

Nonuniform Coercions via Unification Hints

Claudio Sacerdoti Coen

Department of Computer Science, University of Bologna
sacerdot@cs.unibo.it

Enrico Tassi

Microsoft Research-INRIA Joint Center
enrico.tassi@inria.fr

We introduce the notion of *nonuniform* coercion, which is the promotion of a value of one type to an enriched value of a different type via a nonuniform procedure. Nonuniform coercions are a generalization of the (uniform) coercions known in the literature and they arise naturally when formalizing mathematics in an higher order interactive theorem prover using convenient devices like canonical structures, type classes or unification hints. We also show how nonuniform coercions can be naturally implemented at the user level in an interactive theorem prover that allows unification hints.

1 Introduction

In type theory, a coercion is a function k of type $S \rightarrow T$ that can be automatically inserted by an interactive theorem prover to promote a value v of type S to a value $k v$ of type T whenever v is used in a context where it is expected to have type T . The typical example is the promotion of natural numbers to integers.

In a theory with dependent types coercions are better generalized to couples (k, n) where k is a function of type $\prod x_i : S_i(x_1, \dots, x_{i-1}). T(x_1, \dots, x_m)$ and $n \leq m$ is the index of one of the arguments of k . The coercion is automatically inserted to promote a value v of type $S(t_1, \dots, t_{n-1})$ to a value $k t_1 \dots t_{n-1} v t_{n+1} \dots t_m$ of type $T(t_1, \dots, t_m)$. The arguments $t_1, \dots, t_{n-1}, t_{n+1}, \dots, t_m$ are partially inferred from the actual type of v or from the expected type; those that are not inferred become proof obligations for the user. The typical example for the latter situation is the coercion from lists to non-empty lists, that opens a new proof obligation for the non-emptiness of the argument. This kind of parametric coercions have been heavily exploited by Sozeau in the Russell language [16].

All the previous examples of coercions are *uniform* in the sense that, up to the inferred arguments, the very same function k is used to promote the value v . For instance, if k is the promotion from natural numbers to integers, 3 and 5 are promoted respectively to $k 3$ and $k 5$. Nevertheless there are situations where we would like to promote v in a non uniform way, depending on the actual value of v , but the language does not allow one to inspect (i.e. pattern match over) v , only the meta-level allows it.

For example, consider the promotion of a type (the carrier of a semi-group) to a semi-group (the carrier enriched with the operation). Different types are associated (even not uniquely) to different semi-groups on them: the type \mathbb{N} of natural numbers could be promoted to the semi-group $(\mathbb{N}, +)$ whereas the type $\text{List } \mathbb{N}$ could be promoted to $(\text{List } \mathbb{N}, @)$ (where '@' is the append operation). Since no function in type theory can distinguish between \mathbb{N} and $\text{List } \mathbb{N}$, there is no uniform coercion of type $\text{Type} \rightarrow \text{SemiGroup}$ that behaves in the expected way.

As far as we know, nonuniform coercions have not been considered so far in the literature. Nevertheless, they arise quite naturally when devices like canonical structures [9], type classes [17] or unification hints [3] are employed.

In Sect. 2 we introduce a syntax and semantics for nonuniform coercion declarations and in Sect. 3 we present the use of nonuniform coercions as a complementary device to canonical structures like

mechanisms. In Sect. 4 we recall the syntax and semantics of *unification hints* [3]. In Sect. 5 we show how nonuniform coercions can be efficiently implemented at the user level in an interactive theorem prover equipped with a flexible coercion system and unification hints. Sections 6 and 7 are devoted to the solution of two different problems that naturally arise when nonuniform coercions are employed. Conclusions and future works follow in Sect. 8.

2 Syntax and semantics

Let Γ be a well-typed context, i.e. a sequence of variable declarations of the form $x_i : R_i$ where each R_i is a well typed type in the context $x_1 : R_1 \dots x_{i-1} : R_{i-1}$. A nonuniform coercion declaration has the following syntax:

$$\Gamma_1 \vdash \begin{array}{l} S_1 \rightarrow T_1 \\ s_1 \mapsto t_1 \end{array} \quad \dots \quad \Gamma_n \vdash \begin{array}{l} S_n \rightarrow T_n \\ s_n \mapsto t_n \end{array}$$

where each s_i has type S_i in Γ_i ; each t_i has type T_i in Γ_i .

Promotion works as follows: let \bar{s} be a term of type \bar{S} which is expected to have type \bar{T} and let σ be a substitution that instantiates the variables declared in Γ_i and such that $s_i\sigma = \bar{s}$ and $S_i\sigma = \bar{S}$ and $T_i\sigma = \bar{T}$. Then \bar{s} can be promoted to $t_i\sigma$ of type \bar{T} .

Operationally, the substitution σ is determined by looking for the smallest index i such that σ is the most general unifier of s_i with \bar{s} , of S_i with \bar{S} and of T_i with \bar{T} . The substitution σ can be made total by instantiating every still uninstantiated variable $x_j : R_j\sigma$ with the result of a proof obligation for $R_j\sigma$.

Instead of stopping at the first index that yields a result, it could also be possible to try all of them and let the user interactively choose the desired promotion. Stopping at the first index is often desirable since it allows one to add overlapping branches ordering them by number of open proof obligations (see Example 3 below).

