Bisimulation and co-induction

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Text material can be found at

`http://www.cs.unibo.it/~sangio/DOC_public/corsoFL.pdf`

or

`http://www.cs.unibo.it/~sangio/DOC_public/corsoFL.ps`

The semantics of processes:

- usually **operational**: (Labelled Transitions Systems, behavioural equivalences)
- alternative approach could be the **denotational** one: a structure-preserving function would map processes into elements of a given semantic domain.

  Problem: it has often proved very hard to find appropriate semantic domains for these languages

Thse lectures: An introduction to the meaning of behavioural equivalence

We especially discuss bisimulation, as an instance of the co-induction proof method

Outline

- From functions to processes
- Bisimulation
- Induction and Co-induction
- Weak bisimulation
- Other equivalences: failures, testing, trace ...

From processes to functions

Processes?

We can think of sequential computations as mathematical objects, namely **functions**.

Concurrent program are not functions, but **processes**. But what is a process?

No universally-accepted mathematical answer.

Hence we do not find in mathematics tools/concepts for the denotational semantics of concurrent languages, at least not as successful as those for the sequential ones.

Processes are not functions

A sequential imperative language can be viewed as a function from states to states.

These two programs denote the same function from states to states:

$$\texttt{X} := 2 \qquad \text{and} \qquad \texttt{X} := 1;\ \texttt{X} := \texttt{X} + 1$$

But now take a context with parallelism, such as $[\cdot] \mid \texttt{X} := 2$. The program

$$\texttt{X} := 2 \mid \texttt{X} := 2$$

always terminates with $\texttt{X} = 2$. This is not true (why?) for

$$(\texttt{X} := 1;\ \texttt{X} := \texttt{X} + 1) \mid \texttt{X} := 2$$

Therefore: Viewing processes as functions gives us a notion of equivalence that is not a **congruence**. In other words, such a semantics of processes as functions would not be **compositional**.

Furthermore:

- A concurrent program may not terminate, and yet perform meaningful computations (examples: an operating system, the controllers of a nuclear station or of a railway system).
  In sequential languages programs that do not terminate are undesirable; they are ‘wrong’.

- The behaviour of a concurrent program can be non-deterministic.
  Example:
  $$(\ \mathtt{X} := 1;\ \mathtt{X} := \ \mathtt{X} + 1\ )\ |\ \ \mathtt{X} := 2$$
  In a functional approach, non-determinism can be dealt with using powersets and powerdomains.

This works for pure non-determinism, as in $\lambda x.\,(3 \oplus 5)$

But not for parallelism.

What is a process?
When are two processes behaviourally equivalent?

These are basic, fundamental, questions; they have been at the core of the research in concurrency theory for the past 30 years. (They are still so today, although remarkable progress has been made)

Fundamental for a model or a language on top of which we want to make proofs ...

We shall approach these questions from a simple case, in which interactions among processes are just synchronisations, without exchange of values.

Interaction

In the example at page 6

$$\mathtt{X} := 2 \qquad \text{and} \qquad \mathtt{X} := 1;\ \mathtt{X} := \ \mathtt{X} + 1$$

should be distinguished because they interact in a different way with the memory.

Computation is **interaction**. Examples: access to a memory cell, interrogating a data base, selecting a programme in a washing machine, ...

The participants of an interaction are **processes** (a cell, a data base, a washing machine, ...)

The **behaviour** of a process should tell us **when** and **how** a process can interact with its environment

How to represent interaction: labelled transition systems

Definition 1 A **labeled transition system** (LTS) is a triple $(\mathcal{P}, \mathtt{Act}, \mathcal{T})$ where

- $\mathcal{P}$ is the set of **states**, or **processes**;
- $\mathtt{Act}$ is the set of **actions**; (NB: can be infinite)
- $\mathcal{T} \subseteq (\mathcal{P}, \mathtt{Act}, \mathcal{P})$ is the **transition relation**.

