Explanation in Natural Language of $\bar{\lambda}\mu\tilde{\mu}$-terms

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Now at University of Bologna

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Outline

Outline

MKM Formats

- Formulae:
  - Presentation level: MathML Presentation
  - Content level: OpenMath, MathML Content
  - Semantics level: any logic or type theory or set theory
- Proofs:
  - Presentation level: interactive PDF, XHTML + JavaScript
  - Content level: OMDoc
  - Semantics level: proof trees, λ-terms (via Curry-Howard), proof nets, ...
- Theories/Mathematical Documents:
  - Presentation level: PDF, PS, XHTML, ...
  - Content level: OMDoc, Development Graphs/ DocBook
  - Semantics level: locales, functors, dependent records, .../

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OMDoc (Proof Module)

- OMDoc 1.0 (Kohlhase et alt.): not expressive enough for HELM/MoWGLI proofs
- OMDoc 1.2, 2.0 (Asperti, Kohlhase, Sacerdoti Coen): extended to cover HELM/MoWGLI
- Is it a *good* proof format?
  - Set of ad-hoc criteria (e.g. flexibility, embedding of annotations)
  - Kohlhase's motto:
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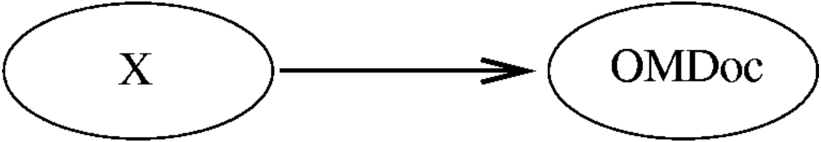
Encoding vs Simulation



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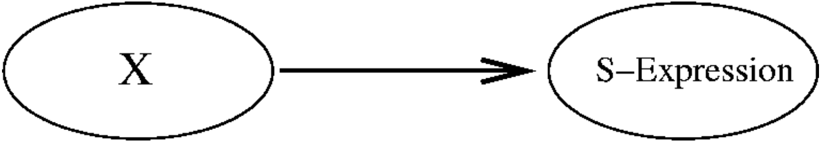

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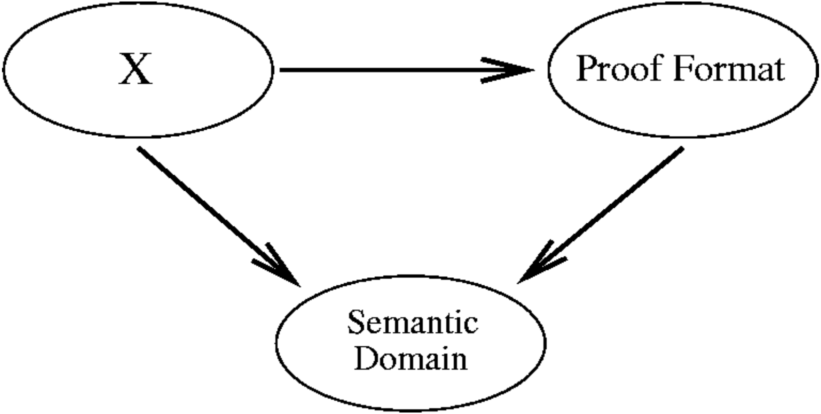
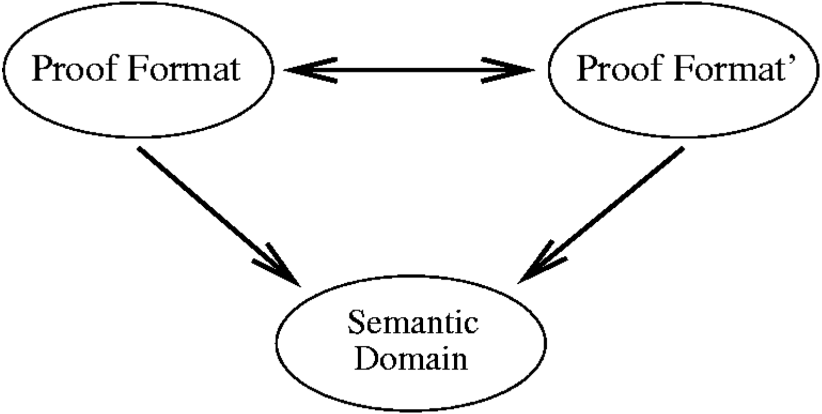
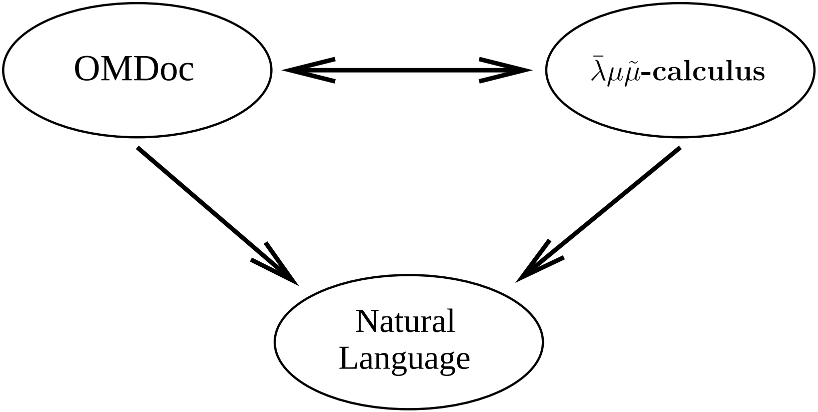