Example 1 (Uniform coercions) *A uniform coercion (k, n) where*

$$k : \overrightarrow{\Pi x_i : S_i(x_1, \dots, x_{i-1})}. T(x_1, \dots, x_m)}$$

is equivalent to a nonuniform coercion

$$\dots x_i : S_i(x_1, \dots, x_{i-1}) \dots \vdash \begin{array}{l} S_n(x_1, \dots, x_{n-1}) \rightarrow T(x_1, \dots, x_m) \\ x_n \mapsto k \ x_1 \ \dots \ x_m \end{array}$$

Conversely, every branch of a nonuniform coercion whose pattern s_i is a variable x_n is equivalent to a nonuniform coercion:

$$\dots x_i : S_i(x_1, \dots, x_{i-1}) \dots \vdash \begin{array}{l} S_n(x_1, \dots, x_{n-1}) \rightarrow T(x_1, \dots, x_m) \\ x_n \mapsto t \end{array}$$

is equivalent to the uniform coercion $(\overrightarrow{\lambda x_i : S_i(x_1, \dots, x_{i-1})}. t, n)$:

$$\overrightarrow{\lambda x_i : S_i(x_1, \dots, x_{i-1})}. t : \overrightarrow{\Pi x_i : S_i(x_1, \dots, x_{i-1})}. T(x_1, \dots, x_m)}$$

Example 2 (Structure enrichment) Let π be a proof of the associativity of $+$ over \mathbb{Z} and $\bar{\pi}$ a proof of the associativity of the append operation over lists.

$$\begin{array}{l} \vdash \text{Type} \rightarrow \text{SemiGroup} \\ \mathbb{Z} \mapsto (\mathbb{Z}, +, \pi) \end{array} \quad \begin{array}{l} X : \text{Type} \vdash \text{Type} \rightarrow \text{SemiGroup} \\ \text{List } X \mapsto (\text{List } X, @ X, \bar{\pi} X) \end{array}$$

Note that the second coercion is polymorphic in the type of the list arguments. Seen as two independent uniform coercions, the two coercions are incoherent [11] since they both promote from and to the same couple of types and they are not $\alpha\beta\eta$ -convertible.

Example 3 (Property enrichment) Let π be a proof that $+$ is associative, `AssocFun` the structure of associative functions and `assoc_comp` a proof that the composition of objects in `AssocFun` is in `AssocFun`.

$$\begin{array}{l} \vdash \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N} \rightarrow \text{AssocFun } \mathbb{N} \\ + \mapsto (+, \pi) \\ \\ X : \text{Type}, * : X \rightarrow X \rightarrow X, \\ p : \forall a, b, c : X. a * (b * c) = (a * b) * c \quad \vdash X \rightarrow X \rightarrow X \rightarrow \text{AssocFun } X \\ * \mapsto (*, p) \\ \\ X : \text{Type}, f, g : \text{AssocFun } X \quad \vdash X \rightarrow X \rightarrow X \rightarrow \text{AssocFun } X \\ f \circ g \mapsto \text{assoc_comp } f g \end{array}$$

When used to promote $+$ to an associative operation, no proof obligation is left open. On the other hand, when used to promote $*$: $\mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$ to an associative operation, the user is left with the proof obligation $p : \forall a, b, c : \mathbb{N}. a * (b * c) = (a * b) * c$. Finally, the composition of two associative operations is promoted to an associative operation applying the theorem `assoc_comp`. The third case can not be expressed as a uniform coercion since the pattern is not a variable.

3 Nonuniform coercions for mathematical structures

Different ITPs provide the user different devices to fill the gap between the high level language used to reason about abstract and complex mathematical theories and the drastically lower level foundational language understandable and checkable by a computer.

Among the most widespread machineries to aid the user we mention decision procedures; general purpose proof searching facilities, usually by means of external tools; model checkers and SMT solvers. All these tools usually fall in the category called, with some abuse, automation. Even if these tools shown to be quite effective in many areas, some radically different techniques were recently successfully employed [4, 9, 5, 18] to aid the user in systems based on higher order languages.

Canonical structures, type classes (and their generalization into unification hints) allow the user to instrument the ITP linking objects and properties by means of structures. Structures pack together axioms that give rise to a theory and that are used to quantify theorems (i.e. once defined the structure `Group` packing together the group axioms, theorems about groups have usually the shape $\forall G : \text{Group}, \dots$). The user is then asked to link models of these structures to their characterizing structure, so that the ITP is able to exploit such link when needed. For example, once proved that integers \mathbb{Z} together with addition $+$, 0 and the inverse $-$ form a group \mathcal{G} , every theorem that holds on groups can be used over expressions laying in the signature $(\mathbb{Z}, +, -, 0)$, even if the group structure \mathcal{G} is not mentioned in any way in the

context of the current conjecture. In addition to that, limited forms of Prolog like proof search allow the ITP to infer derived structures given the basic ones. For example, the pair $(0,0)$ can be transparently considered as the unit of the Cartesian product group $\mathcal{L} \times \mathcal{L}$, given the general result that the Cartesian product of groups forms a group.

These forms of inference, although not as general as other automatic devices, tend to be more predictable and allow the user's formalization and proof strategy to be closer to the widespread and natural mathematical argument: *since X together with Y form a Z , we have $P(X,Y)$* . Moreover they shown to be very effective in building reusable libraries of formalized mathematics [9].

3.1 Mathematical structures in dependently typed languages

We briefly review here how the aforementioned devices are implemented in an interactive theorem prover based on a higher order and dependently typed language.

Suppose that the system library contains the definition of natural numbers \mathbb{N} , of multiplication over them and the proof that multiplication is associative, commutative and it has a neutral element. Suppose also that the library contains the definition of unital semigroups (i.e. semigroups that have a left neutral element), represented as a dependently typed record [14, 15]:

$$\begin{aligned} \text{UnitalSemiGroup} := \\ \{S : \mathbf{Type}; 1 : S; * : S \rightarrow S \rightarrow S; \pi : \text{is_unital_semi_group } S \ 1 \ *\} \end{aligned}$$

The record projection S is declared as a uniform coercion of type

$$\text{UnitalSemiGroup} \rightarrow \mathbf{Type}$$

so that a quantification over a unital semigroup G is automatically promoted to a quantification over the group carrier $(S \ G)$.

