Resource Calculus

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Previously, on Resource Calculus – 1993

- Boudol's lazy calculus.

  Gérard Boudol.
  The lambda-calculus with multiplicities.
  INRIA Research Report 2025, 1993.

- Calculus where arguments may come in limited availability, and mixed together.
- Main motivation: finer observational equivalence on ordinary λ-calculus.
- Boudol *left for future work* links with Girard's Linear Logic. . .

Previously, on Resource Calculus – 2003

- Ehrhard and Regnier's differential λ-calculus.

Thomas Ehrhard and Laurent Regnier.
The differential lambda-calculus.
Theor. Comput. Sci., 309(1):1–41, 2003.

- Calculus with differential operators (linear aproximation).
- Issued from semantical analysis.
- Heavy syntax…
- Link with proof theory? Yes but…

Previously, on Resource Calculus – 2006

- Linear (non lazy) resource calculus (no perpetual arguments).

Thomas Ehrhard and Laurent Regnier.
Differential interaction nets.
Theor. Comput. Sci., 364(2):166–195, 2006.

- The target of the Taylor expansion of ordinary λ-terms.

Thomas Ehrhard and Laurent Regnier.
Böhm trees, Krivine's machine and the Taylor expansion of lambda-terms.
In *CiE*, volume 3988 of *LNCS*, pages 186–197. Springer, 2006.

$$(MN)^* = \sum_{n=0}^{\infty} \frac{1}{n!} M^* [N^*]^k$$

Previously, on Resource Calculus – 2008

- Full (non lazy, non linear) resource calculus.

  Paolo Tranquilli.
  Intuitionistic differential nets and lambda calculus.
  To appear on *Theor. Comput. Sci.*, 2008.

- Convincing link with differential linear logic.
- First proof of confluence (via differential nets).

Now, on Resource Calculus

- Parallel reduction $\implies$ confluence and standardization (current work with Michele),
- Solvability (current work of Michele and Simona)
- ...

The syntax

terms: $M ::= \quad x \quad | \quad \lambda x.M \quad | \quad MN$

bags: $P ::= \quad [M_1, \ldots, M_m, N_1^!, \ldots, N_n^!]$

sums: $\mu ::= \quad M_1 + \cdots + M_m$

But weren't we just into limiting availability of arguments?

$$M[N]^k$$

All the rest follows...

The syntax

terms:	$M ::=$	x	$\mid$	$\lambda x.M$	$\mid$	MP
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All the rest follows. . .

First examples (and some whiteboard!)

- Ordinary λ-calculus: $MN \equiv M[N^!]$.
- Empty bag: 1. $M1$ means M does not get anything.

Informal definition of reduction:

$(\lambda x.M)[N] \to$ "M where N substitutes only a single occurrence of x"

- Need for sum (nondeterminism): $(\lambda x.yxx)[z]\ldots$
- Need for mixed bags: $(\lambda x.y_1(y_2x))[z]\ldots$

Oh my, sums everywhere?

Fortunately, not: sums are pushed to surface only.

$$\lambda x.(M+N) = \lambda x.M + \lambda x.N$$

$$(M+N)P = MP + NP \qquad \text{(function position is linear)}$$

$$M([N+L]\cdot P) = M([N]\cdot P) + M([L]\cdot P)$$

$$M([(N+L)^!]\cdot P) = M([N^!, M^!]\cdot P)$$

…and the zeroary versions

$$\lambda x.0 = 0 \qquad M([0]\cdot P) = 0$$

$$0P = 0 \qquad M([0^!]\cdot P) = MP$$

Something familiar…

$$[(M+N)^!] = [M^!]\cdot[N^!], \qquad [0^!] = 1$$

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Something familiar...

$$\mathrm{e}^{a+b} = \mathrm{e}^a \cdot \mathrm{e}^b \quad , \quad \mathrm{e}^0 = 1$$