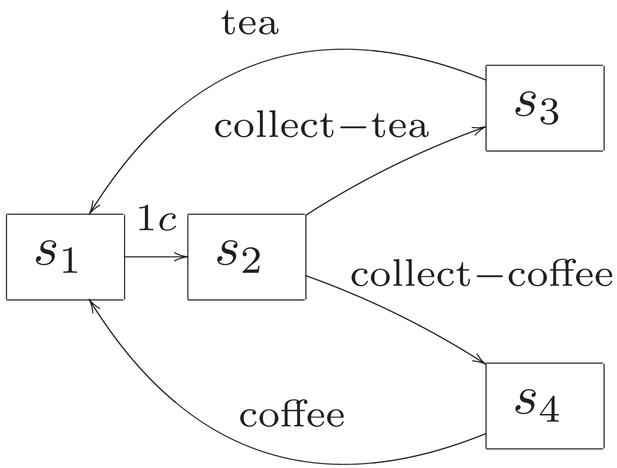
We write $P \xrightarrow{\mu} P'$ if $(P, \mu, P') \in \mathcal{T}$. Meaning: process P accepts an interaction with the environment where P performs action μ and then becomes process P'.

P' is a **derivative** of P if there are $P_1, \ldots, P_n, \mu_1, \ldots, \mu_n$ s.t. $P \xrightarrow{\mu_1} P_1 \ldots \xrightarrow{\mu_n} P_n$ and $P_n = P'$.

Example

A vending machine, capable of dispensing tea or coffee for 1 coin (1c).

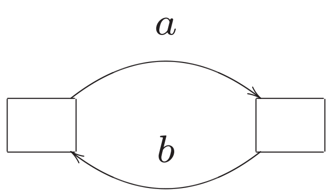
The behaviour of the machine is what we can observe, by interacting with the machine. We can represent such a behaviour as an LTS:

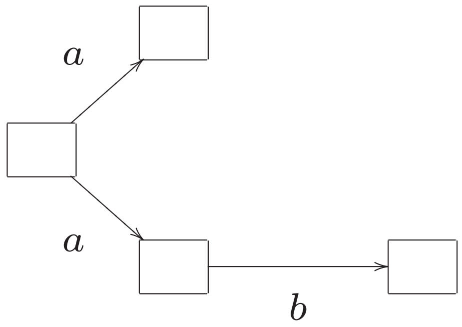


(where s_1 is the initial state)

Other examples of LTS

(we omit the name of the states)

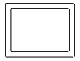




Equivalence of processes

An LTS tells us what is the behaviour of processes. When should two behaviours be considered equal? ie, what does it mean that two processes are equivalent?

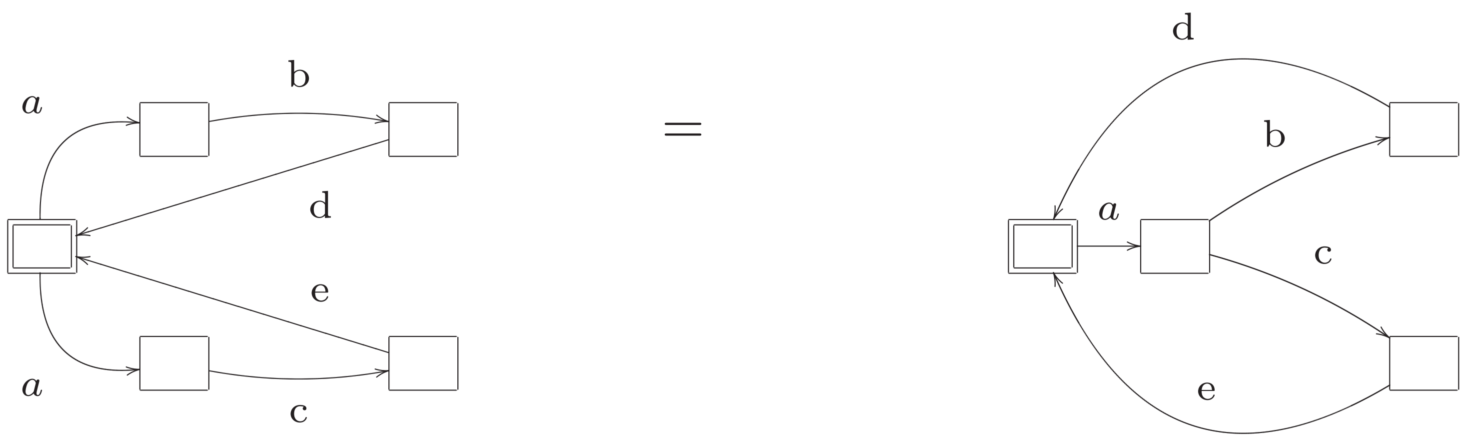
Two processes should be equivalent if we cannot distinguish them by interacting with them.