The library contains all the interesting theorems of the theory of unital semigroups, stated by making the quantifiers range over dependently typed records. One example is the unicity of the neutral element, if it exists:

$$\forall G : \text{UnitalSemiGroup}. \forall x : G. (\forall y : G. y * x = y) \Rightarrow x = 1$$

Suppose now that the user, during a proof over natural numbers, knows the hypothesis $\forall y : \mathbb{N}. y + x = y$ (let's call it H) and she needs to conclude that $x = 0$. In order to do that, she wants to apply the unicity property of the neutral element of a unital semigroup, that is she wants to feed the hypothesis $\forall y : \mathbb{N}. y + x = y$ to the theorem L that proves the unicity and that expects a unital semigroup G , an x in the carrier of G and a proof that $\forall y : G. y + x = y$. The action of invoking the lemma L generates the unification problem

$$\forall y : G. y * x = y \stackrel{?}{\equiv} \forall y : \mathbb{N}. y + x = y$$

that can be solved by choosing for G any unital semigroup over the natural numbers whose operation is addition. The system automatically picks the right group definition if the user has already instructed the system (by means of canonical structures or with the more flexible mechanism of unification hints) to always enrich \mathbb{N} to $(\mathbb{N}, 0, +, \dots)$ under the previous constraints.

More formally, the user has fed the system with the request to apply the proof term $L \ ?_G \ x \ H$ where $?_G$ is a metavariable to be instantiated by unification and the unifier instantiates $?_G$ with $(\mathbb{N}, 0, +, \dots)$. Note that no coercion has been applied in this case: $?_G$ is identified by the system since it occurs in the type of x and H , which are known.

Most of the time, the previous mechanism is sufficient to perform the enrichment. Nevertheless, there are situations where the enriched structure to be discovered does not occur dependently in the type of some other parameter (or in the expected conclusion). For instance, suppose that we already know that the exponential function is injective and suppose that we want to inhabit the type $(\text{InjectiveFun } \mathbb{R} \ \mathbb{R})$ of injective functions over real numbers with the function $\lambda x.e^{e^x}$. Since the previous λ -abstraction is just a type theoretic function from \mathbb{R} to \mathbb{R} , what we need here is an automatic promotion of the function from $(\mathbb{R} \rightarrow \mathbb{R})$ to $(\text{InjectiveFun } \mathbb{R} \ \mathbb{R})$, which can be achieved by means of a nonuniform coercion similar to the one in Example 3 that states that the composition of injective functions is injective.

Without nonuniform coercions, the user is obliged to manually feed the system with the enriched structure itself, as in a traditional procedural language. This is, for instance, what happens currently in [9], where two distinct mechanisms (and notations) are used to enrich structure when the canonical structure mechanism is not triggered. The simplest one overloads every notation defining a scope for every possible enrichment. Then, the explicit scope delimiter can be used to interpret the notation as desired by the user. For example the construction $A \cap B$, even if A and B are groups, returns a set, while $(A \cap B)\%G$ returns a group. This mechanism does not only force to redefine every notation for every structure, but has also some technical limitations that make it fail if one among A and B is not explicitly typed as a group.¹ The second mechanism is way more verbose, but does not suffer from the aforementioned limitations, and is thus used as a fall back for the former. To enrich G to the wanted structure the user types “[the structure of G]” and a rather complex machinery generates behind the scenes an ad-hoc unification problem that triggers canonical structures inference. Note that in both approaches what the users types is what will then be displayed, thus the more verbose an enrichment notation is, the more cluttered the lemmas statements and the intermediate goals of their proofs will be.

In the next section we see that, quite surprisingly, nonuniform coercions can be implemented for free at the user level in a system based on dependent types and unification hints.

4 Unification hints in a nutshell

Unification hints [3] give solutions to (higher order) unification problems that fall outside the domain of the regular unification heuristic implemented by the ITP. They are presented as rules of the following form:

$$\Gamma \vdash \frac{\vec{?}_x := \vec{H}}{P \equiv Q} \text{myhint}$$

where:

1. Γ is a context that declares meta-variables $?_y$;
2. $\vec{?}_x := \vec{H}$ is a telescope of definitions for metavariables undeclared in Γ ;
3. all meta-variables in P and Q are declared in Γ or defined in the telescope;
4. $P \equiv Q$ is a *linear* pattern in the meta-variables defined in the telescope (i.e. every meta-variable $?_x$ occurs just once either in P or in Q);

¹This phenomenon happens, for example, when B is a local definition for $C \cap D$. Unfolding B and then forcing the groups scope would make all instances of \cap to be interpreted at the group level, but if B is not manually unfolded, only one occurrence of \cap is affected by the scope delimiter, thus the full expression fails to be enriched.

5. the type checking rules for meta-variables mimic the corresponding rules for variables; all the terms in Γ , \vec{H} , P and Q must be well typed. We use meta-variables since we expect them to be instantiated by unification².

A hint is *acceptable* if $\Gamma \vdash P[\vec{H}/\vec{?}_x] \equiv Q[\vec{H}/\vec{?}_x]$, i.e. if the two terms obtained by telescopic substitution, are *convertible* in Γ . Since convertibility is (typically) a decidable relation, the system is able to discriminate acceptable hints. As a consequence, for every substitution σ that instantiates only meta-variables in Γ , all unification problems of the form $P\sigma \stackrel{?}{\equiv} Q\sigma$ can be solved by the unifier ρ that assigns to each $?_x$ the term $H\sigma$. As a generalization, if τ is a generic meta-variable substitution, all unification problems of the form $P\tau \stackrel{?}{\equiv} Q\tau$ can be solved by recursively solving the new problems $?_x\tau \stackrel{?}{\equiv} H\tau$ (whose solution is ρ in the simple case where τ behaves as the identity function on the meta-variables in the telescope).

Unification hints are triggered in case of a unification failure. If the failing problem is $\bar{P} \stackrel{?}{\equiv} \bar{Q}$, the system looks for a hint such that there exists a unifier τ of P with \bar{P} and Q with \bar{Q} , and then try to solve the unification problem by triggering the recursive problems $?_x\tau \stackrel{?}{\equiv} H\tau$.

