Computational Complexity of Dynamical Systems: the case of Cellular Automata

Pietro Di Lena, Luciano Margara
dilena@cs.unibo.it, margara@cs.unibo.it

Department of Computer Science, University of Bologna, Mura Anteo Zamboni 7, 40127 Bologna, Italy.

Abstract. Cellular Automata can be considered discrete dynamical systems and at the same time a model of parallel computation. In this paper we investigate the connections between dynamical and computational properties of Cellular Automata. We propose a classification of Cellular Automata according to the language complexities which rise from the basins of attraction of subshift attractors and investigate the intersection classes between our classification and other three topological classifications of Cellular Automata. From the intersection classes we can derive necessary topological properties for a cellular automaton to be computationally universal.

1 Introduction

The concept of computation and computation theory itself are strictly related to Turing Machines. In recent years, however, a new trend of investigation attempts to find connections between dynamical system theory and computation theory. Cellular Automata can be considered discrete dynamical systems and at the same time a model of parallel computation. It is well known that they have the same computational power of Turing Machines.

There's no general agreement on the concept of universality for Cellular Automata. The universality of a cellular automaton is generally proved by showing that such automaton can simulate a universal Turing Machine [13] or some other system which is well known to be computationally universal [2]. A different approach was taken by Wolfram in [14] where the author classifies empirically Cellular Automata in four classes according to the observed (by computer simulation) evolution of the automaton on random configurations. He suggested that Cellular Automata in the fourth class must be capable of universal computation. Several authors have offered formalization to Wolfram classes. We cite just few of them. Gilman [6] proposed a classification based on the concept of *equicontinuity* while Hurley [8] proposed a classification based on the concept of *attractors*. Kůrka [9] refined the Equicontinuity and Attractor classification by using purely topological definitions and investigated the intersection classes between the two classifications and a third one based on the complexity of the languages rising from the column factors of Cellular Automata. All three classifications are based uniquely on topological concepts and it is not evident how this dynamical properties are related to computational properties of Cellular Automata except for the connection with Wolfram's empirical classification.

While it is generally accepted to interpret the evolution of a dynamical system as a process of computation, it is much more less evident how to interpret the input and the output of the computation in the evolution of the system. A possible approach is to see the process of computation in a dynamical system as a flow toward an attractor. The attractor is considered the halting state of the computation. One such approach has been taken in [1] to develop a complexity theory for the set of continuous time dynamical systems defined by differential equations. A more general approach has been taken recently in [3]. The authors rephrase the halting problem as the problem to decide if there exists at least one configuration from some *initial set* whose orbit reaches some *halting set*. Initial and halting sets are intended to be clopen (closed and open) sets of a Cantor space so that they can be described by means of finite information. It is easy to see how these two approaches are related: in a compact metric space the orbit of some configuration converges to an attractor Z if and only if it enters into all clopen invariant sets whose omega limits coincide with Z. The authors of [3] propose a definition of universality which applies to general discrete symbolic (i.e. defined on a Cantor space) dynamical systems and they provide necessary conditions for the universality. According to their model, a universal symbolic dynamical system is not minimal, not equicontinuos and does not satisfy the shadowing property. Moreover they conjecture that a universal dynamical system must have an infinite number of subsystems.

Here we interpret the process of computation in Cellular Automata as a flow toward a subshift attractor. A subshift attractor is an attractor which is invariant under the shift map. Subshift attractors have been investigated in [10] and [5]. We show that it is possible to restate the halting problem as the problem to decide if the omega limit of some clopen set converges to an halting subshift attractor (that is, as the problem to decide if the orbits of all sequences contained in some clopen set converge to the attractor). We say that the computational complexity of a cellular automaton $(A^{\mathbb{Z}}, F)$ with respect to the halting subshift attractor Z is defined as the complexity of clopen sets contained in the basin of attraction of Z. Since a basin of attraction is the countable union of cylinder (clopen) sets and a cylinder set can be univocally described by a word in A^*, we can characterize the complexity of a basin of attraction by using formal language theory. We propose a classification of Cellular Automata according to the complexity of *basin languages* (Section 3). A cellular automaton with highest computational complexity has at least one subshift attractor whose basin language is strictly recursively enumerable.

Since our classification is based on purely topological concepts it is easy to explore the intersection classes with other well known topological classifications of Cellular Automata such as Attractors, Equicontinuity and Languages classifications (Section 4). From the intersection classes we can provide necessary conditions for a cellular automaton to be universal (Section 5). Even in our model a universal cellular automaton is not minimal, not equicontinuous, does not have the shadowing property and, in particular, it is not regular. It is open also in our case the question whether a universal cellular automaton must have an infinite number of subsystems.