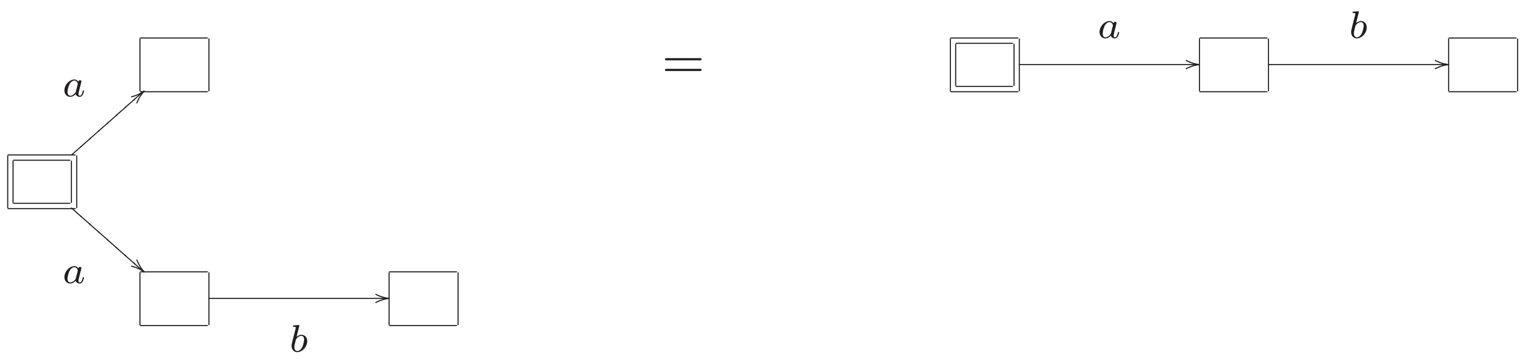
Example (where  indicates the processes we are interested in):

This shows that **graph isomorphism** as behavioural equivalence is too strong.

A natural alternative (from automata theory): **trace equivalence**.

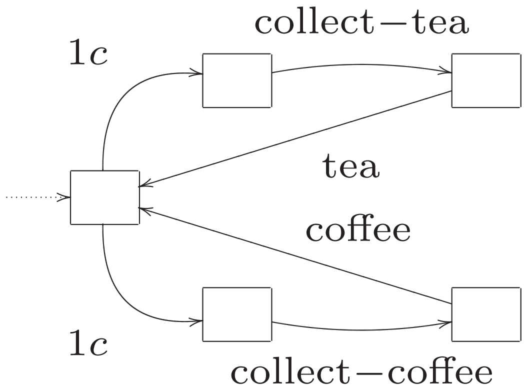
Examples of trace-equivalent processes:

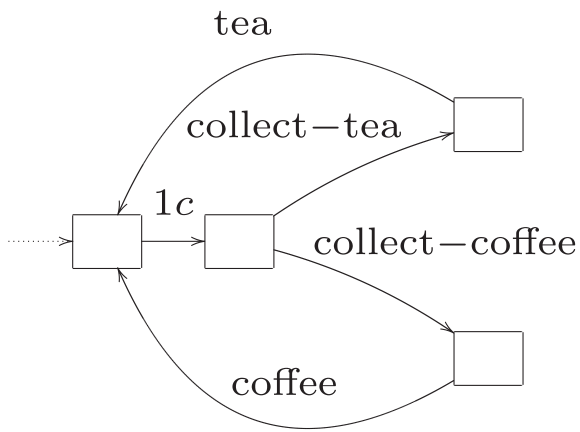




These equalities are OK on automata. But they are not on processes: in one case interacting with the machine can lead to deadlock!