When there are multiple hints matching the failing unification problem, the system can either ask the user about the desired one, or it can pick the first matching hint according to some user defined precedence level (e.g. in order of declaration or by more precise matching).

We give a bird's-eye view on how they allow one to apply a simple property of groups in a context where the group structure is implicit. Consider the following statement and its corresponding form without (part of its) notational sugar.

$$a + 0 = a \quad \text{zplus } a \ 0 = a$$

where a is a point in \mathbb{Z} . The group theoretical property we want to apply is the right-identity law for $+$ and 0 that is

$$\text{grid} : \forall G : \text{Group}, \forall x : \text{carr } G, \text{op } G \ x \ (\text{unit } G) = x$$

Instantiating this axiom to a and leaving G implicit (i.e. considering the term $\text{grid } ?_G \ a$) triggers the unification problem $\text{carr } ?_G \stackrel{?}{\equiv} \mathbb{Z}$, since $a : \mathbb{Z}$ is expected to have type $\text{carr } ?_G$. This problem can be solved only by guessing a model of a group whose carrier is the set \mathbb{Z} .

Similarly, if we use grid to perform the rewriting without instantiating it first (i.e. we use the term $\text{grid } ?_G \ ?_x$ with an expected type whose left hand side is $\text{zplus } a \ 0$), we trigger the unification problem:

$$\text{op } ?_G \ ?_x \ (\text{unit } ?_G) \stackrel{?}{\equiv} \text{zplus } a \ 0$$

If the unification goes from left to right, the system must unify $\text{op } ?_G$ with zplus ; if it goes from right to left, it must unify $\text{unit } ?_G$ with 0 . In both cases we are facing again an unification problem where a projection (op , unit or carr) is applied to an implicit structure $?_G$ and a model of the structure must be guessed by the system. Since any guess would be arbitrary and could involve proof search, we expect the system to fail.

To force a solution to the unification problem, the user can specify the following hints that link the carrier \mathbb{Z} , the operation zplus and the constant 0 to the group structure \mathcal{G} .

² Due to type dependencies, it is not always possible to give all the meta-variable declarations first (in Γ) and then all the meta-variable definitions. Interleaving declarations and definitions poses no problem (and we implicitly assume that it is done). Nevertheless, we prefer to present the hints in this way for clarity purposes.

$$\vdash \frac{?_g := \mathcal{L}}{\text{carr } ?_g \equiv \mathbb{Z}} \quad \vdash \frac{?_g := \mathcal{L}}{\text{op } ?_g \equiv \text{zplus}} \quad \vdash \frac{?_g := \mathcal{L}}{\text{unit } ?_g \equiv 0}$$

Unification hints can also be used to drive proof search with clauses that resemble the ones used in logic programming. For instance, to solve the problem

$$\text{carr } ?_1 \stackrel{?}{\equiv} \mathbb{Z} \times \mathbb{Z}$$

the user can declare the following hint that recursively reduces the problem to simpler ones:

$$?_h, ?_q : \text{Group} \vdash \frac{?_A := \text{carr } ?_h \quad ?_B := \text{carr } ?_q \quad ?_g := ?_h \times ?_q}{\text{carr } ?_g \equiv ?_A \times ?_B}$$

Intuitively, the hint says that, if the carrier of a group $?_g$ is a product $?_A \times ?_B$, where $?_A$ is the carrier of a group $?_h$ and $?_B$ is the carrier of a group $?_q$ then a solution consists in choosing for $?_g$ the group product of $?_h$ and $?_q$.

4.1 Hints and coercions indexing

Another view at unification hints is that they define equivalence classes of convertible terms that are considered indistinguishable by every functionality of the system that works up to unification. Promotion of terms via (uniform) coercions is the typical example: if the user declares a coercion from \mathbb{Z} to \mathbb{Q} , we expect the system to be able to promote also $\text{carr } \mathcal{L}$ to \mathbb{Q} .

This behaviour, however, does not come for free. Consider a failing unification problem $\bar{P} \stackrel{?}{\equiv} \bar{Q}$. In order to retrieve a coercion from \bar{P} to \bar{Q} , it would be inefficient to iterate over the whole set of coercions to find the ones that goes from M to N such that M unifies with \bar{P} and N unifies with \bar{Q} . The usual implementation strategy is to use a discrimination tree [12, 13] (or discrimination nets) to index the coercion and perform a quick approximated search. These data structures, born in the field of automatic theorem proving, are first order oriented and compare terms according to their rigid structure. For example $(\text{carr } \mathcal{L})$ and \mathbb{Z} would be considered different for at least two reasons: both their head constant and head function symbol arity differ.

Thus, in order to retrieve coercions up to hints, the system must do some additional work, for instance to index every coercion multiple times (on all M' and N' such that M and M' are in the same hint-induced equivalence class, and the same for N and N'), or to perform the search multiple times (for every P' and Q' such that \bar{P} and P' are in the same hint-induced equivalence class, and the same for \bar{Q} and Q').

In the rest of the paper we assume our system to implement both unification hints and retrieval of coercions up to unification hints. The Matita interactive theorem prover [1, 2] satisfies these requirements.

5 Non Uniform Coercions via Unification Hints

We analyze at first a particular scenario where it is easier to explain the ideas involved. Suppose that we are interested in implementing the following nonuniform coercion with two branches:

$$\vdash \begin{array}{c} S \rightarrow T \\ s_1 \mapsto t_1 \end{array} \quad \vdash \begin{array}{c} S \rightarrow T \\ s_2 \mapsto t_2 \end{array}$$

The simplified scenario is characterized by the fact that all branches are uniform in the types S and T ; the fact that we consider just two branches and that we analyze only the case where all contexts Γ_i are empty is just to simplify the presentation and dropping these two limitations poses no real problem.

