Complex Systems and Network Science:

Dynamical Systems and Non-Linear Dynamics

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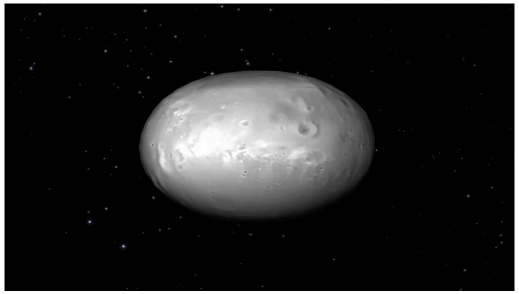
Dynamics

- Study of how systems *change* over time
  - Planetary dynamics — motion of planets and other celestial bodies
  - Fluid dynamics — motion of fluids, turbulence, air flow
  - Crowd dynamics — behavior of groups of people, stampedes
  - Population dynamics — how populations vary over time
  - Climate dynamics — variations in temperature, pressure, hurricanes
  - Financial dynamics — variations in stock prices, exchange rates
  - Social dynamics — conflicts, cooperation

Dynamics of planetary motion

- Consider Nix, one of 5 moons of Pluto



- "If you stood on Nix, the sun might rise in the west and set in the east one day, and rise in the east and set in the north on another"

Dynamical systems theory

- Branch of mathematics for studying how systems change over time
- Gives us a vocabulary and a set of tools for describing dynamics

Dynamical systems theory

- Earliest studies of dynamical systems were focused on celestial motion and date back to ancient Greeks
  - Aristotle (4th century BC) — perfect spheres
  - Claudius Ptolemaeus (2nd century AD) — earth-centered model
- Beginnings of the modern era of dynamical systems
  - Nicolaus Copernicus (15th century AD) — sun-centered model
  - Galileo Galilei (16th century AD) — birth of "scientific method"
  - Isaac Newton (17th century AD) — birth of dynamics
  - Pierre-Simon Laplace (18th century AD) — "clockwork" universe
  - Jules Henri Poincaré (19th century AD) — birth of "chaos theory"

A "clockwork" universe

- "We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes."

—Pierre-Simon Laplace, *A Philosophical Essay on Probabilities* (1840)

Doubts on a "clockwork" universe

- "If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation *approximately*. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that *small differences in the initial conditions produce very great ones in the final phenomena*. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible."

—Jules Henri Poincaré, *Science and Method* (1903)

Butterfly effect

- Poincaré highlights "sensitive dependence on initial conditions"
- "Small differences in the initial conditions produce very great ones in the final phenomena"
- Illustrated by the so-called "butterfly effect"
- A butterfly flapping its wings in California provokes a hurricane in the Philippines some weeks later

Dynamics of iteration

- Dynamics result from some process repeating itself over and over, such as the population of some species
- Consider an extremely simple model for population growth
- At each time step, every member of the population gives birth to some constant number of new members
- Parameters
  - Initial population size
  - Birth rate

Simple model of population growth

- Define the *system state* as the current population size
- *State variables* denote the system state and change with time
- Let n_t denote the size of the population at time t
- Consider *discrete time* model with t assuming the values of natural numbers
- Initial population is n_0
- Let R denote the birth rate — number of offsprings produced by a member at each time step
- In other words, $n_1 = Rn_0$ and $n_2 = Rn_1$, …
- In general, $n_{t+1} = Rn_t$

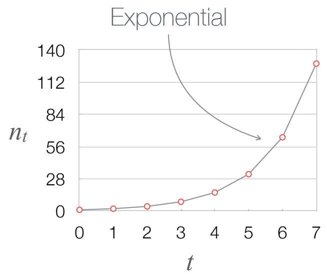
Simple model of population growth

- Assume initial population $n_0 = 1$ and birth rate $R = 2$

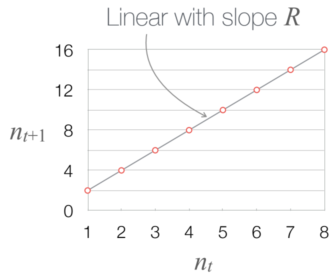
Time t	Population size n_t
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128