2 Notation and Definitions

Let A be a finite alphabet. With $A^{\mathbb{Z}}$ and $A^{\mathbb{N}}$ we denote respectively the set of sequences $(x_i)_{i\in\mathbb{Z}}$ and $(x_i)_{i\in\mathbb{N}}$ where $x_i \in A$. For $x \in A^{\mathbb{Z}}$, let $x_{[i,j]} \in A^{j-i+1}$ denote the word $x_i x_{i+1} ... x_j$. We use the shortcut $w \sqsubseteq x$ to say that $w \in A^*$ is a subword of $x \in A^{\mathbb{Z}}$. Let define a *metric* d on $A^{\mathbb{Z}}$ by $d(x,y) = 2^{-n}$ where $n = min\{|i| \mid x_i \neq y_i\}|$. The set $A^{\mathbb{Z}}$ endowed with metric d is a compact metric space. For $u \in A^*$ and $i \in \mathbb{Z}$, let $[u]_i = \{x \in A^{\mathbb{Z}} \mid x_{[i,i+|u|-1]} = u\}$ denote a *cylinder set*. Sometimes we will refer to the cylinder set $[u]_i$ simply with $[u]$. A cylinder set is a clopen (closed and open) set in $A^{\mathbb{Z}}$. Every clopen set in $A^{\mathbb{Z}}$ is a finite union of cylinder sets. The *shift map* $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $\sigma(x)_i = x_{i+1}$. A *cellular automaton* is a dynamical system $(A^{\mathbb{Z}}, F)$ where $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a continuous and σ-commuting function, i.e. $F\sigma = \sigma F$. According to Curtis-Hedlund-Lyndon theorem [7], $(A^{\mathbb{Z}}, F)$ is a cellular automaton if and only if there exists a *local function* $f : A^{2r+1} \to A$ of *radius* $r > 0$ such that $F(x)_i = f(x_{i-r}, ..., x_{i+r})$.

In the following sections we review Attractor, Equicontinuty and Language classifications for Cellular Automata. The intersection classes between the tree classifications can be found in [9].

2.1 Attractor classification

The ω-limit of a set $U \subseteq A^{\mathbb{Z}}$ is $\omega(U) = \cap_{n>0}\overline{\cup_{m>n}F^n(U)}$. A nonempty set $Z \subseteq A^{\mathbb{Z}}$ is an *attractor* if there exists an F-invariant clopen set $U \subseteq A^{\mathbb{Z}}$ such that $\omega(U) = Z$. A nonempty set is a *quasi-attractor* if it is the countable intersection of attractors. An attractor is minimal if it doesn't contain any proper subset which is also an attractor. The *basin of attraction* of an attractor Z is the set $\mathcal{B}(Z) = \{x \in A^{\mathbb{Z}} \mid \omega(x) \subseteq Z\}$. The basin of attraction is always an open F-invariant set.

This following classification is Kůrka's refinement of Hurley's Attractor classification for Cellular Automata [8].

Corollary 1. *[9] Every $(A^{\mathbb{Z}}, F)$ falls exactly in one of the following classes.*

A1 *There exist two disjoint attractors.*
A2 *There exists a unique minimal quasi-attractor.*
A3 *There exists a unique minimal attractor different from $\omega(A^{\mathbb{Z}})$.*
A4 *There exists a unique minimal attractor $\omega(A^{\mathbb{Z}}) \neq A^{\mathbb{Z}}$.*
A5 *There exists a unique minimal attractor $A^{\mathbb{Z}}$.*

A *subshift attractor* is a σ-invariant attractor. A clopen F-invariant set $U \subseteq A^{\mathbb{Z}}$ is *spreading* if $F^k(U) \subseteq \sigma^{-1}(U) \cap U \cap \sigma(U)$ for some $k > 0$. The following proposition characterizes clopen sets whose omega limits are subshift attractors.

Proposition 1. *[5] Let $(A^{\mathbb{Z}}, F)$ be a cellular automaton and $U \subseteq A^{\mathbb{Z}}$ a clopen F-invariant set. Then $\omega(U)$ is a subshift attractor if and only if U is spreading.*

Every cellular automaton $(A^{\mathbb{Z}}, F)$ has at least one subshift attractor $\omega(A^{\mathbb{Z}})$ but it can have also an infinite number of subshift attractors [12]. For instance, Kůrka [10] shows that, for surjective cellular automata, the full space is the unique subshift attractor. We show two examples which will be useful later. The first example shows an unstable cellular automaton with an infinite number of attractors and with just one subshift attractor

Example 1. The Hurley cellular automaton, whose local rule $f : \{0,1\}^2 \to \{0,1\}$ is defined by $f(a,b) = ab$ has unique minimal quasi-attractor ${}^{\infty}0^{\infty}$ (see [8] or [11]) and unique subshift attractor $\omega(A^{\mathbb{Z}}) = \{x \in A^{\mathbb{Z}} \mid 10^{+}1 \not\sqsubseteq x\}$ (see [5]).

Question 1. Is there a stable cellular automaton with more than one subshift attractor or with an infinite number of subshift attractors?

The second example shows an unstable regular cellular automaton with just two subshift attractors.