Suppose also, to grant the coherence of the coercion graph, that the system only allows one to declare a single uniform coercion between two given (equivalence classes of) types. The only way in which a single uniform coercion can present the expected non uniform behaviour is that it takes t_i in input:

$$\begin{aligned} k : \forall S : \mathbf{Type}. S \rightarrow \forall T : \mathbf{Type}. T \rightarrow T \\ k S s T t := t \end{aligned}$$

so that s_1 can be promoted to $k S s_1 T t_1$ and s_2 can be promoted to $k S s_2 T t_2$. However, when a term \bar{s} of type S is expected to have type T , according to the standard semantics of uniform coercions, the user will be presented with a proof obligation of type T and the term \bar{s} will be promoted to $k S s T t$ (that reduces to t) where t is the proof term provided by the user in the proof obligation. This means that the coercion maps any term of type S to a term of type T after asking the user what term must be chosen. Instead, we would like the system to automatically pick t_1 for t when \bar{s} is s_1 , to pick t_2 for t when \bar{s} is s_2 and to fail in all the other cases, without bothering the user at all.

The idea to achieve the expected result is to use unification hints to automatically suggest the term t . However, unification hints are only triggered in case of a unification failure and apparently the only failing unification problem is the initial one: $S \stackrel{?}{\equiv} T$ where s and t do not occur at all.

5.1 Lifting terms to types

In a dependently typed language, terms can occur in types, or better can be lifted to types when they occur as arguments of a dependently typed function.

The trick we employ is to declare the nonuniform coercion not from S to T , but from S to a type that is convertible to T but that exposes s and t . We can make a first shot with the following definitions:

$$\begin{aligned} \mathbf{force} : \forall S : \mathbf{Type}. S \rightarrow \forall T : \mathbf{Type}. T \rightarrow \mathbf{Type} \\ \mathbf{force} S s T t := T \end{aligned}$$

The \mathbf{force} type has the property that $\mathbf{force} S s T t$ is β -equivalent to T . The coercion k is now redefined as follows:

$$\begin{aligned} k : \forall S : \mathbf{Type}. \forall s : S. \forall T : \mathbf{Type}. \forall t : T. \mathbf{force} S s T t \\ k S s T t := t \end{aligned}$$

With these definitions, when a term \bar{s} of type S is used with type T , the system tries to promote \bar{s} to $k S \bar{s} T ?_t$ (which reduces to $?_t$) yielding a new unification problem

$$\mathbf{force} S \bar{s} T ?_t \stackrel{?}{\equiv} T$$

In this unification problem we have both \bar{s} and $?_t$, and so we have the data for choosing a term for $?_t$ in function of \bar{s} . Moreover, since $?_t$ is also the argument the coercion k is returning, the solution for the unification problem can define the output of the coercion.

However, we also have that the left hand side reduces to $?_t$ and thus the system can easily solve the unification problem by choosing T for $?_t$, without using any unification hint at all and, once again, opening a proof obligation of type T to determine the unconstrained term $?_t$.

5.2 Locked reduction and lockpicking via hints

In order to solve the problem, we need to change our definition of k (and \mathbf{force}) again in order to produce in a similar way a new unification problem whose solution is too difficult to be found by the system without resorting to unification hints.

In other words, we need to change the definition of `force` in such a way that $(\text{force } S \bar{s} ?_T ?_t)$ is no longer always reducible to $?_T$, but only in certain user-defined situations.

The latter behaviour can be achieved by adding an additional parameter to `force` that must take a precise value to unlock the good reduction. This can be achieved in many ways, one of them being to parameterize `force` over an inhabitant of the unit type and using pattern matching on it to block reduction until the canonical inhabitant is passed:

$$\begin{aligned} \text{force} &: \forall S : \mathbf{Type}. S \rightarrow \forall T : \mathbf{Type}. T \rightarrow \mathbf{1} \rightarrow \mathbf{Type} \\ \text{force } S s T t \text{ lock} &:= \text{match } \text{lock} \text{ with } [\star \Rightarrow T] \end{aligned}$$

The `force` type has the property that $(\text{force } S s T t \star)$ is β -equivalent to T (where \star is the canonical inhabitant of $\mathbf{1}$), but reduction of $(\text{force } S s T t ?_l)$ is blocked. The coercion k is now redefined as follows:

$$\begin{aligned} k &: \forall S : \mathbf{Type}. \forall s : S. \forall T : \mathbf{Type}. \forall t : T. \mathbf{1} \rightarrow \text{force } S s T t \\ k S s T t \text{ lock} &:= \text{match } \text{lock} \text{ with } [\star \Rightarrow t] \end{aligned}$$

The system now tries to promote \bar{s} of type S to $(k S s ?_T ?_t ?_l)$ that generates the new failing unification problem

$$\text{force } S \bar{s} ?_T ?_t ?_l \equiv T$$

At last, this is where the unification hints come into play. Indeed, we can define the following two acceptable unification hints:

$$\vdash \frac{\begin{array}{l} ?_T := T \\ ?_t := t_1 \\ ?_l := \star \end{array}}{\text{force } S s_1 ?_T ?_t ?_l \equiv T} \quad \vdash \frac{\begin{array}{l} ?_T := T \\ ?_t := t_2 \\ ?_l := \star \end{array}}{\text{force } S s_2 ?_T ?_t ?_l \equiv T}$$

that suggests respectively the solution t_1 when s_1 is matched and the solution t_2 when s_2 is, unblocking the reduction by fixing $?_l$ to be \star so that $(\text{force } S s_i T t_i \star)$ reduces to T and the promoted term $(k S s_i T t_i \star)$ reduces to t_i as expected.

5.3 The general solution

The simplest scenario of the previous section assumed every branch of the nonuniform coercion to go from S to T . The assumption was necessary to type the very first failing attempts. However, it is useless for the final working solution presented at the end of the section and thus it can be dropped. Similarly, the number of branches in the nonuniform coercion also plays no role since it just corresponds to the number of unification hints to be declared. Also the order in which the branches are listed (and that determines the branch that is triggered when multiple branches can) can be preserved by defining the unification hints in the good order (or by giving an explicit precedence to them). Finally, branches with a non empty context Γ_i also pose no problem since unification hints are also parameterized by a context.