- $n_t = 2^t$ or, in general, $n_t = R^t$
- In other words, the population grows exponentially without bound

Simple model of population growth



Time Series
(n_t as a function of t)



State Space
(n_{t+1} as a function of n_t)

Simple model of population growth

- System is *exponential* in the time series
- System is *linear* in the state space
- Linearity is due to the fact that there are no interactions among the population members — each member acts in isolation
- The whole is indeed the sum of its parts

Linear versus nonlinear

- Note that linearity leads to unbounded population growth (positive feedback)
- Introduce nonlinearity by adding *negative feedback* resulting from interactions among members
- Suppose limited resources result in the death of some offsprings due to overcrowding
- So, the new model becomes $n_{t+1} = R(n_t - d)$ where d is the *death rate*
- Reasonable to assume that the death rate is proportional to the square of the population size (pair-wise interactions)
- NetLogo PopulationGrowthLogistic

Linear versus nonlinear

- Assume $d = n_t^2/k$ where k is the maximum "carrying capacity"
- In other words, $n_{t+1} = R(n_t - n_t^2/k)$
- Rewrite: $$\frac{n_{t+1}}{k} = R(\frac{n_t}{k} - \frac{n_t^2}{k^2})$$
- Change of variables: $$x_{t+1} = R(x_t - x_t^2)$$
- The relation is no longer linear
- Consists of a "positive feedback" term (Rx_t) and a "negative feedback" term ($-Rx_t^2$)
- Resulting equation is known as the "Logistic Map"
- Note that x_t denotes the "normalized" population thus $0 \leq x_t \leq 1$

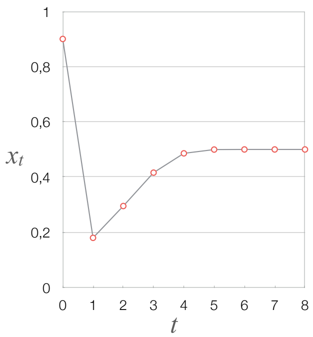
Logistic map

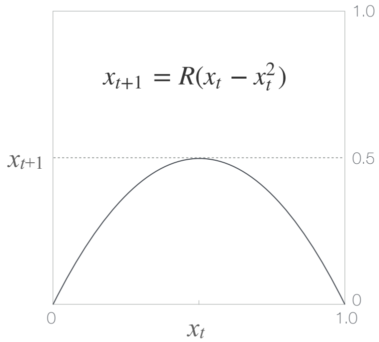
- Suppose $R = 2$ and $x_0 = 0.9$

t	$R(x_t - x_t^2)$
0	0.9
1	0.18
2	0.2952
3	0.41611382
4	0.485926251164
5	0.499603859187
6	0.499999686144
7	0.4999999999998
8	0.5

Logistic map

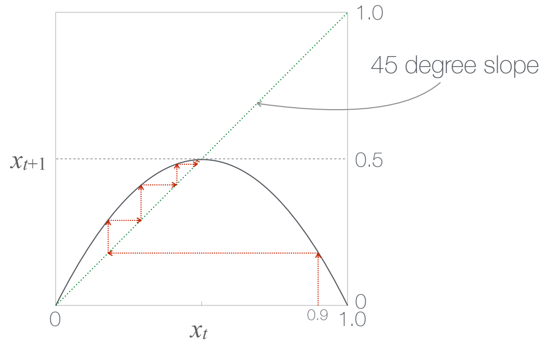
- *Time series diagram* for $R = 2$
- *State space diagram* for $R = 2$