Example 2. The cellular automaton $(A^{\mathbb{Z}}, F)$, whose local rule $f : \{0,1\}^3 \to \{0,1\}$ is defined by $f(x,y,z) = xyz$, has just two subshift attractors $\omega(A^{\mathbb{Z}}) = \{x \in A^{\mathbb{Z}} \mid 10^{+}1 \not\sqsubseteq x\}$ and ${}^{\infty}0^{\infty} \neq \omega(A^{\mathbb{Z}})$. A cellular automaton is regular if and only if Σ_{2r+1} is a sofic shift (see [4]). In this case it is easy to see that Σ_3 is the one-sided sofic shift defined by the σ-closure of the sequences $(111)^{*}x(000)^{\infty}$, where $x = (110) \mid (110)(100) \mid (011) \mid (011)(001) \mid (010)$.

2.2 Equicontinuity classification

We review some topological properties of Cellular Automata.

- *Equicontinuity*:

$$\forall x \in A^{\mathbb{Z}}, \forall \epsilon > 0, \exists \delta > 0, \forall y \in \mathcal{B}_{\epsilon}(x), \forall n \geq 0, d(F^n(x), F^n(y)) < \epsilon$$

- *Almost equicontinuity*:

$$\exists x \in A^{\mathbb{Z}}, \forall \epsilon > 0, \exists \delta > 0, \forall y \in \mathcal{B}_{\epsilon}(x), \forall n \geq 0, d(F^n(x), F^n(y)) < \epsilon$$

- *Sensitivity*:

$$\exists \epsilon > 0, \forall x \in A^{\mathbb{Z}}, \forall \delta > 0, \exists y \in \mathcal{B}_{\epsilon}(x), \exists n \geq 0, d(F^n(x), F^n(y)) \geq \epsilon$$

- *Positively expansiveness*:

$$\exists \epsilon > 0, \forall x, \forall y \neq x, \exists n \geq 0, d(F^n(x), F^n(y)) \geq \epsilon$$

This following classification is Kůrka's modification [9] of Gilman's Equicontinuity classification [6]. Gilman's classification is based on measure-theoretic concepts, while Kůrka's one uses only topological concepts.

Corollary 2. *[9] Every* $(A^{\mathbb{Z}}, F)$ *falls exactly in one of the following classes:*

E1 $(A^{\mathbb{Z}}, F)$ *is equicontinuous.*
E2 $(A^{\mathbb{Z}}, F)$ *is almost equicontinuous but not equicontinuous*
E3 $(A^{\mathbb{Z}}, F)$ *is sensitive but not positively expansive.*
E4 $(A^{\mathbb{Z}}, F)$ *is positively expansive.*

2.3 Language classification

The *column factor* of width $k > 0$ of $(A^{\mathbb{Z}}, F)$ is the set of one-sided infinite sequences $\Sigma_k = \{y \in A^{\mathbb{N}} \mid \exists x \in A^{\mathbb{Z}}, \forall n \geq 0, F^n(x)_{[0,k)} = y_n\}$. A cellular automaton $(A^{\mathbb{Z}}, F)$ is *bounded periodic* if $\forall k > 0, \exists m > 0, \exists n > 0$ such that $\forall x \in \Sigma_k, \forall i \geq m, \sigma^i(x) = \sigma^{i+n}(x)$. A cellular auotmaton is *regular* if $\forall k > 0$ the language $\mathcal{L}(\Sigma_k) = \{w \in (A^k)^* \mid \exists x \in \Sigma_k, x_{[0,|w|-1]} = w\}$ is regular. Obviously, a bounded periodic cellular automaton is regular. Every cellular automaton with the shadowing property is regular [9] while the converse is not true.

The following classification is Kůrka's Language classification of Celular Automata according to the language complexity of column factors.

Corollary 3. *[9] Every* $(A^{\mathbb{Z}}, F)$ *falls exactly in one of the following classes:*

L1 $(A^{\mathbb{Z}}, F)$ *is bounded periodic.*
L2 $(A^{\mathbb{Z}}, F)$ *is regular not bounded periodic.*
L3 $(A^{\mathbb{Z}}, F)$ *is not regular.*

Proposition 2. *[9]* **L1** = **E1**.

3 Basin Language classification and computational complexity of Cellular Automata

In this section we are interested in the basins of attraction of subshift attractors. We study the *complexity* of such basins by using formal language theory.

First we show that the basin of attraction of a subshift attractor is always a dense open set.