For instance, you would not consider these two vending machines 'the same':





Trace equivalence (also called language equivalence) is still important in concurrency.

Examples: confluent processes; liveness properties such as termination

These examples suggest that the notion of equivalence we seek:

- should imply a tighter correspondence between transitions than language equivalence,
- should be based on the informations that the transitions convey, and not on the shape of the diagrams.

Intuitively, what does it mean for an observer that two machines are equivalent?

If you do something with one machine, you must be able to the same with the other, and on the two states which the machines evolve to the same is again true.

This is the idea of equivalence that we are going to formalise; it is called **bisimilarity**.

Bisimulation

References:

Robin Milner, *Communication and Concurrency*, Prentice Hall, 1989.

We define bisimulation on a single LTS, because: the union of two LTSs is an LTS; we will often want to compare derivatives of the same process.

Definition 2 (bisimulation) A relation $\mathcal{R}$ on the states of an LTS is a **bisimulation** if whenever $P \; \mathcal{R} \; Q$:

1. if $P \xrightarrow{\mu} P'$, then there is Q' such that $Q \xrightarrow{\mu} Q'$ and $P' \; \mathcal{R} \; Q'$.
2. if $Q \xrightarrow{\mu} Q'$, then there is P' such that $P \xrightarrow{\mu} P'$ and $P' \; \mathcal{R} \; Q'$.

P and Q are **bisimilar**, written $P \sim Q$, if $P \; \mathcal{R} \; Q$, for some bisimulation $\mathcal{R}$.

The bisimulation diagram:

$$\begin{array}{ccc} P & \mathcal{R} & Q \\ \mu \downarrow & & \mu \downarrow \\ P' & \mathcal{R} & Q' \end{array}$$

Exercises

Exercise 3 Prove that the processes at page 13 are bisimilar. Are the processes at page 14 bisimilar?

Proposition 4 1. $\sim$ is an equivalence relation, i.e. the following hold:

1.1. $p \sim p$ (reflexivity)

1.2. $p \sim q$ implies $q \sim p$ (symmetry)

1.3. $p \sim q$ and $q \sim r$ imply $p \sim r$ (transitivity);

2. $\sim$ itself is a bisimulation.

Proposition 4(2) suggests an alternative definition of $\sim$:

Proposition 5 $\sim$ is the largest relation among the states of the LTS such that $P \sim Q$ implies:

1. if $P \xrightarrow{\mu} P'$, then there is Q' such that $Q \xrightarrow{\mu} Q'$ and $P' \sim Q'$.
2. if $Q \xrightarrow{\mu} Q'$, then there is P' such that $P \xrightarrow{\mu} P'$ and $P' \sim Q'$.

Exercise 6 Prove Propositions 4-5
(for 4.2 you have to show that

$$\cup\{\mathcal{R} \; : \; \mathcal{R} \text{ is a bisimulation } \}$$

is a bisimulation).

We write $P \sim\mathcal{R}\sim Q$ if there are P', Q' s.t. $P \sim P'$, $P' \mathcal{R} Q'$, and $Q' \sim Q$ (and alike for similar notations).

Definition 7 (bisimulation up-to $\sim$) A relation $\mathcal{R}$ on the states of an LTS is a *bisimulation up-to* $\sim$ if $P \mathcal{R} Q$ implies:

1. if $P \xrightarrow{\mu} P'$, then there is Q' such that $Q \xrightarrow{\mu} Q'$ and $P' \sim\mathcal{R}\sim Q'$.
2. if $Q \xrightarrow{\mu} Q'$, then there is P' such that $P \xrightarrow{\mu} P'$ and $P' \sim\mathcal{R}\sim Q'$.

Exercise 8 If $\mathcal{R}$ is a bisimulation up-to $\sim$ then $\mathcal{R} \subseteq\sim$. (Hint: prove that $\sim \mathcal{R} \sim$ is a bisimulation.)

Definition 9 (simulation) A relation $\mathcal{R}$ on the states of an LTS is a *simulation* if $P \mathrel{\mathcal{R}} Q$ implies:

1. if $P \xrightarrow{\mu} P'$, then there is Q' such that $Q \xrightarrow{\mu} Q'$ and $P' \mathrel{\mathcal{R}} Q'$.