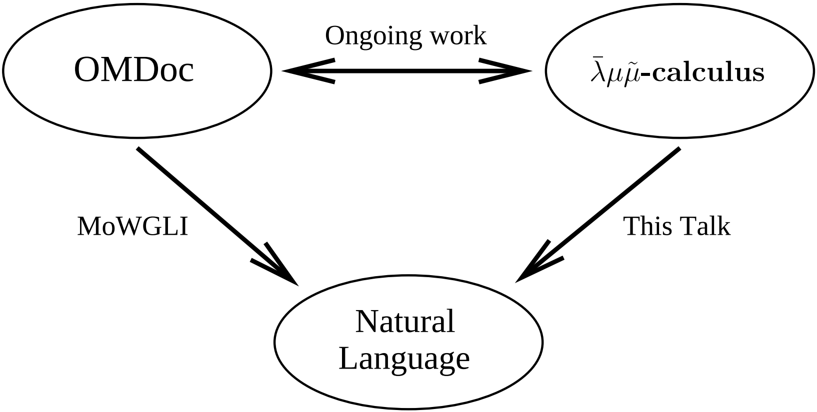
Bi-simulation



OMDoc vs the $\bar{\lambda}\mu\tilde{\mu}$-calculus



OMDoc vs the $\bar{\lambda}\mu\tilde{\mu}$-calculus



Outline

The Curry-Howard Isomorphism

λ-calculus	NJ
type	statement
term	derivation
reduction	proof normalization

- The λ-calculus is naturally a proof format
+ It has a clear semantics . . .
+ . . . that we can easily trade off for flexibility
- Y. Coscoy equipped it with a rendering semantics
- We can associate user-provided annotations to override the generated text

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- The λ-calculus is naturally a proof format
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- \- Y. Coscoy equipped it with a **rendering semantics**
- \- We can associate user-provided **annotations** to override the generated text

Implicit Information: the Inner Types

$$
\cfrac{\cfrac{\cfrac{\cfrac{[K : B \to C] \qquad \cfrac{[H : A \to B] \qquad [a : A]}{H\ a : B}}{(K\ (H\ a)) : C}}{\lambda a : A.(K\ (H\ a)) : A \to C}}{\lambda K : B \to C.\lambda a : A.(K\ (H\ a)) : (B \to C) \to A \to C}}{\lambda H : A \to B.\lambda K : B \to C.\lambda a : A.(K\ (H\ a)) : (A \to B) \to (B \to C) \to A \to C}
$$

Implicit Information: the Inner Types

$$\dfrac{\dfrac{\dfrac{[K : B \to C] \qquad \dfrac{[H : A \to B] \qquad [a : A]}{H\ a : B}}{(K\ (H\ a)) : C}}{\lambda a : A.(K\ (H\ a)) : A \to C}}{\dfrac{\lambda K : B \to C.\lambda a : A.(K\ (H\ a)) : (B \to C) \to A \to C}{\lambda H : A \to B.\lambda K : B \to C.\lambda a : A.(K\ (H\ a)) : (A \to B) \to (B \to C) \to A \to C}}$$

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Implicit Information: the Inner Types

$$\lambda H : A \to B.\lambda K : B \to C.\lambda a : A.(K(Ha))$$

$$\begin{array}{l}
\lambda H : A \to B. \\
\quad (\lambda K : B \to C. \\
\quad\quad (\lambda a : A. \\
\quad\quad\quad (((K\ ((H\ a) :: B)) :: C) \\
\quad\quad :: A \to C) \\
\quad :: (B \to C) \to A \to C) \\
:: (A \to B) \to (B \to C) \to A \to C
\end{array}$$

