The following text replaces the part of section 2.3 that begins at the line preceding lemma 2.3.19 and goes till the end of the section

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The differences from the text in the book are:
-- Lemma 2.3.19,
-- the proof of Lemma 2.3.21 and the text following it,
-- the text in brackets preceding 2.3.24,
-- Section 2.3.4.
```

Clearly, $\mathcal{F}_{\mathrm{ni1}}(\mathcal{R}) \subseteq \mathcal{F}_{\mathrm{ni}}(\mathcal{R})$ for any $\mathcal{R}$. We also have (we recall that $\mathcal{F}_{\iota}$ is the identity relation):

Lemma 2.3.19 $\mathcal{F}_{\mathrm{ni}} \subseteq (\mathcal{F}_{\mathrm{ni1}})^* \cup \mathcal{F}_{\iota}$. □

As an example of Lemma 2.3.19, consider the relation $\mathcal{R} = \{(P_1, Q_1), (P_2, Q_2)\}$, and a context C with two holes, each of which appears once. Writing $\eta_{(R_1,R_2)}$ for the instantiation that maps the two holes to the processes R_1 and R_2, we have

$$(C\eta_{(P_1,P_2)}, C\eta_{(Q_1,Q_2)}) \in \mathcal{F}_{\mathrm{ni}}(\mathcal{R}).$$

Since $(\mathcal{F}_{\mathrm{ni1}})^*(\mathcal{R})$ is the transitive closure of $\mathcal{F}_{\mathrm{ni1}}(\mathcal{R})$, the inclusion $(C\eta_{(P_1,P_2)}, C\eta_{(Q_1,Q_2)}) \in (\mathcal{F}_{\mathrm{ni1}})^*(\mathcal{R})$ is proved thus:

$$C\eta_{(P_1,P_2)}\ \mathcal{F}_{\mathrm{ni1}}(\mathcal{R})\ C\eta_{(P_1,Q_2)}\ \mathcal{F}_{\mathrm{ni1}}(\mathcal{R})\ C\eta_{(Q_1,Q_2)}\,.$$

We need the identity relation in Lemma 2.3.19 because $\mathcal{F}_{\mathrm{ni}}$ includes 0-ary contexts. To see that the inclusion in the lemma is strict, consider the relation

$$\mathcal{R} = \{(a.\mathbf{0}, b.\mathbf{0}), (b.\mathbf{0}, c.\mathbf{0})\}$$

We have $(a.\mathbf{0}, c.\mathbf{0}) \in (\mathcal{F}_{\mathrm{ni1}})^*(\mathcal{R})$, but not $(a.\mathbf{0}, c.\mathbf{0}) \in \mathcal{F}_{\mathrm{ni}}(\mathcal{R})$.

Exercise 2.3.20 Prove Lemma 2.3.19. □

We want to use the $\mathcal{F}_{\mathrm{ni}}$ function as an up-to technique, and we want to combine it with other up-to techniques. For this, one can directly prove that $\mathcal{F}_{\mathrm{ni}}$ is strongly safe.

Lemma 2.3.21 $\mathcal{F}_{\mathrm{ni}}$ is strongly safe.

Proof The proof is similar to that of Lemma 2.3.29, which shows the safety of $(\mathcal{F}_{\mathrm{ni1}})^*$, but more complex because we have to use multi-hole contexts, rather than just single-hole contexts. □

It is actually possible to avoid having to prove Lemma 2.3.21, by exploiting the (simpler) proof of the safety of $(\mathcal{F}_{\mathrm{ni1}})^*$ (see Lemma 2.3.29). This alternative strategy is reported in Section 2.3.4. It involves a strengthening of the method based on safe functions. The new method uses the *quasi-safe* functions. Their importance is however limited. In the book, they are used only to show the alternative to Lemma 2.3.21 (for instance, this alternative is only feasible in the case of strong equivalences; for weak equivalences we will be forced to stick to Lemma 2.3.21). In the remainder of the book we freely refer to Lemma 2.3.21 and avoid references to results concerning quasi-safe functions. However, all references to Lemma 2.3.21 could be replaced by its alternative, Corollary 2.3.30. (The proof of the safety of $(\mathcal{F}_{\mathrm{ni1}})^*$, Lemma 2.3.29, is however instructive, and we encourage the reader to look at it; it also allows one to understand better the definition of safe function: see the discussion in the paragraph following the lemma.)