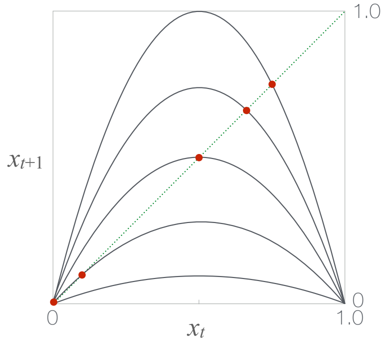
Logistic map

- For $R = 2$, the value 0.5 is a *fixed-point attractor* — regardless of the initial value, the system always reaches this value and remains there



Logistic map

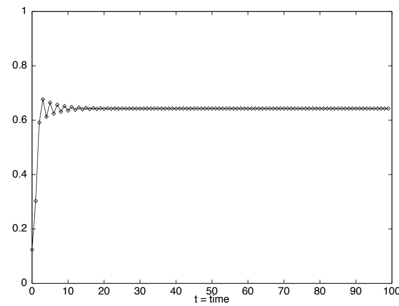
- Observe what happens as we increase R

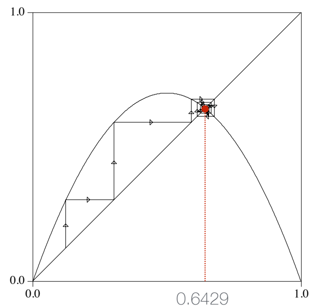


- 0 always remains a (unstable) fixed point, with additional intersections of diagonal line as we increase R

Fixed-point dynamics

- What if $R = 2.8$?

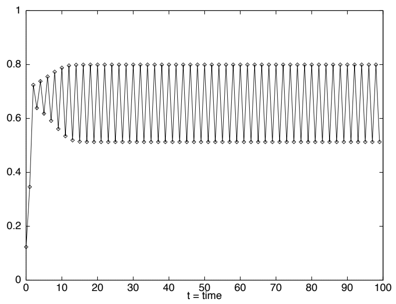


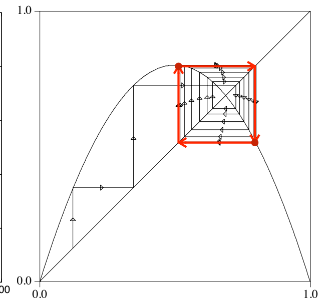


- Still (larger) fixed point
- Defined by the intersection of parabola with diagonal line

Periodic dynamics

- What if $R = 3.2$?

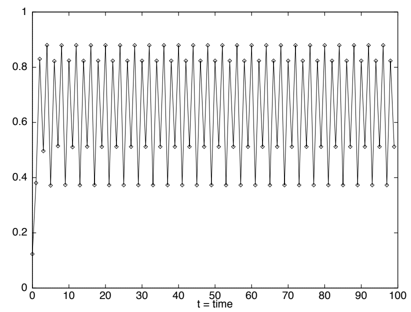


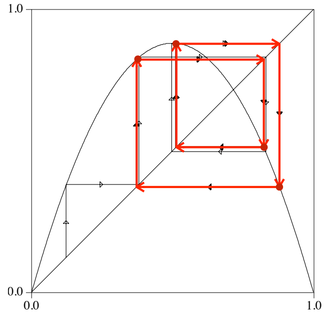


- The trajectory converges to a limit cycle of period 2 and oscillates between the values 0.513 and 0.7995

Period doubling

- What if $R = 3.52$?