Non Structural Translation

$\lambda H : A \to B.$
$(\lambda K : B \to C.$
$(\lambda a : A.$
$((K\ ((H\ a) :: B)) :: C)$
$:: A \to C)$
$:: (B \to C) \to A \to C)$
$:: (A \to B) \to (B \to C) \to A \to C$

suppose $H : A \to B$
suppose $K : B \to C$
suppose $a : A$
by H and a we prove B $(*)$
by K and $*$ we prove C
thus we proved $A \to C$
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Non Structural Translation

$((J\ ((K\ ((H\ a) :: B)) :: C)) :: D)$

by H and a we prove B $(*)$
by K and $*$ we prove C $(\dagger)$
by J and $\dagger$ we prove D

Non Structural Translation

$((J\ ((K\ ((H\ a) :: B)) :: C)) :: D)$

by H and a we prove B $(*)$
by K and $*$ we prove C $(\dagger)$
by J and $\dagger$ we prove D

Non Structural Translation

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by H and a we prove B $(*)$
by K and $*$ we prove C $(\dagger)$
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$((J\ ((K\ ((H\ a) :: B)) :: C)) :: D)$

by H and a we prove B $(*)$
obviously we have C $(\dagger)$
by J and $\dagger$ we prove D

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$((J \fbox{((K \fbox{((H a) :: B)}) :: C)}) :: D)$

by *H* and *a* we prove *B* $(*)$
obviously we have *C* $(\dagger)$
by *J* and $\dagger$ we prove *D*

Non Structural Translation

$((J \; \boxed{((K \; \boxed{((H\ a) :: B)}) :: C)}) :: D)$

by H and a we prove B $(*)$
obviously we have C $(\dagger)$
by J and $\dagger$ we prove D

Non Structural Translation

$$((J\ ((K\ ((H\ a) :: B)) :: C)) :: D)$$

$$\begin{array}{l}\texttt{let}\ * : B\ := (H\ A)\ \texttt{in}\\ \texttt{let}\ \dagger : C\ := (K\ *)\ \texttt{in}\\ \ (J\ \dagger)\end{array}$$

Top Down vs Bottom Up

$$C$$

$$A \to C$$

$$(B \to C) \to A \to C$$

$$\lambda H : A \to B.\lambda K : B \to C.\lambda a : A.(K\ (H\ a)) :$$
$$(A \to B) \to (B \to C) \to A \to C$$

Top Down vs Bottom Up

$$
\dfrac{\dfrac{\dfrac{\dfrac{[B \to C] \qquad B}{C}}{A \to C}}{(B \to C) \to A \to C}}{\lambda H : A \to B.\lambda K : B \to C.\lambda a : A.(K\ (H\ a)) : (A \to B) \to (B \to C) \to A \to C}
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Outline

Sequent Calculus: Bottom Up Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \to C;\ A \vdash A} \qquad \dfrac{\dfrac{\dfrac{}{A;\ B \vdash B} \qquad \dfrac{}{A;\ B;\ C \vdash C}}{B \to C;\ A;\ B \vdash C}}{B \to C;\ A;\ B \vdash C}}{A \to B;\ B \to C;\ A \vdash C}}{A \to B;\ B \to C;\ A \vdash C}}{\dfrac{A \to B;\ B \to C \vdash A \to C}{A \to B \vdash (B \to C) \to A \to C}}}{\vdash (A \to B) \to (B \to C) \to A \to C}$$

Sequent Calculus: Top Down Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A} \qquad \overline{A;\ B \vdash B}}{A \to B;\ A \vdash B}}{A \to B;\ A \vdash B} \qquad \overline{A \to B;\ A;\ C \vdash C}}{A \to B;\ B \to C;\ A \vdash C}}{A \to B;\ B \to C;\ A \vdash C}}{A \to B;\ B \to C \vdash A \to C}}{A \to B \vdash (B \to C) \to A \to C}}{\vdash (A \to B) \to (B \to C) \to A \to C}$$