Time to get the hands on substitutions

- $M\{N/x\}$: good old capture free substitution.
- $M\langle N/x\rangle$: linear substitution

$$y\langle N/x\rangle := \begin{cases} N & \text{if } y = x, \\ 0 & \text{otherwise,} \end{cases}$$

$$(MP)\langle N/x\rangle := M\langle N/x\rangle P + M(P\langle N/x\rangle),$$

$$(P \cdot R)\langle N/x\rangle := P\langle N/x\rangle \cdot R + P \cdot R\langle N/x\rangle.$$

$$[M]\langle N/x\rangle := [M\langle N/x\rangle],$$

$$[M^!]\langle N/x\rangle := [M\langle N/x\rangle, M^!],$$

Again, something familiar…

$$(P \cdot R)\langle N/x\rangle = P\langle N/x\rangle \cdot R + P \cdot R\langle N/x\rangle,$$

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Again, something familiar...

$$
\begin{aligned}
\frac{\partial u \cdot v}{\partial x} &= \frac{\partial u}{\partial x} \cdot v + u \cdot \frac{\partial v}{\partial x} \qquad , \\
\frac{\partial \mathrm{e}^u}{\partial x} &= \frac{\partial u}{\partial x} \cdot \mathrm{e}^u \qquad .
\end{aligned}
$$

Other tidbits on substitutions

Lemma (Schwartz)

For u regular enough

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}u\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}u\right).$$

Reduction will be

$$\begin{aligned}
(\lambda x.M)[N] \cdot P &\to (\lambda x.M\langle N/x\rangle)P, \\
(\lambda x.M)[N^!] \cdot P &\to (\lambda x.M\{x + N/x\})P, \\
(\lambda x.M)1 &\to M\{0/x\}.
\end{aligned}$$

Other tidbits on substitutions

Lemma (Schwartz)

For $x \notin \mathrm{FV}(L)$, $y \notin \mathrm{FV}(N)$ (also when $x = y$)

$$M\langle N/x\rangle\langle L/y\rangle = M\langle L/y\rangle\langle N/x\rangle.$$

Reduction will be

$$\begin{aligned}(\lambda x.M)[N] \cdot P &\to (\lambda x.M\langle N/x\rangle)P,\\ (\lambda x.M)[N^!] \cdot P &\to (\lambda x.M\{x + N/x\})P,\\ (\lambda x.M)1 &\to M\{0/x\}.\end{aligned}$$

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Other tidbits on substitutions

Lemma (Schwartz)

Linear and partial substitution commute (even on the same variable).

Reduction will be

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Examples

- $x\{x + N_1/x\} \cdots \{x + N_k/x\} = x + \sum_i N_i.$
- $x\langle L_1/x\rangle \cdots \langle L_h/x\rangle\{x + \sum_i N_i/x\} = \begin{cases} 0 & \text{if } h > 1, \\ L_1 & \text{if } h = 1, \\ x + \sum_i N_i & \text{if } h = 0. \end{cases}$
- $$\begin{aligned} x[x]\langle L/x\rangle &= L[x] + x[L] \\ x[x]\langle L/x\rangle\langle N/x\rangle &= L[N] + N[L]. \\ x[x]\langle L/x\rangle\langle N/x\rangle\langle K/x\rangle &= 0 \end{aligned}$$
- $$\begin{aligned} x[x^!]\langle L/x\rangle &= L[x^!] + x[L, x^!] \\ x[x^!]\langle L/x\rangle\langle N/x\rangle &= L[N, x^!] + N[L, x^!] + x[L, N, x^!]. \\ x[x^!]\langle L/x\rangle\langle N/x\rangle\langle K/x\rangle &= L[N, K, x^!] + N[L, K, x^!] \\ &\quad + K[L, N, x^!] + x[L, N, K, x^!]. \end{aligned}$$
- $x[x^!]\{x + N/x\} = (N + x)[(N + x)^!] = N[N^!, x^!] + x[N^!, x^!].$

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- $x[x^!]\{x+N/x\} = (N+x)[(N+x)^!] = N[N^!,x^!]+x[N^!,x^!]$.