Thus, to summarize, the final general solution is the following. First of all, we declare at the beginning of the library the definition of `force` and k

$$\begin{aligned} \text{force} &: \forall S : \mathbf{Type}. S \rightarrow \forall T : \mathbf{Type}. T \rightarrow \mathbf{1} \rightarrow \mathbf{Type} \\ \text{force } S s T t \text{ lock} &:= \text{match } \text{lock} \text{ with } [\star \Rightarrow T] \end{aligned}$$

$$\begin{aligned} k &: \forall S : \mathbf{Type}. \forall s : S. \forall T : \mathbf{Type}. \forall t : T. \mathbf{1} \rightarrow \text{force } S s T t \\ k S s T t \text{ lock} &:= \text{match } \text{lock} \text{ with } [\star \Rightarrow t] \end{aligned}$$

and we declare k as the only (uniform) coercion. Then, to declare a nonuniform coercion whose branches are all of the form

$$\Gamma_i \vdash \begin{array}{l} S_i \rightarrow T_i \\ s_i \mapsto t_i \end{array}$$

the user declares for each branch the following acceptable unification hint:

$$\Gamma'_i \vdash \frac{?_T := T'_i \quad ?_t := t'_i \quad ?_l := \star}{\text{force } S'_i s'_i ?_T ?_t ?_l \equiv T'_i}$$

where Γ'_i is obtained from Γ_i by turning every variable x in a meta-variable $?_x$, and the same holds for T'_i , S'_i and s'_i and t'_i .

Of course, the system could just implement the standard syntax for nonuniform coercions as syntactic sugar for the unification hint itself.

In particular, since only one uniform coercion (k) is declared, the ITP code that implements coercion can be simplified, for example dropping all optimizations for fast indexing/retrieval. Actually, since the coercion k must be able to promote any type to `force`, the ITP must allow the declaration of coercions going from any type to something. This feature can be considered quite unusual since it does not allow any form of efficient indexing. For instance, the Coq system would not accept k as a coercion. Matita, instead, does not have this restriction. In any case, with a bit of work, we can avoid the limitation when it is there by redeclaring k multiple times as a coercion from T to `force` for every type T whose elements can be promoted.

Remember that, in our implementation of nonuniform coercions via unification hints, we have assumed the ITP to only allow a coherent set of uniform coercions. With our implementation, uniform coercions can now be handled as special cases of nonuniform coercions, with the side effect of having the possibility to declare incoherent sets. Incoherence is then handled for free by systematically choosing the first matching hint (or the one with the greatest priority).

6 Reasoning about Curried morphisms

During the last year the Matita ITP was redesigned, and one of the novelties was the introduction of unification hints to support the mathematical structures technique. Even if the old system (version 0.5.x) was lacking any kind of structure inference support, we formalized with it a hierarchy of algebraic and topological structures. It was clear that porting this formalization to the new system embracing the mathematical structures approach was a good test bench. The following example, extracted from the formalization mentioned above, convinced us to study the notion of nonuniform coercions presented in this paper.

The setting of the formalization is deeply extensional: points belongs to types equipped with an equivalence relation, usually called setoids. Functions respecting the equivalence relation of their source and target types are called morphisms and are denoted with $A \Rightarrow B$ where A and B are setoids:

$$\begin{aligned} \text{setoid} &:= \{ T : \mathbf{Type}; \approx : T \rightarrow T \rightarrow \mathbf{Prop}; \pi_{\approx} : \text{is_equiv_relation } T \approx \} \\ A \Rightarrow B &:= \{ f : A \rightarrow B; \pi_f : \forall x, y : A. x \approx_A y \rightarrow f x \approx_B f y \} \end{aligned}$$

One may expect that the concept of unary morphism extends in a straightforward way via currying to n-ary morphisms as the concept of unary function extends, in an higher order languages, to n-ary functions. This is not the case for morphisms, since currying comes at an extra price: in order to form

$A \Rightarrow (B \Rightarrow C)$, the target $B \Rightarrow C$ must be a setoid, while $B \Rightarrow C$ is a type. We can obtain a setoid of carrier $B \Rightarrow C$ equipping the carrier with the following extensional equality relation over morphisms:

$$A, B : \text{setoid}; f, g : A \Rightarrow B \vdash f \approx_{\Rightarrow} g := \forall x : A. f \ x \approx_B g \ x$$

so that $(B \Rightarrow C, \approx_{\Rightarrow} BC, \dots)$ is a setoid (that we denote as $B \Rightarrow_{\approx} C$). It is now possible, but cumbersome and distracting, to write a curried morphism as $A \Rightarrow (B \Rightarrow_{\approx} C)$. A much better solution is to exploit the following nonuniform coercion to allow the implicit promotion of $B \Rightarrow C$ to $B \Rightarrow_{\approx} C$:

$$A : \text{setoid}, B : \text{setoid} \vdash \begin{array}{l} \text{Type} \quad \rightarrow \quad \text{setoid} \\ A \Rightarrow B \quad \mapsto \quad (A \Rightarrow B, \approx_{\Rightarrow}, \pi_{\Rightarrow}) \end{array}$$

where π_{\Rightarrow} is a proof that \approx_{\Rightarrow} is an equivalence relation over morphisms.

Before developing the concept of non uniform coercions we tried other solutions, but they turned out to be quite unsatisfactory.

First, note that a uniform coercion cannot distinguish the type $A \Rightarrow B$ from another type, thus would apply in unexpected contexts too.