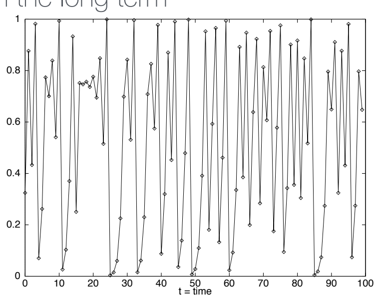


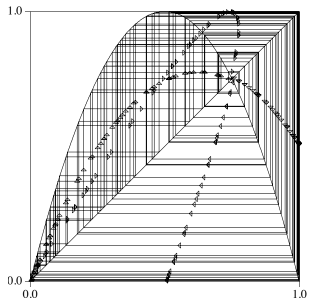


- The trajectory converges to a limit cycle of period 4

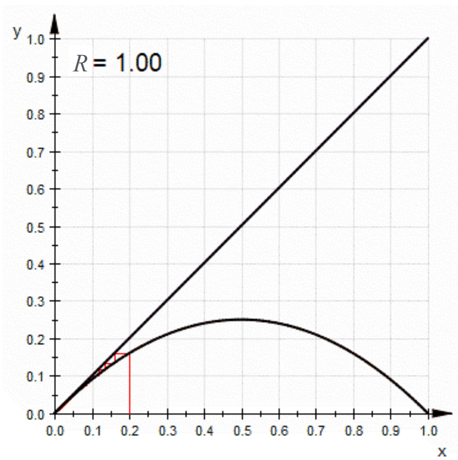
Onset of chaos

- Period doubling continues as R increases towards 4 when the dynamics becomes an infinite-period limit cycle
- This is the onset of *chaos* — seemingly random behavior that is difficult to predict in the long term



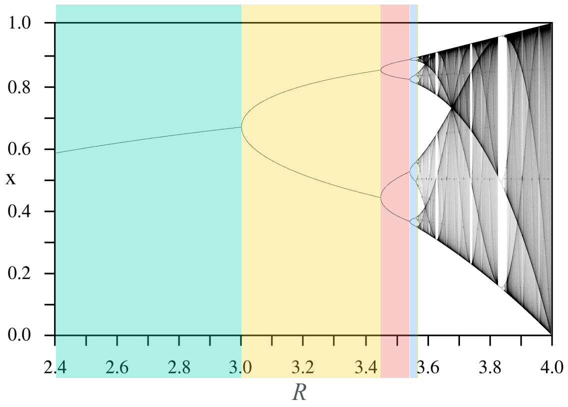


Logistic Map summary



Bifurcation diagram

- Attractor value(s) as a function of R



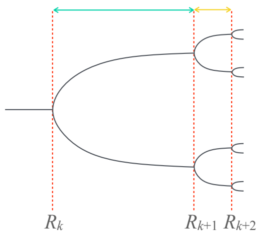
Bifurcations and universality

- Note that bifurcations occur at shorter and shorter distances
- How much shorter?

Bifurcation	*R*	*Behavior*	*Difference*
1	3.0	period 2 cycle	3.0
2	3.44949	period 4 cycle	0.44949
3	3.54409	period 8 cycle	0.0946
4	3.564407	period 16 cycle	0.020317
5	3.568759	period 32 cycle	0.004352
∞	4.0	chaos	

Bifurcations and universality

- Look at three consecutive bifurcation points



Bifurcation	*R*	*Ratio*
1	3.0	6.67423079
2	3.44949	4.75147992
3	3.54409	4.65619924
4	3.564407	4.66842831
5	3.568759	

- Compute the ratio of the distances
- Take the limit as the number of bifurcations tends to infinity

$$\lim_{k\to\infty} \frac{R_{k+1} - R_k}{R_{k+2} - R_{k+1}} \longrightarrow 4.6692016\ldots \text{ Feigenbaum's constant}$$

Bifurcations and universality

- Feigenbaum proved that the result applies to any dynamical system that is characterized through a "one-humped" map
  - Economics
  - Fluid dynamics
  - Electrical circuits
  - Chemical reactions
  - Brain activity (EEG)
  - Heart activity (EKG)
  - Solar system orbits
  - Dripping faucet
- all have the same rate of increase in bifurcations

Chaos versus randomness

- "The fact that the simple and deterministic equation — the Logistic Map — can possess dynamical trajectories which look like some sort of random noise has disturbing practical implications. It means, for example, that apparently erratic fluctuations in the census data for an animal population need not necessarily betoken either the vagaries of an unpredictable environment or sampling errors; they may simply derive from a rigidly deterministic population growth relationship … Alternatively, it may be observed that in the chaotic regime, *arbitrarily close initial conditions can lead to trajectories which, after a sufficiently long time, diverge widely*. This means that, even if we have a simple model in which all the parameters are determined exactly, long-term prediction is nevertheless impossible"

