An introduction to Linear Logic and proof nets

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Abstract

In this article we will give an introduction to Linear Logic from a point of view of a computer scientist. First we will introduce the natural deduction and a classical example, made by J.Y. Girard, in order to understand the idea of resource.

Then we will introduce the Gentzen representation of the classical logic and we will discuss about structural rules. We shall show how new connectives will arise if we consider only a subset of structural rules.

We will introduce the Linear Logic and the MLL fragment. At the end we will focus on proof-nets in MLL.

A short introduction

Logic, from greek λογος (word, discussion, thought), can be defined as the science of reasoning, studying its rules and its principles.

Since ancient times, logic has been studied a lot. For the ancient Greek, it was considered as one of the most important subject because it was consider as the base of the art of speaking and writing effectively. This approach, nearer to the scientific study of language, it has been for long time the most followed and studied.

1 Problems in classical logic

Classic Logic (**CL**) can express, in a easy way, sentences with stable truth. Indeed, if we say "is the sun high now?" the answer could be "yes" or "no". No other kind of answer is allowed. True or False are the only values that we can associate, in **CL** to a sentence.

In this note we shall not discuss about fuzzy logic or temporal logic (and some other non-classical logic); we are interested to express, as well as we can, the idea of resource.

It is difficult to find a formal definition of what it is a "resource". The following one could be a "good" definition.

Definition A resource is any physical or virtual entity of limited availability that can be used, consumed, replaced, copied and renewed.

Example Example of resources are easy to show. If we are talking abour computer science, a resource could be a variable in our program or some phisical devices (ethernet card, hard-disk, . . .).

Apart from the strict topic of computer science, money are a tipical example of resource that we use in everyday life. Money are used to buy something, an exchange between our money and the desired object.

1.1 Natural deduction in classic logic

1.2 Handeling the resources

Consider the following example, that we shall discuss:

$$\dfrac{\dfrac{\dfrac{[A] \qquad A \Rightarrow B}{B}\,(mp) \qquad \dfrac{[A] \qquad A \Rightarrow C}{C}\,(mp)}{B \wedge C}\,\wedge}{A \Rightarrow (B \wedge C)}\,[A]$$

The previous example is a correct proof in natural deduction. Starting from the hypotesis that $A \Rightarrow B$ (from the element A i can have B) and $A \Rightarrow C$ (from element A i can have C) we deduce $A \Rightarrow (B \wedge C)$ (from the element A i can have both B and C).

Suppone now to susbstitute the element A with the sentence "3euro", B with "a packet of cigarettes" and C with "a sandwitch"; we obtain the following proof:

$$\dfrac{\dfrac{\dfrac{[2euro] \qquad 2euro \Rightarrow cig}{cig}\,(mp) \qquad \dfrac{[2euro] \qquad 2euro \Rightarrow sandw}{sandw}\,(mp)}{cig \wedge sandw}\,\wedge}{2euro \Rightarrow (cig \wedge sandw)}\,[2euro]$$

As the reader could easly understand, the proof is no more correct. Or, well, we have to say that from the point of view of the applied rules, the proof is correct, but its meaning is not. The example in equation 1.2 express a real life example. Suppose to have two euro and be in front of two object, whose price is exactely two euro. Obviously, we cannot buy all of them, we can have just only one.

The proof in 1.2 tells us a surprised result: *with two euro we are able to buy both objects*! The solution to this enigma lead us to two considerations:

- We are living in a non-capitalistic world, where money does not have importance.
- Classi Logic is not able to handle the idea of resource.

Obviously, everybody knows that the first consideration is not correct: once we have used our two euros we will no more have them.

However, saying that Classic Logic is not able to handle the idea of resouce does not solve our problem. For this reason we will investigate, deeply, this issue.

Considering the example 1.2, implicitly we have duplicated our money. Indeed, in the proof, the element "2 euro" appears two times. The main problem is that classic logic let us to do such things.

2 From Gentzen representation to Linear Logic

In 1936, Gerard Gentzen, a german mathematician and logician, published ****. He introduce a new representation for calculus.

2.1 Sequent Calculus

In sequent Calculus the sentences are expressed in the following way:

$$\Gamma \vdash \Delta$$

Γ and Δ express an ordered set of formulas.

Definition dd

eg: Just to make easier the comprehension of the topic we are going to introduce, consider the following example. Γ could be the set $A \wedge B, C \Rightarrow A, D, E \vee F$.

2.2 Logiske Klassic Calkulus

3 Additive, Multiplicative connectives and exponetials

4 Proof Nets

References