We call $\mathcal{R}$ a *bisimulation up to non-input context* if $\mathcal{R} \rightsquigarrow \mathcal{F}_{\mathrm{ni}}(\mathcal{R})$. By definition, if $\mathcal{R}$ is a bisimulation up to non-input context, $(P, Q) \in \mathcal{R}$, and $P \xrightarrow{\alpha} P'$, then there are Q' such that $Q \xrightarrow{\alpha} Q'$, instantiations η and η' such that $(\eta_i, \eta'_i) \in \mathcal{R}$ for all i, and a non-input multi-hole context C such that $P' = C\eta$ and $Q' = C\eta'$. Since a common multi-hole context can be factored out of matching derivatives, a bisimulation up to non-input context need not be a large relation. For instance, as we will see below, given a process P, the singleton relation $\{(\,!P \mid !P,\, !P)\}$ is a bisimulation up to non-input context. On the other hand, the smallest bisimulation containing $(\,!P \mid !P,\, !P)$ is infinite if P has any transition.

A special case of bisimulation up to non-input context that we will use later is bisimulation up to restriction. We call $\mathcal{R}$ a *bisimulation up to restriction* if $\mathcal{R} \rightsquigarrow \mathcal{F}_{\boldsymbol{\nu}}(\mathcal{R})$, where the safe function $\mathcal{F}_{\boldsymbol{\nu}}$ is defined by

$$\mathcal{F}_{\boldsymbol{\nu}}(\mathcal{R}) = \{((\boldsymbol{\nu}\widetilde{z})\,P, (\boldsymbol{\nu}\widetilde{z})\,Q) \;\mid\; (P, Q) \in \mathcal{R},\ \widetilde{z} \text{ arbitrary}\}\,.$$

Thus far we have considered proving bisimilarity. We now consider showing full bisimilarity. Recall that by definition, $P \sim^{\mathrm{c}} Q$ if $P\sigma \sim Q\sigma$ for all σ. Hence by Lemma 2.3.8:

Lemma 2.3.22 If $(P, Q) \in \mathcal{R}$ where $\mathcal{R}$ is closed under substitution, and $\mathcal{R} \rightsquigarrow \mathcal{F}(\mathcal{R})$ where $\mathcal{F}$ is strongly safe, then $P \sim^{\mathrm{c}} Q$. □

In particular by Lemma 2.3.21:

Corollary 2.3.23 If $(P, Q) \in \mathcal{R}$ where $\mathcal{R}$ is closed under substitution and $\mathcal{R} \rightsquigarrow \mathcal{F}_{\mathrm{ni}}(\mathcal{R})$, then $P \sim^{\mathrm{c}} Q$. □

We finish with two results about contexts in which some hole occurs under an input prefix. The first can be proved using a strategy similar to that used to prove Lemma 2.3.21 (here again, one could just prove quasi-safety, rather than safety, proceeding as in Section 2.3.4; the proof is a bit simpler).

Lemma 2.3.24 $\mathcal{F}_C$ is strongly safe, where $\mathcal{F}_C(\mathcal{R})$ is the set of all pairs of the form $(C\eta, C\eta')$ with C a multi-hole context and η, η' such that

(1) $(\eta_j, \eta'_j) \in \mathcal{R}$ for all j, and
(2) if $[\cdot]_i$ occurs under an input prefix in C, then $(\eta_i\sigma, \eta'_i\sigma) \in \mathcal{R}$ for all σ. $\square$

A consequence of Lemma 2.3.24 is the following strengthening of Corollary 2.3.23 for proving processes strong full bisimilar. This result differs from Corollary 2.3.23 in that arbitrary contexts can be used to show the progression.