Proposition 3. *The basin of every subshift attractor is a dense open set.*

Proof. Let Z be a subshift attractor of $(A^{\mathbb{Z}}, F)$. Then $\mathcal{B}(Z)$ is open so we just need to show that every $x \in A^{\mathbb{Z}}$ belongs to the closure of $\mathcal{B}(Z)$. Let consider a clopen set $V \subseteq \mathcal{B}(Z)$ and let $\epsilon > 0$. Since $A^{\mathbb{Z}}$ is mixing, there exists $n > 0$ such that $\emptyset \neq \sigma^n(\mathscr{B}_\epsilon(x)) \cap V \subseteq \sigma^n(\mathscr{B}_\epsilon(x)) \cap \mathcal{B}(Z)$. Since Z is a subshift, for all $n \in \mathbb{Z}, \sigma^{-n}(V) \subseteq \mathcal{B}(Z)$ and $\emptyset \neq \mathscr{B}_\epsilon(x) \cap \sigma^{-n}(V) \subseteq \mathscr{B}_\epsilon(x) \cap \mathcal{B}(Z)$. Then $x \in cl(\mathcal{B}(Z))$. □

A *qualitative characterization* of basins of attraction is provided by formal language theory. By Proposition 3, the basin $\mathcal{B}(Z)$ of a subshift attractor Z is defined by the countable union of cylinder sets. A cylinder set can be (univocally) identified by some word in A^*. Considering basins of subshift attractors offers some advantages respect to basins of general attractors. Since the basin of a subshift attractor is σ-invariant, we don't need to take care of the coordinate of the cylinder in the space $A^{\mathbb{Z}}$. This means that if a cylinder $[u]_i$ is contained in the basin of some subshift attractor Z, then for every $j \in \mathbb{Z}$, $[u]_j$ is contained in $\mathcal{B}(Z)$ (this implies that the orbit of every configuration which contains the word u will converge to Z).

Definition 1. *Let denote with*

$$\mathcal{L}_Z = \{u \in A^* \mid [u] \subseteq \mathcal{B}(Z)\}$$

the basin language *of the subshift attractor Z of $(A^{\mathbb{Z}}, F)$.*

The language complexity of $\mathcal{L}_Z$ is a qualitative measure of the *complexity* of $\mathcal{B}(Z)$. We show that the language $\mathcal{L}_Z$ can be at most recursively enumerable. Next we show that $\mathcal{L}_Z$ can be strictly recursively enumerable.

Lemma 1. *Let $(A^{\mathbb{Z}}, F)$ be a cellular automaton. Let $V \subseteq A^{\mathbb{Z}}$ be a clopen F-invariant spreading set and let $U \subseteq A^{\mathbb{Z}}$ be a clopen set such that $\omega(U) \subseteq V$. Then $\exists n \in \mathbb{N}$ such that $F^n(U) \subseteq V$.*

Proof. Since V is clopen, $\overline{V} = A^{\mathbb{Z}} \setminus V$ is clopen and compact. For $n \in \mathbb{N}$, let define $X_n = \{x \in U \mid F^n(x) \notin V\} \subseteq U \cap \overline{V}$. Since U is clopen, every X_n is clopen. Moreover, since V is F-invariant, $\forall n \in \mathbb{N}, X_{n+1} \subseteq X_n$. Assume for absurd that, $\forall n \in \mathbb{N}, X_n \neq \emptyset$. Then, by compactness, $X = \cap_{n \in \mathbb{N}} X_n \subseteq U \cap \overline{V}$ is not empty and $\omega(X) \cap \overline{V} \neq \emptyset$ which is a contradiction. □

Proposition 4. *Let Z be a subshift attractor of $(A^{\mathbb{Z}}, F)$. Then $\mathcal{L}_Z$ is r.e.*

Proof. Let $U \subseteq A^{\mathbb{Z}}$ be a clopen F-invariant spreading set such that $\omega(U) = Z$. By Lemma 1, for every $u \in A^*$, $[u] \in \mathcal{B}(Z)$ if and only if $\exists n \in \mathbb{N}$ such that $F^n([u]) \subseteq U$. Since U is a finite union of cylinder sets, given some $v \in A^*$ and $k \in \mathbb{N}$, the property $F^k([v]) \subseteq U$ is decidable. This implies that $[u] \subseteq \mathcal{B}(Z)$ is a semidecidable question. Then $\mathcal{L}_Z$ is at most recursively enumerable. □

The following proposition shows that every r.e. language recognition problem is Turing-reducible to the basin language recognition problem for some cellular automaton. In particular we show that the *halting problem* for Turing Machines can be rephrased in terms of *reachability of a subshift attractor* for Cellular Automata.

Proposition 5. *Let $\mathcal{L} \subseteq B^*$ be a r.e. language. Then there is a cellular automaton $(A^{\mathbb{Z}}, F)$ with a subshift attractor Z and an injective computable mapping $\varphi : B^* \to A^*$ such that $u \in \mathcal{L}$ if and only if $\varphi(u) \in \mathcal{L}_Z$.*

Proof. Let $M = (B, Q, \delta, q_0, F)$ be a Turing machine recognizing the language $\mathcal{L}$. Let define $(A^{\mathbb{Z}}, F)$ where $A = B \cup Q \cup \{S, L, R\}$. The particle S is a spreading state. The particle L moves to left one step at time and erases everything on its path except when it encounters S and/or R: in that case generates an S particle. The R particle behaves exactly like L but it moves on the right. The other particles simulate the computation of the Turing machine M (the tape alphabet symbols are always quiescent). When some erroneous step occurs (unknown transition, two states collide, ..) then it is generated a particle S. If a final state is reached, then it is generated a particle S. Note that ${}^\infty S^\infty$ is a subshift attractor.