—Robert May, *Simple mathematical models with very complicated dynamics.* Nature (1976)

Chaos and initial conditions

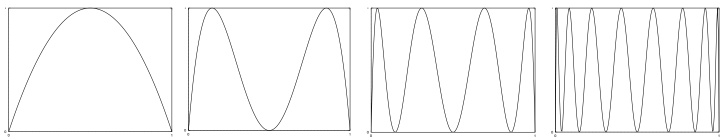
- Chaos is characterized not so much by randomness but by extreme sensitivity to initial conditions
- NetLogo SensitiveDependence with initial values 0.987654321 and 0.987654320 (difference in 10*th* decimal place)
- How many time steps necessary for the two trajectories to diverge significantly?
- Transform Logistic Map to a "Yes/No" function as follows:

$$x_t \leq 0.5 \implies \text{No}$$
$$x_t > 0.5 \implies \text{Yes}$$

- Define "diverge significantly" as "one answer Yes, the other No"

Chaos and limits to prediction

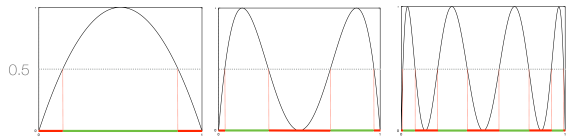
- Let $f(x) = 4(x - x^2)$
- Then, $x_1 = f(x_0) = f^1$
  - $x_2 = f(x_1) = f(f(x_0)) = f^2$
  - $x_3 = f(x_2) = f(f(f(x_0))) = f^3$
  - $x_4 = f(x_3) = f(f(f(f(x_0)))) = f^4$



f^1 1 hump | f^2 2 humps | f^3 4 humps | f^4 8 humps

Chaos and limits to prediction

- Consider the "Yes/No" formulation of the Logistic Map
- Will the answer at time step 1 be a "Yes" or a "No"?
- Look at f^1



f^1 | f^2 | f^3

- If the initial value is in a green region, the answer will be "Yes", if it is in a red region, the answer will be "No"
- What about m time steps into the future? Look at f^m

Chaos and limits to prediction

- In general, f^m will have 2^{m-1} humps
- The number of regions is twice the number of humps plus 1
- In other words, f^m will have $2 \times 2^{m-1}+1 = 2^m+1$ regions
- To predict f^m we need to distinguish which of the 2^m+1 regions the initial value falls into
- This requires that the initial value be encoded with at least $m+1$ bits of accuracy
- If we use fewer bits, the prediction can be no better than a random guess (flip a coin to decide between "Yes" and "No")
- Each time step into the future "consumes" one bit of information
- 0.987654321 requires roughly $9 \times 3 = 27$ bits to encode
- This explains why the two trajectories diverged after 27 steps

Characteristics of chaos

- All chaotic systems have the following properties:
  - *Deterministic*: given its history, the future of a chaotic system is not random but completely determined
  - *Sensitive*: chaotic systems are extremely sensitive to initial conditions (butterfly effect)
  - *Ergodic*: the state space trajectory of a chaotic system will always return to the local region of previous point on the trajectory
- These properties are necessary but not sufficient

Characteristics of chaos

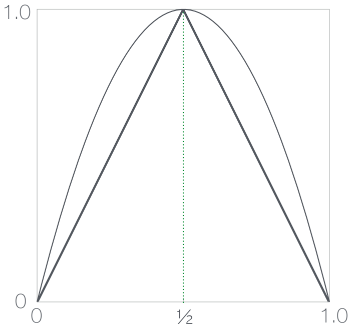
- Ergodic property implies that a *continuous time* system with fewer than 3 state variables cannot be chaotic (the Logistic map is one-dimensional and chaotic but it is a *discrete time* system)
- For contradiction, suppose that a *continuous time* system with only 2 state variables is chaotic
- State space (of 2 variables) can be seen as a plane
- Ergodicity requires that each point in this plane be reached, with no point ever being revisited
- Equivalent to covering the entire plane with ink without ever crossing a line or lifting the pen — impossible
- To not cross an existing line, must jump over it — 3rd dimension