Corollary 2.3.25 If $(P, Q) \in \mathcal{R}$ where $\mathcal{R}$ is closed under substitution and $\mathcal{R} \rightsquigarrow \mathcal{F}_C(\mathcal{R})$, then $P \sim^{\mathrm{c}} Q$. $\square$

2.3.3 Deferred proofs

We now use some of these techniques to give the proofs deferred from Section 2.2.

Proof (of Lemma 2.2.3)
Suppose $\mathcal{R}$ is a bisimulation with $(P, Q) \in \mathcal{R}$. Then since $\mathcal{R} \subseteq \mathcal{F}_\sigma(\mathcal{R})$ we have $\mathcal{R} \rightsquigarrow \mathcal{F}_\sigma(\mathcal{R})$, and the assertion follows by Lemma 2.3.10 and Lemma 2.3.8. $\square$

Proof (of Theorem 2.2.8(1))
This follows from Corollary 2.3.9 and Lemma 2.3.21. $\square$

Proof (of Theorem 2.2.8(2))
If $\mathcal{R}$ is a bisimulation containing $(P\{y/z\}, Q\{y/z\})$ for each $y \in \mathsf{fn}(P, Q, z)$, then

$$\{(x(z).\,P, x(z).\,Q)\} \cup \mathcal{R}$$

is a bisimulation up to injective substitution. $\square$

Proof (of Lemma 2.2.23)
We prove just the first assertion; the proofs of the other two are similar and are left as exercises for the interested reader.
Let $\mathcal{R}$ be the relation

$$\{(!P \mid !P,\ !P) \ \mid\ P \text{ arbitrary}\}\ .$$

Let $\mathcal{G}$ be the safe function $\mathcal{F}_{\equiv}\mathcal{F}_{\mathrm{ni}}\mathcal{F}_{\equiv}$. Since $\mathcal{R}$ is closed under substitution, it suffices by Lemma 2.3.22 to show that $\mathcal{R} \rightsquigarrow \mathcal{G}(\mathcal{R})$.

Suppose that $!P \mid !P \xrightarrow{\alpha} Q$. We have to show that there are R such that $!P \xrightarrow{\alpha} R$ and a multi-hole context C and instantiations η and η' such that $Q \equiv C\eta$ and $R \equiv C\eta'$ and $(\eta_i, \eta'_i) \in \mathcal{R}$ for all i. There are four cases.

(1) Suppose $Q = R \mid !P$ where $!P \xrightarrow{\alpha} R$. Then $R = S \mid !P$ where either $P \xrightarrow{\alpha} S$ or $\alpha = \tau$ and $P \mid P \xrightarrow{\alpha} S$. In either case $!P \xrightarrow{\alpha} R$, and $(Q, R) \in \mathcal{G}(\mathcal{R})$ because $Q \equiv S \mid (!P \mid !P)$.

(2) If $Q = !P \mid R$ where $!P \xrightarrow{\alpha} R$, then the argument is similar.

(3) If $Q = R_1 \mid R_2$ where $!P \xrightarrow{\overline{x}y} R_1$ and $!P \xrightarrow{xy} R_2$ and $\alpha = \tau$ (or similarly with the roles of sender and receiver swapped), then $R_1 = S_1 \mid !P$ where $P \xrightarrow{\overline{x}y} S_1$, and $R_2 = S_2 \mid !P$ where $P \xrightarrow{xy} S_2$. Hence by transition rule REP-COMM, $!P \xrightarrow{\alpha} R \stackrel{\mathrm{def}}{=} S_1 \mid S_2 \mid !P$, and $(Q, R) \in \mathcal{G}(\mathcal{R})$ because $Q \equiv S_1 \mid S_2 \mid (!P \mid !P)$.

