\vdash V:\tau\to\tau\quad \Gamma\vdash W:\tau}{\Gamma\vdash \mathtt{iter}(V,W):\mathsf{Num}\to\tau}$$
and to the following one-step reduction rules:
$$\mathtt{iter}(V,W)\;0\to\delta(W)\qquad \mathtt{iter}(V,W)\;(n+1)\to\delta(\mathtt{let}\;(\mathtt{iter}(V,W)\;n)=y\;\mathtt{in}\;Vy)$$
Does this change anything, as far as termination is concerned?

Exercise 4.
Complete the definition of logical relations for $\mathsf{ST}_{\oplus}$ from the slides, and in particular, fill the details of the definition of the $\mathbb{RCT}_\tau$ relation. Prove the fundamental lemma.

Exercise 5.
In the calculus $\Lambda_{\oplus}$, the sets $\mathbb{CT}$ and $\mathbb{CV}$ stand for the set of closed terms and values, respectively. The definition of an applicative bisimulation for $\Lambda_{\oplus}$ we saw can be equivalently spelled out as consisting of *two* equivalence relations $\mathcal{R}_{\mathbb{T}}\subseteq\mathbb{CT}\times\mathbb{CT}$ and $\mathcal{R}_{\mathbb{V}}\subseteq\mathbb{CV}\times\mathbb{CV}$ such that

$$(\lambda x.M,\lambda x.N)\in\mathcal{R}_{\mathbb{V}}\Longrightarrow\forall V\in\mathbb{CV}.(M[V/x],N[V/x])\in\mathcal{R}_{\mathbb{C}}$$
$$(M,N)\in\mathcal{R}_{\mathbb{T}}\Longrightarrow\forall E\in\mathbb{CV}/\mathcal{R}_{\mathbb{V}}.\langle M\rangle(E)=\langle N\rangle(E)$$

where $\mathbb{CV}/\mathcal{R}_{\mathbb{V}}$ is the set of all equivalence classes modulo $\mathcal{R}_{\mathbb{V}}$ and with notations like $\mathcal{D}(X)$ we indicate the probability of observing any element from X in the distribution $\mathcal{D}$. Now, prove that the following two terms are observationally equivalent by building an applicative bisimulation relation which contains them:

$$(\lambda f.\lambda x.ff)(\lambda f.\lambda x.ff)\qquad\qquad(\lambda f.(\lambda x.ff)\oplus(\lambda x.ff))(\lambda f.(\lambda x.ff)\oplus(\lambda x.ff))$$