It would be also hard to achieve the same result relaying on the notational support the interactive theorem prover offers to overload the \Rightarrow and \Rightarrow_{\approx} notations. Indeed, in the very same expression $A \Rightarrow (B \Rightarrow C)$ the first occurrence of \Rightarrow is intended to define a morphism, while the other occurrence must define a setoid, but interpreting all the occurrences as setoids, possibly inserting a projection in front, makes sense too (even if it is not the intended meaning). This approach would thus require the system to employ a non trivial mechanism to disambiguate notational overloading.

Last, it is always possible to define different record types for unary, binary, etc. morphisms, but it turns out to be rather inconvenient as soon as one has to reason about partially instantiated morphisms. For example a partially instantiated binary morphism of type $A \Rightarrow B \Rightarrow C$ would have type $B \rightarrow C$, since functions belonging to that morphism class have type $A \rightarrow B \rightarrow C$ and not $A \rightarrow B \Rightarrow C$. The real shortcoming is not that the system has to enrich partially instantiated morphism, but that the user has to link every n-ary morphism to $n - 1$ possible enrichments (and all of them comprise a different proof for the π_f component of the morphism structure).

7 Lifting types, not notations

Every ITP allows some degree of user configurable notational support. Consider the following conjecture (where we used the respectively infix notation $+$ for addition over integers and prefix notation $-$ for inverse)

$$x, y : \mathbb{Z} \vdash x + -(y + x) = -y$$

and the following well known fact:

$$\text{invmul} : \forall G : \text{Group}. \forall a, b : G. (a * b)^{-1} = b^{-1} * a^{-1}$$

where $*$ is a notation for ($\text{op } G$) and \cdot^{-1} for ($\text{inv } G$). Thanks to unification hints, we can rewrite the conjecture using `invmul` since the system knows that \mathbb{Z} together with $+$ and $-$ forms the Group \mathcal{Z} . The result, however, is quite confusing:

$$x, y : \mathbb{Z} \vdash x + x^{-1} * y^{-1} = -y$$

To understand it, it is better to de-activate the notation for group, obtaining

$$x, y : \mathbb{Z} \vdash x + \text{op } \mathcal{L} (\text{inv } \mathcal{L} x) (\text{inv } \mathcal{L} y) = -y$$

The result is clearly correct, since $(\text{op } \mathcal{L})$ reduces to $+$ and $(\text{inv } \mathcal{L})$ to $-$, but, until we perform the reduction, the notation we get is wrong.

This problem is already extremely frequent when we use the formalization approach described in Section 3, and it becomes worse with nonuniform coercions, that may promote even an atomic term written by the user (like `zplus`, or `+`) to a richer operation, like multiplication in a group (i.e. denoted with `*`).

A possible solution that avoids reducing the term is to overload the notation for $+$ and $-$ over $(\text{op } \mathcal{L})$ and $(\text{inv } \mathcal{L})$. This is usually possible since the pattern $(\text{op } \mathcal{L})$ is more precise than the pattern $(\text{op } _)$ which is matched in the generic group notation. Nevertheless, this is a bad solution for two reasons: it requires to declare new notations every time we define a new model of a given structure; and terms like $(\text{op } \mathcal{L} x y)$ are left in the proof term where we would expect to find the simpler $x + y$.

A superior solution is forcing the needed reductions, without requiring any user intervention. We achieve this in two steps. First of all, we change the way we declare our hints so that the term that is left by unification is no longer a projection of the form $(\text{op } \mathcal{L})$, but a redex of the form

$$\text{op } \langle \mathbb{Z}, +, -, \text{assoc } \mathcal{L}, \text{inv_op_cancel } \mathcal{L}, \text{unit_law } \mathcal{L}, \text{closed_op } \mathcal{L} \rangle$$

Then we change the implementation of our ITP so that redexes of this kind are automatically reduced as soon as they are formed. In particular, if the projection in the redex retrieves the carrier or an operation of the structure (like `op` or `inv`), the reduction yields the plain operation of the structure ($+$ or $-$); if the projection retrieves a property, the reduction yields a projection, like `assoc` \mathcal{L} which is a compact proof term for the associativity of addition.

The strategy of reducing this form of redexes is consistent with the usually implemented one for β -redexes. For example, in a system like Coq or Matita, every rewriting step with an equation $a = b$ is performed by first changing the conjecture P to the β -redex $((\lambda x.P[x/a])a)$ and then applying the elimination principle for equality. The rewritten conjecture would be $((\lambda x.P[x/a])b)$, but, since β -redex are immediately reduced, we obtain $P[b/a]$.

The following is the hint able to generate the redexes in the case of the group of integers. Compare it with the similar hint given in Section 4:

$$\vdash \frac{?_g := \langle \mathbb{Z}, +, -, 0, \text{assoc } \mathcal{L}, \text{inv_op_cancel } \mathcal{L}, \text{unit_law } \mathcal{L}, \text{closed_op } \mathcal{L} \rangle}{\text{op } ?_g \equiv +}$$

It is worth observing that the expanded form of the hinted solution can be obtained mechanically by separating fields that hold properties, that are kept as projections of the \mathcal{L} structure, from fields holding types or operations, that on the contrary are expanded.

It is natural to compare this approach with the alternative one consisting in implementing an ad hoc simplification tactic that reduces only redexes involving projections applied to concrete instances of a structure. While this approach does not necessarily require any deep modification to the interactive theorem prover,³ it has to be triggered manually and its effect is not recorded in the proof term. Reduction is not a proof step in CIC, thus the effect of any reduction tactic is left implicit in the resulting proof term. Hardwiring the greedy reduction strategy we propose in the proof engine allows to obtain proof term in

³Assuming the ITP features a language to let the user define new tactics, like \mathcal{L} -tac.

which the aforementioned redexes are not present. This affects in a positive way not only the type checking time, thanks to the smaller size of proof terms, but also the output quality of any procedure manipulating proof terms.