(4) If $Q = \boldsymbol{\nu} z\,(R_1 \mid R_2)$ where $!P \xrightarrow{\overline{x}(z)} R_1$ and $!P \xrightarrow{xz} R_2$ and $z \notin \mathsf{fn}(P)$ and $\alpha = \tau$ (or similarly with the roles of sender and receiver swapped), then $R_1 = S_1 \mid !P$ where $P \xrightarrow{\overline{x}(z)} S_1$, and $R_2 = S_2 \mid !P$ where $P \xrightarrow{xz} S_2$. Hence by transition rule REP-CLOSE, $!P \xrightarrow{\alpha} R \stackrel{\mathrm{def}}{=} \boldsymbol{\nu} z\,(S_1 \mid S_2) \mid !P$, and $(Q, R) \in \mathcal{G}(\mathcal{R})$ because $Q \equiv \boldsymbol{\nu} z\,(S_1 \mid S_2) \mid (!P \mid !P)$.

Similarly, if $!P \xrightarrow{\alpha} R$ then $!P \mid !P \xrightarrow{\alpha} Q$ with $(Q, R) \in \mathcal{G}(\mathcal{R})$, as the reader may care to check. $\square$

Proof (of Theorem 2.2.25)

Let $\mathcal{R}$ be the relation containing all pairs of processes of the forms

$$\boldsymbol{\nu} x\,(P_1 \mid P_2 \mid !x(z).\,R) \quad \text{and} \quad \boldsymbol{\nu} x\,(P_1 \mid !x(z).\,R) \mid \boldsymbol{\nu} x\,(P_2 \mid !x(z).\,R)$$

such that x occurs in P_1, P_2, and R only in output subject position. Let $\mathcal{G}$ be the safe function $\mathcal{F}_{\sim}\mathcal{F}_{\mathrm{ni}}\mathcal{F}_{\sim}$. Since $\mathcal{R}$ is closed under substitution, it suffices by Lemma 2.3.22 to show that $\mathcal{R} \rightsquigarrow \mathcal{G}(\mathcal{R})$. An important theme in the proof is that the property that x occurs in a process only in output subject position is invariant provided the process does not receive x, and that the processes P_1 and P_2 cannot receive x because x is restricted.

Suppose $\boldsymbol{\nu} x\,(P_1 \mid P_2 \mid !x(z).\,R) \xrightarrow{\alpha} Q$. There are eight cases. They are:

(1) $Q = \boldsymbol{\nu} x\,(P'_1 \mid P_2 \mid !x(z).\,R)$ where $P_1 \xrightarrow{\alpha} P'_1$, noting that α cannot be of the form $\overline{a}x$ because x occurs in P_1 only in output subject position

(2) $Q = \boldsymbol{\nu} x\,(P'_1 \mid P'_2 \mid !x(z).\,R)$ where $P_1 \mid P_2 \xrightarrow{\alpha} P'_1 \mid P'_2$ and $\alpha = \tau$

(3) $Q = \boldsymbol{\nu} x\,(\boldsymbol{\nu} y\,(P'_1 \mid P'_2) \mid !x(z).\,R)$ where $P_1 \mid P_2 \xrightarrow{\alpha} \boldsymbol{\nu} y\,(P'_1 \mid P'_2)$ and $\alpha = \tau$

(4) $Q = \boldsymbol{\nu}x\,(P_1' \mid P_2 \mid (R\{y/z\} \mid !x(z).\,R))$ where $P_1 \xrightarrow{\overline{x}y} P_1'$

(5) $Q = \boldsymbol{\nu}x\,(\boldsymbol{\nu}y\,(P_1' \mid P_2 \mid (R\{y/z\} \mid !x(z).\,R)))$ where $P_1 \xrightarrow{\overline{x}(y)} P_1'$ and y is fresh

and the three cases similar to (1), (4), and (5) in which P_2 rather than P_1 acts. We give just cases (1), (3), and (5), leaving the rest to the reader.

For (1) we have

$$\begin{array}{cl} & \boldsymbol{\nu}x\,(P_1 \mid !x(z).\,R) \mid \boldsymbol{\nu}x\,(P_2 \mid !x(z).\,R) \\ \xrightarrow{\alpha} & Q' \stackrel{\text{def}}{=} \boldsymbol{\nu}x\,(P_1' \mid !x(z).\,R) \mid \boldsymbol{\nu}x\,(P_2 \mid !x(z).\,R)\,, \end{array}$$

and $(Q, Q') \in \mathcal{G}(\mathcal{R})$ since x occurs in P_1' only in output subject position.