P is *simulated by* Q, written $P < Q$, if $P \mathrel{\mathcal{R}} Q$, for some simulation $\mathcal{R}$.

Exercise* 10 Does $P \sim Q$ imply $P < Q$ and $Q < P$? What about the converse? (Hint for the second point: think about the 2nd equality at page 14.)

- Bisimulation has been introduced in Computer Science by Park (1981) and made popular by Milner.

- Bisimulation is a robust notion: characterisations of bisimulation have been given in terms of non-well-founded-sets, modal logic, final coalgebras, open maps in category theory, etc.

- But the most important feature of bisimulation is the associated **co-inductive** proof technique.

Induction and co-induction

Co-inductive definitions and co-inductive proofs

Consider this definition of $\sim$ (Proposition 5):

$\sim$ is the largest relation such that $P \sim Q$ implies:

1. if $P \xrightarrow{\mu} P'$, then there is Q' such that $Q \xrightarrow{\mu} Q'$ and $P' \sim Q'$.
2. if $Q \xrightarrow{\mu} Q'$, then there is P' such that $P \xrightarrow{\mu} P'$ and $P' \sim Q'$.

It is a circular definition; does it make sense?

We claimed that we can prove $(P, Q) \in \sim$ by showing that $(P, Q) \in \mathcal{R}$ and $\mathcal{R}$ is a *bisimulation relation*, that is a relation that satisfies the same clauses as $\sim$. Does such a proof technique make sense?

Contrast all this with the usual, familiar *inductive definitions* and *inductive proofs*.

The above definition of $\sim$, and the above proof technique for $\sim$ are examples of *co-inductive definition* and of *co-inductive proof technique*.

Bisimulation and co-induction: what are we talking about?

Has *co*-induction anything to do with induction?

An example of an inductive definition: reduction to a value in the λ-calculus

The set Λ of λ-terms (an inductive def!)

$$e ::= x \;\Big|\; \lambda x.\, e \;\Big|\; e_1(e_2)$$

Consider the definition of $\Downarrow_n$ in λ-calculus (convergence to a value):

$$\frac{}{\lambda x.\, e \Downarrow_n \lambda x.\, e} \qquad \frac{e_1 \Downarrow_n \lambda x.\, e_0 \qquad e_0\{{}^{e_2}\!/\!_x\} \Downarrow_n e'}{e_1(e_2) \Downarrow_n e'}$$

$\Downarrow_n$ is the *smallest* relation on λ-terms that is closed forwards under these rules; i.e., the smallest subset C of $\Lambda \times \Lambda$ s.t.

- $\lambda x.\, e C \lambda x.\, e$ for all abstractions,
- if $e_1 \; C \; \lambda x.\, e_0$ and $e_0\{{}^{e_2}\!/\!_x\} \; C \; e'$ then also $e_1(e_2) \; C \; e'$.

An example of a co-inductive definition: divergence in the λ-calculus

Consider the definition of $\Uparrow^n$ (divergence) in λ-calculus :

$$\frac{e_1 \Uparrow^n}{e_1(e_2) \Uparrow^n} \qquad\qquad \frac{e_1 \Downarrow_n \lambda x.\, e_0 \qquad e_0\{^{e_2}\!/\!_x\} \Uparrow^n}{e_1(e_2) \Uparrow^n}$$

$\Uparrow^n$ is the *largest* predicate on λ-terms that is closed backwards under these rules; i.e., the largest subset D of Λ s.t. if $e \in D$ then

- either $e = e_1(e_2)$ and $e_1 \in D$,
- or $e = e_1(e_2)$, $e_1 \Downarrow_n \lambda x.\, e_0$ and $e_0\{^{e_2}\!/\!_x\} \in D$.

Hence: to prove e is divergent it suffices to find $E \subseteq \Lambda$ that is closed backwards and with $e \in E$ (co-induction proof technique).

What is the smallest predicate closed backwards?