Reductions

Two flavours

baby-step $\begin{cases} (\lambda x.M)[N] \cdot P \xrightarrow{\mathrm{b}} (\lambda x.M\langle N/x\rangle)P \\ (\lambda x.M)[N^!] \cdot P \xrightarrow{\mathrm{b}} (\lambda x.M\{x + N/x\})P \\ (\lambda x.M)1 \xrightarrow{\mathrm{b}} M\{0/x\}. \end{cases}$

giant-step $(\lambda x.M)[L_1, \dots, L_h, N_1^!, \dots, N_k^!] \xrightarrow{\mathrm{g}}$
$M\langle L_1/x\rangle \cdots \langle L_h/x\rangle\{\sum_i N_i/x\}$

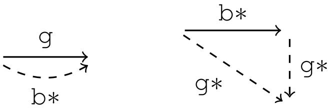
Sums: either one addend at a time or in parallel (up to taste)

$M+\nu \to \mu+\nu$ or $M_1 + \cdots + M_k + \nu \to \mu_1 + \cdots + \mu_k + \nu \quad (k \geq 1)$

Pros and cons of the two

baby-step : more atomic, more readable, closer to Boudol's original calculus.

giant-step : redex elimination, closer to ordinary λ-calculus, Curry-Howard with differential linear logic.



Examples, examples!

- $\Delta = \lambda x.xx = \lambda x.x[x^!]$
  - $\Delta[\Delta^!]$
  - $\Delta[\Delta]$
  - $\Delta[F, M^!]$
  - $(\lambda x.x[\Delta^!])[\Delta, I]$
- fixed point operator $Y \to \lambda f.f(Yf)$
  - $Y1 \to 0,$
  - $Y[F] \to F1,$
  - $Y[F]^2 \to 2F[F1],$
  - $Y[F]^3 \to 3!F[F[F1]] + 3F[F1]^2,$
  - $Y[F]^4 \to 4!F[F[F[F1]]] + 4!F[F[F1]^2] + 4!F[F[F1], F1] + 4F[F1]^3,$
  - $Y[F]^5 \to 5!F[F[F[F[F1]]]] + 5!F[F[F[F1]^2]] + 5!F[F[F[F1], F1]] + 20F[F[F1]^3] + 5!F[F[F[F1], F1]] + 60F[F[F1]^2, F1] + 60F[F[F1]]^2 + 60F[F[F1]] \cdot [F1]^2 + 5F[F1]^4$
  - $Y[F, G] \to F[G1] + G[F1].$
  - $Y[F^!, G^!] \to F[(Y[F^!, G^!])^!] + G[(Y[F^!, G^!])^!]$
  - $Y[(\lambda d.\mathtt{c}_0)^!, \mathtt{succ}^!] \to \mathtt{c}_0 + \mathtt{c}_1 + \mathtt{c}_2 + \cdots$
  - $Y[\lambda g.\lambda k.\mathtt{iszero}\, k\, \mathtt{c}_1\, (\mathtt{mult}\, (g\, (\mathtt{pred}\, k))\, \mathtt{c}_2)]^h$

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  - $Y[F]^4 \to 4!F[F[F[F1]]] + 4!F[F[F1]^2] + 4!F[F[F1], F1] + 4F[F1]^3,$
  - $$\begin{aligned} Y[F]^5 \to & 5!F[F[F[F[F1]]]] + 5!F[F[F[F1]^2]] + 5!F[F[F[F1], F1]] \\ & + 20F[F[F1]^3] + 5!F[F[F[F1], F1]] + 60F[F[F1]^2, F1] \\ & + 60F[F[F1]]^2 + 60F[F[F1]] \cdot [F1]^2 + 5F[F1]^4 \end{aligned}$$
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Confluence

Theorem

Both baby steps and giant steps are confluent.

With parallel reductions.

Q: Nondeterminism and confluence?

A: By modelling nondeterminism with sums while having confluence, one is assured that nondeterminism is internal: results are not precluded by evaluator.

Standardization

- We know in λ-calculus the properties of head reduction.
- Resource calculus suggests its founding property to be reducing in linear position.
- Place taken by $\xrightarrow{\circ}$, outer reduction, which does not reduce inside $M^!$ ($\xrightarrow{\mathtt{i}}$, inner reduction).

Theorem (Standardization)

If $M \xrightarrow{*} \mu$, then there is ν with $M \xrightarrow{\circ *} \nu \xrightarrow{\mathtt{i} *} \mu$.

And now, on to the whiteboard!

- Resource Calculus and other calculi
  - Boudol's calculus
  - Nondeterministic λ-calculus
  - Differential λ-calculus
- RC and linear algebra
- RC and typing