For (3) we have

$$\begin{array}{cl} & \boldsymbol{\nu}x\,(P_1 \mid !x(z).\,R) \mid \boldsymbol{\nu}x\,(P_2 \mid !x(z).\,R) \\ \xrightarrow{\alpha} & Q' \stackrel{\text{def}}{=} \boldsymbol{\nu}y\,(\boldsymbol{\nu}x\,(P_1' \mid !x(z).\,R) \mid \boldsymbol{\nu}x\,(P_2' \mid !x(z).\,R))\,, \end{array}$$

and $(Q, Q') \in \mathcal{G}(\mathcal{R})$ because $Q \sim \boldsymbol{\nu}y\,\boldsymbol{\nu}x\,(P_1' \mid P_2' \mid !x(z).\,R)$ and x occurs in P_1' and P_2' only in output subject position.

For (5) we have

$$\begin{array}{cl} & \boldsymbol{\nu}x\,(P_1 \mid !x(z).\,R) \mid \boldsymbol{\nu}x\,(P_2 \mid !x(z).\,R) \\ \xrightarrow{\alpha} & Q' \stackrel{\text{def}}{=} \boldsymbol{\nu}x\,(\boldsymbol{\nu}y\,(P_1' \mid (R\{y/z\} \mid !x(z).\,R))) \mid \boldsymbol{\nu}x\,(P_2 \mid !x(z).\,R)\,, \end{array}$$

and $(Q, Q') \in \mathcal{G}(\mathcal{R})$ because

$$Q \sim \boldsymbol{\nu}x\,(\boldsymbol{\nu}y\,(P_1' \mid R\{y/z\}) \mid P_2 \mid !x(z).\,R)$$

and

$$Q' \sim \boldsymbol{\nu}x\,(\boldsymbol{\nu}y\,(P_1' \mid R\{y/z\}) \mid !x(z).\,R) \mid \boldsymbol{\nu}x\,(P_2 \mid !x(z).\,R)$$

and x occurs in $P_1' \mid R\{y/z\}$ and P_2' only in output subject position.

Similarly if $\boldsymbol{\nu}x\,(P_1 \mid !x(z).\,R) \mid \boldsymbol{\nu}x\,(P_2 \mid !x(z).\,R) \xrightarrow{\alpha} Q'$, then there is Q such that $\boldsymbol{\nu}x\,(P_1 \mid P_2 \mid !x(z).\,R) \xrightarrow{\alpha} Q$ and $(Q, Q') \in \mathcal{G}(\mathcal{R})$. $\square$

Proof (of Theorem 2.2.26)

The proof involves a similar kind of case analysis to that of Theorem 2.2.25, and uses the same safe function. The proof also appeals to Theorem 2.2.25. It is an instructive exercise. $\square$

2.3.4 Quasi-safe functions

The intuition for the strengthening of the safe-function method reported in this section is as follows. In the safe-function method we can only combine safe functions. The strengthening is obtained by observing that it is possible to combine functions that are *smaller* than safe functions. The only further requirement is that all functions and operators should be monotone – but this is always true in cases of practical interest. We call the new functions *quasi-safe*. Formally, there is no need to use such functions: any combination involving quasi-safe functions yields a function that is smaller than (and therefore, as a proof technique, less powerful than) an appropriate combination of safe functions. However, a quasi-safe function might have a simpler definition and therefore be simpler to use in combinations with other functions. An example of this is $\mathcal{F}_{\mathrm{ni}}$, which is smaller, but simpler, than $(\mathcal{F}_{\mathrm{ni1}})^*$ (see Lemmas 2.3.19 and Lemma 2.3.29). Further, the proof of the strong quasi-safety of $\mathcal{F}_{\mathrm{ni}}$ is simpler than that of its strong safety (Lemma 2.3.21). The quasi-safe functions discussed in this section, and their corresponding results, will not be used elsewhere in the book.