Let define the computable mapping $\varphi : B^* \to A^*$ by $\varphi(u) = Lq_0uR$. It is easy to see that if $a \in B$ is some tape symbol of the Turing Machine then $\omega({}^\infty aLq_0uRa^\infty) = {}^\infty S^\infty$ if and only if $u \in \mathcal{L}$. Then u is accepted by M if and only if $\omega([Lq_0uR]) = {}^\infty S^\infty$. □

We can classify Cellular Automata according to basin languages complexity.

Corollary 4. *Every $(A^{\mathbb{Z}}, F)$ falls exactly in one of the following classes:*

B1 $\exists Z, \mathcal{L}_Z = A^*$
B2 $\forall Z, \mathcal{L}_Z \neq A^*$ *is recursive*
B3 $\exists Z, \mathcal{L}_Z$ *is strictly r.e.*

By Proposition 5, class **B3** is not empty and it contains Cellular Automata capable of universal computation. By the existence of intermediate Turing degrees we cannot affirm that all Cellular Automata in class **B3** are universal so if we can provide some characterization for class **B3** we just have necessary conditions for the universality. Several natural questions easily arise.

Question 2. Is the membership in Basin Language classes decidable?

Is it possible to characterize classes **B1**, **B2**, **B3** in terms of the cardinality of subshift attractors? For instance, every cellular automaton in **B1** has just one subshift attractor.

Question 3. Is there some cellular automaton with an infinite number of subshift attractors in **B2**?

Question 4. Is there some cellular automaton with a finite number of subshift attractors in **B3**?

4 Classes comparison

In this section we compare Basin Language classification with Attractors, Equicontinuity and Language classifications. First we show two techniques to build Cellular Automata with nice properties. These two constructions will be useful to investigate the intersection classes.

The first construction is the *product cellular automaton.*

Definition 2. *The* product cellular automaton $(A^{\mathbb{Z}} \times B^{\mathbb{Z}}, F \times G)$ *of* $(A^{\mathbb{Z}}, F)$ *with* $(B^{\mathbb{Z}}, G)$ *is defined by* $\forall (x, y) \in A^{\mathbb{Z}} \times B^{\mathbb{Z}}, (F \times G)(x, y) = (F(x), G(y))$.

The proof of the following lemmas are trivial.

Lemma 2. *Let* $(A^{\mathbb{Z}} \times B^{\mathbb{Z}}, F \times G)$ *be a product cellular automaton. Then* $(Z', Z'') \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ *is a (subshift) attractor of* $(A^{\mathbb{Z}} \times B^{\mathbb{Z}}, F \times G)$ *if and only if* Z' *and* Z'' *are (subshift) attractors of* $(A^{\mathbb{Z}}, F)$ *and* $(B^{\mathbb{Z}}, G)$*, respectively.*

Lemma 3. *Let* $(A^{\mathbb{Z}}, F) \in \mathbf{A}i$ *and let* $(B^{\mathbb{Z}}, G) \in \mathbf{A}j$ *for* $1 \leq i, j \leq 5$*. Then* $(A^{\mathbb{Z}} \times B^{\mathbb{Z}}, F \times G) \in \mathbf{A}k, k = Min\{i, j\}$.

Lemma 4. *Let* $(A^{\mathbb{Z}}, F) \in \mathbf{E3}$*. Then* $(A^{\mathbb{Z}} \times B^{\mathbb{Z}}, F \times G) \in \mathbf{E3}$ *for every cellular automaton* $(B^{\mathbb{Z}}, G)$.

Lemma 5. *Let* $(A^{\mathbb{Z}}, F) \in \mathbf{L3}$*. Then* $(A^{\mathbb{Z}} \times B^{\mathbb{Z}}, F \times G) \in \mathbf{L3}$ *for every cellular automaton* $(B^{\mathbb{Z}}, G)$.

Lemma 6. *Let* $(A^{\mathbb{Z}}, F) \in \mathbf{B}i$ *and let* $(B^{\mathbb{Z}}, G) \in \mathbf{B}j$ *for* $1 \leq i, j \leq 3$*. Then* $(A^{\mathbb{Z}} \times B^{\mathbb{Z}}, F \times G) \in \mathbf{B}k, k = Max\{i, j\}$.