For example in [10] and [6] the authors reconstruct a proof script, respectively based on a procedural and a declarative tactic language, starting from a proof term. Another example is [8, 7] where an explanation of a proof in natural language is obtained solely processing a proof term. In all these cases a proof term were operations are of the form $(\text{op } \mathcal{L})$ would be clearly explained with the nomenclature of abstract group theory, while the original proof was carried on the concrete setting of integers, and the user did resort to abstract group theory only to justify some of his proof steps.

8 Conclusions

The most powerful tool of modern mathematics is abstraction in the following sense: the mathematical corpus is no longer a flat collection of facts, but a hierarchy of algebraic, topological and geometrical theories, all characterized by its own set of axioms and often very rich in models. Every concrete mathematical object belongs, directly or via isomorphisms, to several models and the mathematician effortlessly mixes lemmas from all the relative theories.

This paper aims at making the formalization technique pioneered by Gonthier et al. [4, 9] and the general machinery behind it [3] run smoothly, allowing the unification heuristic of ITPs to behave in a more consistent way with respect to the inference of the mathematical structure a model belongs to. In particular, the inference is not triggered only when projections of the structure are involved in a unification problem, but more generally whenever a term (or type) has to be promoted to a richer structure. The promotion is expressed in terms of nonuniform coercions, a novel notion to the authors knowledge, that are shown to be implementable in terms of uniform coercions and unification hints, in an higher order logic equipped with dependent types and pattern matching.

Acknowledgments We would like to thank Jeremy Avigad for proofreading preliminary versions of this paper.

References

- [1] A. Asperti, W. Ricciotti, C. Sacerdoti Coen & E. Tassi (2009): *A compact kernel for the Calculus of Inductive Constructions*. *Sadhana* 34(1), pp. 71–144, doi:10.1007/s12046-009-0003-3.
- [2] A. Asperti, C. Sacerdoti Coen, E. Tassi & S. Zacchiroli (2007): *User Interaction with the Matita Proof Assistant*. *Journal of Automated Reasoning* 39(2), pp. 109–139, doi:10.1007/s10817-007-9070-5.
- [3] Andrea Asperti, Wilmer Ricciotti, Claudio Sacerdoti Coen & Enrico Tassi (2009): *Hints in unification*. In: *TPHOLs 2009. LNCS 5674/2009*, Springer-Verlag, pp. 84–98, doi:10.1007/978-3-642-03359-9_8.
- [4] Yves Bertot, Georges Gonthier, Sidi Ould Biha & Ioana Pasca (2008): *Canonical Big Operators*. In: *TPHOLs*. pp. 86–101, doi:10.1007/978-3-540-71067-7_11.
- [5] Thomas Braibant & Damien Pous (2009): *A Tactic for Deciding Kleene Algebras*. 1st Coq Workshop, doi:10.1007/978-3-642-14052-5_13. Available at <http://hal.archives-ouvertes.fr/hal-00383070/fr/>. (Available as a HAL report).
- [6] Claudio Sacerdoti Coen (2010): *Declarative Representation of Proof Terms*. *J. Autom. Reasoning* 44(1-2), pp. 25–52, doi:10.1007/s10817-009-9136-7.

- [7] Yann Coscoy (2000): *Explication textuelle de preuves pour le Calcul des Constructions Inductives*. Ph.D. thesis, Université de Nice-Sophia Antipolis.
- [8] Yann Coscoy, Gilles Kahn & Laurent Thery (1995): *Extracting Text from Proofs*. Technical Report RR-2459, Inria (Institut National de Recherche en Informatique et en Automatique), France, doi:10.1007/BFb0014048.
- [9] F. Garillot, G. Gonthier, A. Mahboubi & L. Rideau (2009): *Packaging mathematical structures*. In: *TPHOLs 2009*. fixme, p. fixme, doi:10.1007/978-3-642-03359-9_23.
- [10] Ferruccio Guidi (2007): *Procedural Representation of CIC Proof Terms*. In: *PLMMS, Workshop on Programming Languages for Mechanized Mathematics*. doi:10.1007/s10817-009-9137-6. To appear.
- [11] Zhaohui Luo (1999): *Coercive Subtyping*. *J. Logic and Computation* 9(1), pp. 105–130, doi:10.1093/logcom/9.1.105. Available at http://www.oup.co.uk/logcom/hdb/Volume_09/Issue_01/090105.sgm.abs.html.
- [12] W. McCune (1992): *Experiments with discrimination tree indexing and path indexing for term retrieval*. *Journal of Automated Reasoning* 9(2), pp. 147–167, doi:10.1007/BF00245458.
- [13] R. Nieuwenhuis, T. Hillenbrand, A. Riazanov & A. Voronkov (2001): *On the Evaluation of Indexing Techniques for Theorem Proving*. *LNCS 2083*, pp. 257–271, doi:10.1007/3-540-45744-5_19. Available at citeseer.ist.psu.edu/nieuwenhuis03evaluation.html.
- [14] Robert Pollack (2002): *Dependently Typed Records in Type Theory*. *Formal Aspects of Computing* 13, pp. 386–402, doi:10.1007/s001650200018. Available at <http://homepages.inf.ed.ac.uk/rpollack/export/recordsFAC.ps.gz>.
- [15] C. Sacerdoti Coen & E. Tassi (2007): *Working with Mathematical Structures in Type Theory*. In: *TYPES*. *LNCS 4941/2008*, pp. 157–172.
- [16] M. Sozeau (2006): *Subset Coercions in Coq*. In: *Types for Proofs and Programs*. *LNCS 4502/2007*, Springer-Verlag, pp. 237–252, doi:10.1007/978-3-540-74464-1_16.
- [17] M. Sozeau & N. Oury (2008): *First-Class Type Classes*. In: *TPHOLs*. pp. 278–293, doi:10.1007/978-3-540-71067-7_23.
- [18] Matthieu Sozeau (2009): *A New Look at Generalized Rewriting in Type Theory*. *Journal of Formalized Reasoning* 2(1), pp. 41–62.