We recall that, for functions $\mathcal{F}$ and $\mathcal{G}$, we write $\mathcal{F} \subseteq \mathcal{G}$ if for all $\mathcal{R}$ it holds that $\mathcal{F}(\mathcal{R}) \subseteq \mathcal{G}(\mathcal{R})$).

Definition 2.3.26 (Strongly quasi-safe function) A function $\mathcal{F}$ is *strongly quasi-safe* it there is a strongly safe function $\mathcal{G}$ such that $\mathcal{F} \subseteq \mathcal{G}$. $\square$

Of course, all safe functions are also quasi-safe. The lemma below strengthens the soundness of safe functions (Lemma 2.3.8) to quasi-safe functions.

Lemma 2.3.27 If $\mathcal{F}$ is strongly quasi-safe and $\mathcal{R} \rightsquigarrow \mathcal{F}(\mathcal{R})$, then $\mathcal{R}$ and $\mathcal{F}(\mathcal{R})$ are included in $\sim$.

Proof Follows from Lemma 2.3.8. $\square$

A function $\mathcal{F}$ is *monotone* if $\mathcal{R} \subseteq \mathcal{S}$ implies $\mathcal{F}(\mathcal{R}) \subseteq \mathcal{F}(\mathcal{S})$. Similarly, an operator $\mathcal{OP}$ of arity n is *monotone* if whenever $\mathcal{F}_i \subseteq \mathcal{G}_i$ $(1 \leq i \leq n)$ then also $\mathcal{OP}(\mathcal{F}_1, \ldots, \mathcal{F}_n) \subseteq \mathcal{OP}(\mathcal{G}_1, \ldots, \mathcal{G}_n)$.

Lemma 2.3.28 Suppose a function $\mathcal{F}$ is the result of a combination of functions that are strongly quasi-safe and monotone, and of operators that are strongly secure and monotone. Then $\mathcal{F}$ is strongly quasi-safe. $\square$

All functions and operators considered in the book (notably those of Example 2.3.7, Lemmas 2.3.21 and 2.3.14) are monotone.

Lemma 2.3.29 $(\mathcal{F}_{\mathrm{ni1}})^*$ is strongly safe.

Proof Suppose $\mathcal{R} \subseteq \mathcal{S}$ and $\mathcal{R} \rightsquigarrow \mathcal{S}$. To prove the assertion, it suffices, by Lemma 2.3.19 and Lemma 2.3.16, to show that $\mathcal{F}_{\mathrm{ni1}}(\mathcal{R}) \subseteq \mathcal{F}_{\mathrm{ni}}(\mathcal{S})$ and $\mathcal{F}_{\mathrm{ni1}}(\mathcal{R}) \rightsquigarrow \mathcal{F}_{\mathrm{ni}}(\mathcal{S})$.

The inclusion is immediate from the definitions of the functions and the assumption that $\mathcal{R} \subseteq \mathcal{S}$. For the progression, we have to show that if C is a 1-ary non-input context, $(P, Q) \in \mathcal{R}$, and $C[P] \xrightarrow{\alpha} P'$, then $C[Q] \xrightarrow{\alpha} Q'$ with $(P', Q') \in \mathcal{F}_{\mathrm{ni}}(\mathcal{S})$. This is done by induction on C.

(1) Suppose C is $[\cdot]$, so $P \xrightarrow{\alpha} P'$. Then since $(P, Q) \in \mathcal{R}$ and $\mathcal{R} \rightsquigarrow \mathcal{S}$, we have $Q \xrightarrow{\alpha} Q'$ with $(P', Q') \in \mathcal{S}$, as required since $\mathcal{S} \subseteq \mathcal{F}_{\mathrm{ni}}(\mathcal{S})$.

(2) Suppose C is $\pi. C'$. Then $P' = C'[P]$, because C is non-input. Moreover, $C[Q] \xrightarrow{\alpha} C'[Q]$ and $(C'[P], C'[Q]) \in \mathcal{F}_{\mathrm{ni}}(\mathcal{S})$ as required. The assumption $\mathcal{R} \subseteq \mathcal{S}$ licenses this conclusion.