Proof. By Lemma 2, the language $\mathcal{L}_Z$ of the subshift attractor $Z = (Z', Z'')$ of $(A^{\mathbb{Z}} \times B^{\mathbb{Z}}, F \times G)$ is $\mathcal{L}_Z = \mathcal{L}_{Z'} \times \mathcal{L}_{Z''}$. Then, since $\mathcal{L}_Z$ can be at most recursively enumerable, the language complexity of $\mathcal{L}_Z$ is trivially the highest between the complexities of languages $\mathcal{L}_{Z'}$ and $\mathcal{L}_{Z''}$. □

The second construction consists in adding a spreading state to a cellular automaton.

Definition 3. *Let* $(A^{\mathbb{Z}}, F)$ *be of radius* r *and let* $A_s = A \cup \{s\}$ *where* $s \notin A$*. Let* $(A_s^{\mathbb{Z}}, F_s)$ *denote the CA whose local rule* $f_s : A_s^{2r+1} \to A_s$ *is defined by*

$$f_s(x_{-r}, ..., x_r) = s \textit{ if } \exists x_i = s \textit{ and } f_s(x_{-r}, ..., x_r) = f(x_{-r}, ..., x_r) \textit{ otherwise.}$$

Lemma 7. *Let* $(A^{\mathbb{Z}}, F)$ *be a cellular automaton and let* $s \notin A$*. Let consider* $(A_s^{\mathbb{Z}}, F_s)$*. Then* $(A_s^{\mathbb{Z}}, F_s) \in \mathbf{E2} \cap \mathbf{A3} \cap (\mathbf{B2} \cup \mathbf{B3})$*. Moreover,* $(A_s^{\mathbb{Z}}, F_s) \in \mathbf{B2}$ *if and only if* $(A^{\mathbb{Z}}, F) \in \mathbf{B1} \cup \mathbf{B2}$.

Proof. By definition, s is a blocking word. Moreover, $Z_s = \{^\infty s^\infty\} \neq \omega(A_s^{\mathbb{Z}})$ is a fixed point attractor. Then $(A_s^{\mathbb{Z}}, F_s) \in \mathbf{E2} \cap \mathbf{A3}$ and $(A_s^{\mathbb{Z}}, F_s) \notin \mathbf{B1}$. We now show that adding a spreading state doesn't affect the complexity of the basin languages of $(A^{\mathbb{Z}}, F)$. The basin of attraction of Z_s consists of the set of all biinfinite sequences which contain at least one occurrence of s, that is $\mathcal{B}(Z_s) = \{x \in A_s^{\mathbb{Z}} \mid \exists i \in Z, x_i = s\}$. Then, the basin language $\mathcal{L}_{Z_s} = \{w \in A_s^* \mid \exists i, w_i = s\}$ is recursive. It is easy to see that Z is a subshift attractor of $(A_s^{\mathbb{Z}}, F_s)$ if and only if $Z = \omega(U \cup [s])$ where $U \subseteq A^{\mathbb{Z}}$ is a clopen F-invariant spreading set for $(A^{\mathbb{Z}}, F)$. Let $Z' = \omega(U) \subset A^{\mathbb{Z}}$ be a subshift attractor of $(A^{\mathbb{Z}}, F)$. Then $\mathcal{L}_Z = \mathcal{L}_{Z'} \cup \mathcal{L}_{Z_s}$ and $\mathcal{L}_{Z'} \cap \mathcal{L}_{Z_s} = \emptyset$ which implies that $\mathcal{L}_Z$ is strictly recursively enumerable if and only if $\mathcal{L}_{Z'}$ is strictly recursively enumerable. □

4.1 Comparison with Language classification

By Proposition 2, the class **L1** of bounded periodic Cellular Automata coincides with the class **E1** of equicontinuous Cellular Automata. We show that every equicontinuous cellular automaton has exactly one subshift attractor.

Proposition 6. *Every equicontinuous cellular automaton has a unique subshift attractor which is a mixing shift of finite type.*

Proof. Since $(A^{\mathbb{Z}}, F)$ is stable, then $Z = \omega(A^{\mathbb{Z}}) = F^n(A^{\mathbb{Z}})$ for some $n \in \mathbb{N}$. Then Z is a mixing sofic shift. We show that Z is actually a SFT. Since $(A^{\mathbb{Z}}, F)$ is equicontinuous, there exists $p > 0$ such that $\forall x \in Z, \forall i \in \mathbb{N}, F^{ip}(x) = x$. (see [9]). Let r be the radius of $(A^{\mathbb{Z}}, F)$ and let consider the shift of finite type defined by $Z^{(2rp+1)} = \{x \in A^{\mathbb{Z}} \mid \forall i \in \mathbb{Z}, x_{[i,2rp+i]} \in \mathcal{L}_{2rp+1}(Z)\}$, i.e. the shift of finite type identified by the set of legal $(2rp+1)$-blocks of Z. Obviously, $Z \subseteq Z^{(2rp+1)}$. Moreover, F^p is the identity on $Z^{(2rp+1)}$, then $Z^{(2rp+1)} \subseteq Z$.
Now, assume for absurd that there exists a subshift attractor $Z' \subset Z$. Let U be a clopen spreading set such that $\omega(U) = Z'$. Since $U \neq Z$, $U \cap Z \neq \emptyset$ and Z is mixing, there exists $y \in Z$ and $m \in \mathbb{Z}$ such that $y \in U$ and $\sigma^m(y) \notin U$. Then, for every $i \in \mathbb{N}$, $F^{ip}(\sigma^m(x)) = \sigma^m(x) \notin U$ contradicting the fact that U is spreading. □

More generally, the basins of attraction of regular Cellular Automata give rise only to recursive basin languages.