Sequent Calculus: Bottom Up Proof

$$
\dfrac{
\dfrac{
\dfrac{}{K : B \longrightarrow C;\ a : A \vdash a : A}
\qquad
\dfrac{
\dfrac{
\dfrac{}{a : A;\ b : B \vdash b : B}
\qquad
\dfrac{}{a : A;\ b : B;\ \gamma : C \vdash \gamma : C}
}{b \circ \gamma : B \longrightarrow C;\ a : A;\ b : B \vdash \gamma : C}
}{K : B \longrightarrow C;\ a : A;\ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle : B \vdash \gamma : C}
}{
\dfrac{a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle : A \longrightarrow B;\ K : B \longrightarrow C;\ a : A \vdash \gamma : C}{
\dfrac{H : A \longrightarrow B;\ K : B \longrightarrow C;\ a : A \vdash \mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle \rangle : C}{
H : A \longrightarrow B;\ K : B \longrightarrow C \vdash \lambda a : A.\mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle \rangle : A \longrightarrow C}}}
}{
\dfrac{H : A \longrightarrow B \vdash \lambda K : B \longrightarrow C.\lambda a : A.\mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle \rangle : (B \longrightarrow C) \longrightarrow A \longrightarrow C}{
\vdash \lambda H : A \longrightarrow B.\lambda K : B \longrightarrow C.\lambda a : A.\mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle \rangle : (A \longrightarrow B) \longrightarrow (B \longrightarrow C) \longrightarrow A \longrightarrow C}}
$$

Sequent Calculus: Bottom Up Proof

$$
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\qquad
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}{H : A \to B;\ K : B \to C;\ a : A \vdash \mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma\rangle\rangle : C}
}{H : A \to B;\ K : B \to C \vdash \lambda a : A.\mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma\rangle\rangle : A \to C}
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}{\vdash \lambda H : A \to B.\lambda K : B \to C.\lambda a : A.\mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma\rangle\rangle : (A \to B) \to (B \to C) \to A \to C}
$$

Sequent Calculus: Bottom Up Proof

$$\lambda H : A \to B.\lambda K : B \to C.\lambda a : A.\mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma\rangle\rangle$$

suppose $A \to B$ (H)	$\lambda H : A \to B.$
suppose $B \to C$ (K)	$\lambda K : B \to C.$
suppose A (a)	$\lambda a : A.$
we need to prove C	$\mu\gamma : C.$
by H and a	$\langle H \| a$
we have B (b)	$\circ\tilde{\mu} b : B.$
by K and b	$\langle K \| b$
qed	$\circ\gamma\rangle\rangle$

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we have B (b)	$\circ\tilde{\mu} b : B.$
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$$\lambda H : A \rightarrow B.\lambda K : B \rightarrow C.\lambda a : A.\mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle\rangle$$

suppose $A \rightarrow B$ (H)	$\lambda H : A \rightarrow B.$
suppose $B \rightarrow C$ (K)	$\lambda K : B \rightarrow C.$
suppose A (a)	$\lambda a : A.$
we need to prove C	$\mu\gamma : C.$
by H and a	$\langle H \vert\vert a$
we have B (b)	$\circ\tilde{\mu} b : B.$
by K and b	$\langle K \vert\vert b$
qed	$\circ\gamma\rangle\rangle$

Sequent Calculus: Bottom Up Proof

$$a : A;\ b : B \vdash b : B \qquad a : A;\ b : B;\ \gamma : C \vdash \gamma : C$$

$$b \circ \gamma : B \to C;\ a : A;\ b : B \vdash \gamma : C$$

$$K : B \to C;\ a : A \vdash a : A \qquad K : B \to C;\ a : A;\ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle : B \vdash \gamma : C$$

$$a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle : A \to B;\ K : B \to C;\ a : A \vdash \gamma : C$$

$$H : A \to B;\ K : B \to C;\ a : A \vdash \mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle\rangle : C$$

$$H : A \to B;\ K : B \to C \vdash \lambda a : A.\mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle\rangle : A \to C$$

$$H : A \to B \vdash \lambda K : B \to C.\lambda a : A.\mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle\rangle : \quad (B \to C) \to A \to C$$

$$\vdash \lambda H : A \to B.\lambda K : B \to C.\lambda a : A.\mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle\rangle : (A \to B) \to (B \to C) \to A \to C$$