An example of an inductive definition: finite lists over a set A

$$\frac{}{\texttt{nil} \in \mathcal{L}} \qquad\qquad \frac{\ell \in \mathcal{L} \qquad a \in A}{\texttt{cons}(a, \ell) \in \mathcal{L}}$$

Finite lists: the set generated by these rules; i.e., the smallest set closed forwards under these rules.

Inductive proof technique for lists: Let $\mathcal{P}$ be a predicate (a property) on lists. To prove that $\mathcal{P}$ holds on all lists, prove that

- $\texttt{nil} \in \mathcal{P}$;
- $\ell \in \mathcal{P}$ implies $\texttt{cons}(a, \ell) \in \mathcal{P}$, for all $a \in A$.

An example of an co-inductive definition: finite and infinite lists over a set A

$$\frac{}{\texttt{nil} \in \mathcal{L}} \qquad \frac{\ell \in \mathcal{L} \qquad a \in A}{\texttt{cons}(a, \ell) \in \mathcal{L}}$$

Finite and infinite lists: the largest set closed backwards under these rules.

To explain finite and infinite lists as a set, we need *non-well-founded sets* (Forti-Honsell, Aczel).

– * An inductive definition tells us what are the *constructors* for generating all the elements (cf: closure forwards).
  * A co-inductive definition tells us what are the *destructors* for decomposing the elements (cf: closure backwards).
    The destructors show what we can *observe* of the elements (think of the elements as black boxes; the destructors tell us what we can do with them; this is clear in the case of infinite lists).

– When a definition is give by means of some rules:
  * if the definition is inductive, we look for the smallest universe in which such rules live.
  * if it is co-inductive, we look for the largest universe.

The duality

constructors	destructors
inductive defs	co-inductive defs
induction technique	co-inductive technique
congruence	bisimulation
least fixed-points	greatest fixed-points

(The dual of a *bisimulation* is a *congruence*: intuitively: a bisimulation is a relation “closed backwards”, a congruence is “closed forwards”)

- In what sense are $\Downarrow_n, \Uparrow^n, \sim$ fixed-points?
- What is the co-induction proof technique?
- In what sense is co-induction dual to the familiar induction technique?

What follows answers these questions. It is a simple application of fixed-point theory.

To make things simpler, we work on *powersets*. (It is possible to be more general, working with universal algebras or category theory.)

For a given set S, the powerset of S, written $\wp(S)$, is

$$\wp(S) \triangleq \{T \ : \ T \subseteq S\}$$

$\wp(S)$ is a *complete lattice*.

Complete lattices are "dualisable" structures: reverse the arrows and you get another complete lattice.

From Knaster-Tarski's theorem for complete lattices, we know that if $\mathcal{F} : \wp(S) \rightarrow \wp(S)$ is monotone, then $\mathcal{F}$ has a least fixed-point (lfp), namely:

$$\mathcal{F}_{\mathrm{lfp}} \triangleq \bigcap \{A \ : \ \mathcal{F}(A) \subseteq A\}$$

As we are on a complete lattice, we can dualise the statement:

If $\mathcal{F} : \wp(S) \rightarrow \wp(S)$ is monotone, then $\mathcal{F}$ has a greatest fixed-point (gfp), namely:

$$\mathcal{F}^{\mathrm{gfp}} \triangleq \bigcup \{A \ : \ A \subseteq \mathcal{F}(A)\}$$

These results give us proof techniques for $\mathcal{F}_{\mathrm{lfp}}$ and $\mathcal{F}^{\mathrm{gfp}}$:

$$\text{if } \mathcal{F}(A) \subseteq A \text{ then } \mathcal{F}_{\mathrm{lfp}} \subseteq A \quad (1)$$

$$\text{if } A \subseteq \mathcal{F}(A) \text{ then } A \subseteq \mathcal{F}^{\mathrm{gfp}} \quad (2)$$

- Inductive definitions give us lfp's (precisely: an inductive definition tells us how to construct the lfp). Co-inductive definitions give us gfp's.
- On inductively-defined sets (1) is the same as the familiar induction technique (cf: example of lists). (2) gives us the co-inductive proof technique.