The cases when the main operator of C is $+$ or $|$ or $\boldsymbol{\nu}$ are left as exercises for the interested reader.

(3) Suppose C is $!C'$. There are three cases depending on which replication rule is used to infer the transition $C[P] \xrightarrow{\alpha} P'$.

(a) Suppose $P' = P'' \mid C[P]$ where $C'[P] \xrightarrow{\alpha} P''$. Then $C'[Q] \xrightarrow{\alpha} Q''$ with $(P'', Q'') \in \mathcal{F}_{\mathrm{ni}}(\mathcal{S})$, by induction hypothesis. Thus $P'' = C''\eta$ and $Q'' = C''\eta'$ for some C'', η, and η' with $(\eta_i, \eta'_i) \in \mathcal{S}$ for all i; we may assume that $\eta_0 = P$ and $\eta'_0 = Q$ (we can rename the holes of the context, if necessary). Then $C[Q] \xrightarrow{\alpha} Q' \stackrel{\text{def}}{=} Q'' \mid C[Q]$, and $(P', Q') \in \mathcal{F}_{\mathrm{ni}}(\mathcal{S})$ because $P' = C'''\eta$ and $Q' = C'''\eta'$ where $C''' \stackrel{\text{def}}{=} C'' \mid C$.

(b) Suppose α is τ and $P' = P'' \mid P''' \mid C[P]$ where $C'[P] \xrightarrow{\overline{x}y} P''$ and $C'[P] \xrightarrow{xy} P'''$. Then $C'[Q] \xrightarrow{\overline{x}y} Q''$ and $C'[Q] \xrightarrow{xy} Q'''$ with $(P'', Q'') \in \mathcal{F}_{\mathrm{ni}}(\mathcal{S})$ and $(P''', Q''') \in \mathcal{F}_{\mathrm{ni}}(\mathcal{S})$, by induction hypothesis. Then $P'' = C''\eta$ and $Q'' = C''\eta'$, for some C'', η, and η' with $(\eta_i, \eta'_i) \in \mathcal{S}$ for all i. Similarly, $P''' = C'''\eta''$ and $Q''' = C'''\eta'''$ for some C''', η'', and η''' with $(\eta''_i, \eta'''_i) \in \mathcal{S}$ for all i. We may assume that no hole of C'' occurs in C''', and vice versa. Therefore, since the values of η and η' (respectively η'' and η''') on the indices of holes not in C'' (respectively C''') are not important, we may assume that $\eta = \eta''$ and $\eta' = \eta'''$. We may also assume, as in case (a) above, that $\eta_0 = \eta''_0 = P$ and $\eta'_0 = \eta'''_0 = Q$. Finally, we have $C[Q] \xrightarrow{\tau} Q' \stackrel{\text{def}}{=} Q'' \mid Q''' \mid C[Q]$, and $(P', Q') \in \mathcal{F}_{\mathrm{ni}}(\mathcal{S})$ because $P' = C''''\eta''$ and $Q' = C''''\eta'''$ where $C'''' \stackrel{\text{def}}{=} C'' \mid C''' \mid C$.

(c) The argument when the rule is REP-CLOSE is similar.

This completes the proof. □

Note that in case (2) in the proof of Lemma 2.3.29, we see why the assumption $\mathcal{R} \subseteq \mathcal{S}$ appears in the definition of safe function (Definition 2.3.6): in the case $C' = [\cdot]$, for instance, we need to be able to conclude $(P, Q) \in \mathcal{F}_{\mathrm{ni}}(\mathcal{S})$ from the hypothesis $(P, Q) \in \mathcal{R}$.

Corollary 2.3.30 $\mathcal{F}_{\mathrm{ni}}$ is quasi-safe. $\square$

We do not use this result in the book. It could however be used everywhere in place of the safety of $\mathcal{F}_{\mathrm{ni}}$ (Lemma 2.3.21).