Proposition 7. *If $(A^{\mathbb{Z}}, F)$ is regular then $\forall Z, \mathcal{L}_Z$ is recursive.*

Proof. We show that for every $u \in A^*$ the question $[u] \subseteq \mathcal{B}(Z)$ is decidable.

Let $U \subseteq A^{\mathbb{Z}}$ be a clopen F-invariant spreading set such that $\omega(U) = Z$. Let $k = \max\{|u| \mid [u] \subseteq U\}$ and let $v \in A^*$. Since $(A^{\mathbb{Z}}, F)$ is regular, it is possible to compute a labeled graph representation $\mathcal{G}$ of its column factor Σ_N where $N = \max\{k, |v|\}$ (see [4]). Then $\omega([u]) \not\subseteq Z$ if and only if there exists in $\mathcal{G}$ an infinite path $q_1 \overset{w_1}{\to} q_2 \overset{w_2}{\to} q_3 ...$ such that $u \sqsubseteq w_1$ and $[w_i] \not\subseteq U, \forall i \in \mathbb{N}$. Given a labeled graph $\mathcal{G}$ this property is easily decidable. □

Corollary 5. $\mathbf{L1} \subset \mathbf{B1}$*,* $\mathbf{L2} \cap \mathbf{B1} \neq \emptyset$*,* $\mathbf{L3} \cap \mathbf{B1} \neq \emptyset$*.*

Proof. Since every surjective cellular automaton is in **B1**, the proof follows from the nonemptiness of the intersection classes $\mathbf{L}i \cap \mathbf{A5} \neq \emptyset, 1 \leq i \leq 3$ (see [9]) and from $\mathbf{L1} = \mathbf{E1} \subset \mathbf{B1}$ (see Proposition 2 and Proposition 6). □

Corollary 6. $\mathbf{L2} \subset \mathbf{B1} \cup \mathbf{B2}$

Proof. The automaton of Example 2 has two subshift attractors and it is regular. Then $\mathbf{L2} \cap \mathbf{B2} \neq \emptyset$. The conclusion follows from Proposition 7. □

Corollary 7. $\mathbf{L3} \cap \mathbf{B2} \neq \emptyset$*,* $\mathbf{B3} \subset \mathbf{L3}$*.*

Proof. Let $(A^{\mathbb{Z}}, F) \in \mathbf{L3} \cap \mathbf{B1}$ and let $(B^{\mathbb{Z}}, G) \in \mathbf{L2} \cap \mathbf{B2}$. Then, by Lemma 5 and Lemma 6, $(A^{\mathbb{Z}} \times B^{\mathbb{Z}}, F \times G) \in \mathbf{L3} \cap \mathbf{B2}$. The inclusion $\mathbf{B3} \subset \mathbf{L3}$ follows from Corollary 6. □

	L1	L2	L3
B1	X	X	X
B2		X	X
B3			X

Fig. 1. Basin Language and Languages classifications

4.2 Comparison with Equicontinuity classification

Corollary 8. $\mathbf{E1} \subset \mathbf{B1}$*,* $\mathbf{E2} \cap \mathbf{B1} \neq \emptyset$*,* $\mathbf{E3} \cap \mathbf{B1} \neq \emptyset$*,* $\mathbf{E4} \subset \mathbf{B1}$*.*

Proof. By Proposition 6, $\mathbf{E1} \subset \mathbf{B1}$. Moreover $\mathbf{E4} \subset \mathbf{A5} \subset \mathbf{B1}$. For the other two cases, the proof follows from the nonemptiness of the intersection classes $\mathbf{E}i \cap \mathbf{A5} \neq \emptyset, 2 \leq i \leq 4$ (see [9]). □

Corollary 9. $\mathbf{E2} \cap \mathbf{B2} \neq \emptyset$*,* $\mathbf{E2} \cap \mathbf{B3} \neq \emptyset$*.*

Proof. Let $(A^{\mathbb{Z}}, F) \in \mathbf{B}i, 2 \leq i \leq 3$, and let $s \notin A$. Then, by Lemma 7, $(A_s^{\mathbb{Z}}, F_s) \in \mathbf{E2} \cap \mathbf{B}i$. □

Corollary 10. $\mathbf{E3} \cap \mathbf{B2} \neq \emptyset$*,* $\mathbf{E3} \cap \mathbf{B3} \neq \emptyset$*.*