$\Downarrow_n$ and $\Uparrow^n$ as fixed-points

A set R of rules on a set S give us a function $\mathcal{R}$: $\wp(S) \to \wp(S)$, so defined:

$$\mathcal{R}(A) \triangleq \{a \ : \ \text{there are } a_1, \ldots, a_n \in A \ \text{ and a rule in } R \text{ so that using } a_1, \ldots, a_n \text{ as premises in the rule we can derive } a\}$$

$\mathcal{R}$ is monotone, and therefore (by Tarsky) has lfp and gfp.

In this way, the definitions of $\Downarrow_n$ and $\Uparrow^n$ can be formulated as lfp and gfp of functions.

For instance, in the case of $\Uparrow^n$, take $S = \Lambda$. Then

$$\mathcal{R}(A) = \{e_1(e_2) \ : \ e_1 \in A, \quad \text{or } e_1 \Downarrow_n \lambda x.\, e_0 \text{ and } e_0\{e_2/x\} \in A\}.$$

The co-induction proof technique for $\Uparrow^n$ mentioned at page 28 is just an instance of (2).

Bisimulation as a fixed-point

Let $(\mathcal{P}, \mathtt{Act}, \mathcal{T})$ be an LTS. Consider the function $\mathcal{F} : \wp(\mathcal{P} \times \mathcal{P}) \to \wp(\mathcal{P} \times \mathcal{P})$ so defined.

$\mathcal{F}(R)$ is the set of all pairs (P, Q) s.t.:

1. if $P \xrightarrow{\mu} P'$, then there is Q' such that $Q \xrightarrow{\mu} Q'$ and $(P', Q') \in \mathcal{R}$.
2. if $Q \xrightarrow{\mu} Q'$, then there is P' such that $P \xrightarrow{\mu} P'$ and $(P', Q') \in \mathcal{R}$.

Proposition 11 1. $\mathcal{F}$ is monotone;

2. $\sim = \mathcal{F}^{\mathrm{gfp}}$;

3. R is a bisimulation iff $R \subseteq \mathcal{F}(\mathcal{R})$.

The induction technique as a fixed-point technique: the example of finite lists

Let $\mathcal{F}$ be this function (from sets to sets):

$$\mathcal{F}(S) \triangleq \{\texttt{nil}\} \cup \{\texttt{cons}(a, s) \ : \ a \in A, s \in S\}$$

$\mathcal{F}$ is monotone, and $\text{finLists} = \mathcal{F}_{\text{lfp}}$. (i.e., finLists is the smallest set solution to the equation $\mathcal{L} = \texttt{nil} + \texttt{cons}(A, \mathcal{L})$).

From (1), we infer: Suppose $\mathcal{P} \subseteq \text{finLists}$. If $\mathcal{F}(\mathcal{P}) \subseteq \mathcal{P}$ then $\mathcal{P} \subseteq \text{finLists}$ (hence $\mathcal{P} = \text{finLists}$).

Proving $\mathcal{F}(\mathcal{P}) \subseteq \mathcal{P}$ requires proving

- $\texttt{nil} \in \mathcal{P}$;
- $\ell \in \text{finLists} \cap \mathcal{P}$ implies $\texttt{cons}(a, \ell) \in \mathcal{P}$, for all $a \in A$.

This is the same as the familiar induction technique for lists (page 29).

Note: $\mathcal{F}$ is defined the class of all sets, rather than on a powerset; the class of all sets is not a complete lattice (because of paradoxes such as Russel’s), but the constructions that we have seen for lfp and gfp of monotone functions apply.

Continuity

Another important theorem of fixed-point theory: if $\mathcal{F} : \wp(S) \to \wp(S)$ is continuous, then

$$\mathcal{F}_{\mathrm{lfp}} = \bigcup_n \mathcal{F}^n(\bot)$$

This has, of course, a dual, for gfp (also the definition of continuity has to be dualised), but: the function $\mathcal{F}$ of which bisimilarity is the gfp may not be continuous! (This is usually the case for *weak* bisimilarity, that we shall introduce later.)

It is continuous only if the LTS is finite-branching, meaning that for all P the set $\{P' \ : \ P \xrightarrow{\mu} P', \text{ for some } \mu\}$ is finite.