Chaos and randomness

- Prior to chaos theory, we believed that determinism and randomness were mutually exclusive
- We believed randomness was possible only through physical processes related to quantum-level events (e.g., alpha decay)
- Chaos showed that it is possible to create a behavior that is effectively random through a deterministic process

Chaos and randomness
Tent map

- Let $L(x)=4(x-x^2)$ denote the Logistic Map with $R=4$
- Define the "Tent Map" $T(x) = \begin{cases} 2x & \text{for } 0 \le x < \frac{1}{2} \\ 2(1-x) & \text{for } \frac{1}{2} \le x \le 1 \end{cases}$



Chaos and randomness
Tent map

- Note: $T(x)$ is "piecewise linear" and discontinuous at the point ½
- Claim: the Logistic Map and the Tent Map are *equivalent*
- Need to find a transformation function $g(x)$ and its inverse $g^{-1}(x)$ such that $L(x)=g^{-1}(T(g(x)))$ and $T(x)=g(L(g^{-1}(x)))$
- Starting from an initial value, we can compute the entire trajectory using only $T(x)$, $g(x)$ and $g^{-1}(x)$ without ever needing $L(x)$
- Because of equivalence, whatever we prove about one map applies directly to the other

Chaos and randomness
Tent map

- How does the Tent Map $T(x)$ behave?
- Rewrite x in its binary form (recall $0 \le x \le 1$): $x=0.b_1b_2b_3\ldots$
- To compute $T(x)$ need to know if x is less than ½
- If x is less than ½, then b_1 must be 0: $x=0.0b_2b_3\ldots$
- If x is at least ½, then b_1 must be 1: $x=0.1b_2b_3\ldots$
- In other words, it suffices to examine b_1 to decide which of the two linear pieces of $T(x)$ to compute: $2x$ or $2(1-x)$
- $2x$ is easy — just left shift x one bit to the left: $x=0.b_2b_3\ldots$
- What about $2(1-x)$?

Chaos and randomness
Tent map

- Do $(1-x)$ first:

$$\begin{array}{r} 0.1\,1\;1\;1 \\ -0.1b_2b_3b_4 \\ =0.0\bar{b}_2\bar{b}_3\bar{b}_4 \end{array}$$

- Then, double by shifting left one bit to the left: $0.\bar{b}_2\bar{b}_3\bar{b}_4$
- Putting it all together:

$$T(x) = \begin{cases} 0.b_2b_3b_4 & \text{for } 0 \le x < \frac{1}{2} \\ 0.\bar{b}_2\bar{b}_3\bar{b}_4 & \text{for } \frac{1}{2} \le x \le 1 \end{cases}$$

- Each iteration "consumes" one bit of the input (left shift one)
- Complement or not the remaining bits depending on if x is less than ½

Chaos and randomness
Tent map

- What can we say about the long-term trajectory of $T(x)$?
- Three cases to consider depending on the choice of x_0
- *Case 1*: x_0 is a rational number with a finite binary representation such as $1/2 + 1/32 + 1/1024 = 0.1000100001$
- *Case 2*: x_0 is a rational number with an infinite but repeating binary representation such as $0.10111011101110111011\ldots$
- *Case 3*: x_0 is an irrational number with an infinite binary representation that never repeats such as
  $\pi/10 = 0.01010000011011\ldots$

Chaos and randomness
Logistic map

- Case 1 $\Longrightarrow$ “fixed point”
- Case 2 $\Longrightarrow$ “periodic”
- Case 3 $\Longrightarrow$ “chaotic”
- The Logistic Map at chaos can be turned into a “random bit generator”

$$x_t \leq 0.5 \Longrightarrow \text{"0"}$$
$$x_t > 0.5 \Longrightarrow \text{"1"}$$

- NetLogo LogisticMapInformationContent