Sequent Calculus: Bottom Up Proof

$$\dfrac{}{a : A;\ b : B \vdash b : B} \qquad \dfrac{}{a : A;\ b : B;\ \gamma : C \vdash \gamma : C}$$

$$\dfrac{}{b \circ \gamma : B \longrightarrow C;\ a : A;\ b : B \vdash \gamma : C}$$

$$\dfrac{}{K : B \longrightarrow C;\ a : A \vdash a : A} \qquad \dfrac{}{K : B \longrightarrow C;\ a : A;\ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle : B \vdash \gamma : C}$$

$$\dfrac{}{a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle : A \longrightarrow B;\ K : B \longrightarrow C;\ a : A \vdash \gamma : C}$$

$$\dfrac{}{H : A \longrightarrow B;\ K : B \longrightarrow C;\ a : A \vdash \mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle \rangle : C}$$

$$\dfrac{}{H : A \longrightarrow B;\ K : B \longrightarrow C \vdash \lambda a : A.\mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle \rangle : A \longrightarrow C}$$

$$\dfrac{}{H : A \longrightarrow B \vdash \lambda K : B \longrightarrow C.\lambda a : A.\mu\gamma : C.\langle H || a \circ \tilde{\mu} b : B.\langle K || b \circ \gamma \rangle \rangle : (B \longrightarrow C) \longrightarrow A \longrightarrow C}$$

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Sequent Calculus: Top Down Proof

suppose $A \rightarrow B$ (H)
suppose $B \rightarrow C$ (K)
suppose A (a)
we need to prove C
  by K
  we reduce the thesis to B
    by H
    we reduce the thesis to A
      by a
    qed
  qed
qed

$$\begin{array}{l}
\lambda H : A \rightarrow B. \\
\lambda K : B \rightarrow C. \\
\lambda a : A. \\
\mu\gamma : C. \\
\quad \langle K || \\
\quad \mu\beta : B. \\
\qquad \langle H || \\
\qquad \mu\alpha : A. \\
\qquad\quad \langle a || \\
\qquad \alpha\rangle \\
\quad \circ\beta\rangle \\
\circ\gamma\rangle
\end{array}$$

Sequent Calculus: Top Down Proof

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suppose $B \to C$ (K)	$\lambda K : B \to C.$
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by K	$\langle K \|\|$
we reduce the thesis to B	$\mu\beta : B.$
by H	$\langle H \|\|$
we reduce the thesis to A	$\mu\alpha : A.$
by a	$\langle a \|\|$
qed	$\alpha\rangle$
qed	$\circ\beta\rangle$
qed	$\circ\gamma\rangle$

Sequent Calculus: Top Down Proof

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qed

$\lambda H : A \rightarrow B.$
$\lambda K : B \rightarrow C.$
$\lambda a : A.$
$\mu\gamma : C.$
$\langle K ||$
$\mu\beta : B.$
$\langle H ||$
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$\langle a ||$
$\alpha\rangle$
$\circ\beta\rangle$
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Sequent Calculus: Top Down Proof

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we need to prove C	$\mu\gamma : C.$
by K	$\langle K \mid\mid$
we reduce the thesis to B	$\mu\beta : B.$
by H	$\langle H \mid\mid$
we reduce the thesis to A	$\mu\alpha : A.$
by a	$\langle a \mid\mid$
qed	$\alpha\rangle$
qed	$\circ\beta\rangle$
qed	$\circ\gamma\rangle$

Some Intuitions (a posteriori)

- Isomorphic to Sequent calculus $\Rightarrow$ bottom-up vs top-down proofs
- Explicit manipulation of continuations $\Rightarrow$ it allows to state and operate on the future (i.e. what we need to prove)
- Continuations as first class objects and multiple continuations $\Rightarrow$ classical logic
- Syntactic reification of call-by-value vs call-by-name $\Rightarrow$ bottom-up vs top-down proofs
- Explicit labelling of each value computed in the past/continuation $\Rightarrow$ the CPS style $\Rightarrow$ Göedel's A-translation $\Rightarrow$ classical logic

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The Great Mistery (no intuition yet)

The function that associates to a term its natural language is fully structural (from left to right, from top to bottom, only immediate recursion).

Summary

- Natural language is NOT isomorphic to natural deduction: it is isomorphic to sequent calculus . . .
- . . . and via Curry-Howard to the $\bar{\lambda}\mu\tilde{\mu}$-calculus
- The isomorphism between the $\bar{\lambda}\mu\tilde{\mu}$-calculus and the natural language is
  - fully explicit (no need for type inference, labelling, etc.) (as for OMDoc)
  - fully compositional (top to bottom, left to right) (as for OMDoc)
- The $\bar{\lambda}\mu\tilde{\mu}$-calculus has very nice theoretical properties (e.g. cut elimination) (cfr. OMDoc)
- We are looking for a bisimulation between the $\bar{\lambda}\mu\tilde{\mu}$-calculus and OMDoc
- Fellowship is a proof assistant for first order subtractive logic ($\forall, \exists, \wedge, \vee, \rightarrow, -$) with $\bar{\lambda}\mu\tilde{\mu}$-calculus proof objects