On a finite branching LTS, it is indeed the case that

$$\sim = \bigcap_n \mathcal{F}^n(\mathcal{P} \times \mathcal{P})$$

where $\mathcal{P}$ is the set of all processes.

Stratification of bisimilarity

Continuity, operationally:

Consider the following sequence of equivalences, inductively defined:

$\sim_0 \triangleq \mathcal{P} \times \mathcal{P}$

$P \sim_{n+1} Q \;\triangleq\;$:

1. if $P \xrightarrow{\mu} P'$, then there is Q' such that $Q \xrightarrow{\mu} Q'$ and $P' \sim_n Q'$.
2. if $Q \xrightarrow{\mu} Q'$, then there is P' such that $P \xrightarrow{\mu} P'$ and $P' \sim_n Q'$.

Then set:

$\sim_\omega \triangleq \bigcap_n \sim_n$

Theorem 12 On processes that are image-finite: $\sim = \sim_\omega$

Image-finite processes :

each reachable state can only perform a finite set of transitions.

Abbreviation: $a^n \triangleq a. \ldots a. 0$ (n times)

Example: $\sum_{1 \leq i \leq n} a^n$ (note: n is fixed)

Non-example: $P \triangleq \sum_{1 \leq i < \omega} a^n$

In the theorem, image-finiteness is necessary:

$P \sim_\omega P + a^\omega$ but $P \not\sim P + a^\omega$

The stratification of bisimilarity given by continuity is also the basis for **algorithms** for mechanically checking bisimilarity and for minimisation of the state-space of a process

These algorithms work on processes that are **finite-state** (ie, each process has only a finite number of possible derivaties)

They proceed by progressively refining a partition of all processes

In the initial partition, all processes are in the same set

Bisimulation: P-complete

[Alvarez, Balcazar, Gabarro, Santha, ’91]

With m transitions, n states:

O(m log n) time and O(m + n) space **[Paige, Tarjan, ’87]**

Trace equivalence, testing: PSPACE-complete

[Kannelakis, Smolka, ’90; Huynh, Tian, 95]

The success of bisimulation

and co-induction

Bisimulation in Computer Science

- One of the most important contributions of concurrency theory to CS
- It has spurred the study of coinduction
- In concurrency: probably the most studied equivalence
  - ... in a plethora of equivalences **(see van Glabbeek 93)**
  - Why?

Bisimulation in concurrency

- **Clear** meaning of equality
- **Natural**
- The **finest** extensional equality

  Extensional: – "whenever it does an output at b
  it will also do an input at a"

  Non-extensional: – "Has 8 states"
  – "Has an Hamiltonian circuit"
- An associated powerful **proof technique**
- **Robust**

  Characterisations: logical, algebraic, set-theoretical, categorical, game-theoretical, ...
- Several **separation results** from other equivalences

Bisimulation in concurrency, today

- To **define equality** on processes (fundamental !!)
- To **prove equalities**
  - ∗ even if bisimilarity is not the chosen equivalence
    - · trying bisimilarity first
    - · coinductive characterisations of the chosen equivalence
- To **justify algebraic laws**
- To **minimise** the state space
- To **abstract** from certain details

Coinduction in programming languages

- **Bisimilarity in functional languages and OO languages**

  [Abramsky, Ong]

  A major factor in the movement towards operationally-based techniques in PL semantics in the 90s

- **Program analysis** **(see Nielson, Nielson, Hankin 's book)**

  Noninterference (security) properties

- **Verification tools**: algorithms for computing gfp (for modal and temporal logics), tactics and heuristics

- **Types** **[Tofte]**
  - type soundness
  - coinductive types and definition by corecursion

    Infinite proofs in Coq **[Coquand, Gimenez]**
  - recursive types (equality, subtyping, ...)

    A coinductive rule:

$$\frac{\Gamma, \langle p_1, q_1 \rangle \sim \langle p_2, q_2 \rangle \vdash p_i \sim q_i}{\Gamma \vdash \langle p_1, q_1 \rangle \sim \langle p_2, q_2 \rangle}$$

- **Recursively defined data types and domains** **[Fiore, Pitts]**
- **Databases** **[Buneman]**
- **Compiler correctness** **[Jones]**