Proof. Let $(A^{\mathbb{Z}}, F) \in \mathbf{E3} \cap \mathbf{B1}$ and let $(B^{\mathbb{Z}}, G) \in \mathbf{E2} \cap \mathbf{B}i, 2 \leq i \leq 3$. Then, by Lemma 4 and Lemma 6, $(A^{\mathbb{Z}} \times B^{\mathbb{Z}}, F \times G) \in \mathbf{E3} \cap \mathbf{B}i$. □

	E1	E2	E3	E4
B1	X	X	X	X
B2		X	X	
B3		X	X	

Fig. 2. Basin Language and Equicontinuity classifications

4.3 Comparison with Attractor classification

Corollary 11. $\mathbf{A1} \cap \mathbf{B1} \neq \emptyset$*,* $\mathbf{A1} \cap \mathbf{B2} \neq \emptyset$*,* $\mathbf{A1} \cap \mathbf{B3} \neq \emptyset$*.*

Proof. The identity cellular automaton $(\{0,1\}^{\mathbb{Z}}, I)$ has two disjoint attractors $\omega([0]), \omega([1])$ and, since it is surjective its unique subshift attractor is the full space. Then $\mathbf{A1} \cap \mathbf{B1} \neq \emptyset$. Let $(B^{\mathbb{Z}}, G) \in \mathbf{B}i, 1 \leq i \leq 3$. Then, by Lemma 3 and Lemma 6, $(A^{\mathbb{Z}} \times B^{\mathbb{Z}}, I \times G) \in \mathbf{A1} \cap \mathbf{B}i$. □

Corollary 12. $\mathbf{A2} \cap \mathbf{B1} \neq \emptyset$, $\mathbf{A2} \cap \mathbf{B2} \neq \emptyset$, $\mathbf{A2} \cap \mathbf{B3} \neq \emptyset$.

Proof. Let $(A^{\mathbb{Z}}, F) \in \mathbf{A2} \cap \mathbf{B1}$ be the Hurley cellular automaton of Example 1. Let $(B^{\mathbb{Z}}, G) \in \mathbf{B}i, 2 \leq i \leq 3$ and let $s \notin B$. By Lemma 7, $(B_s^{\mathbb{Z}}, G_s) \in \mathbf{A3} \cap \mathbf{B}i$. Then, by Lemma 3 and Lemma 6, $(A^{\mathbb{Z}} \times B_s^{\mathbb{Z}}, F \times G_s) \in \mathbf{A2} \cap \mathbf{B}i$. □

Corollary 13. $\mathbf{A3} \cap \mathbf{B1} = \emptyset$, $\mathbf{A3} \cap \mathbf{B2} \neq \emptyset$, $\mathbf{A3} \cap \mathbf{B3} \neq \emptyset$.

Proof. If $(A^{\mathbb{Z}}, F) \in \mathbf{A3}$ then it has at least two subshift attractors: the minimal attractor and the maximal attractor. Then $\mathbf{A3} \cap \mathbf{B1} = \emptyset$. Let $(A^{\mathbb{Z}}, F) \in \mathbf{B}i, 2 \leq i \leq 3$ and $s \notin A$. Then, by Lemma 7, $(A_s^{\mathbb{Z}}, F_s) \in \mathbf{A3} \cap \mathbf{B}i \neq \emptyset$. □

To conclude, since a cellular automaton in $\mathbf{A4} \cup \mathbf{A5}$ has only one attractor, we can easily derive the intersection classes for **A4** and **A5**.

Corollary 14. $\mathbf{A4} \cup \mathbf{A5} \subset \mathbf{B1}$.

	A1	A2	A3	A4	A5
B1	X	X		X	X
B2	X	X	X		
B3	X	X	X		

Fig. 3. Basin Language and Attractors classifications

5 Conclusions

We investigated the connections between dynamical and computational properties of Cellular Automata. We classified Cellular Automata according to the complexity of the languages rising from the basins of attraction of subshift attractors (see Corollary 4). According to our classification, Cellular Automata capable of universal computation are in our highest complexity class. We investigated the intersection classes between our classification and Languages, Equicontinuity and Attractors classifications (see figures 1, 2 and 3). By exploring intersection classes we can provide necessary conditions for Cellular Automata to be universal. Like in [3], according to our model, a universal cellular automaton is not regular (then it is not equicontinuous, not positively expansive and does not satisfy the shadowing property) and is not minimal (minimal Cellular Automata cannot have two distinct subshift attractors so they belong to our lowest complexity class). Several questions remain open:

1. Is there some stable cellular automaton with an infinite number of subshift attractors?
2. Is the membership in our classes decidable?
3. Is there some cellular automaton with an infinite number of subshift attractors in class **B2**?
4. Is there some cellular automaton with a finite number of subshift attractors in class **B3**?

References

1. A. Ben-Hur, H. Siegelmann, S. Fishman. A theory of complexity for continuous time systems. J. Complexity 18 (2002), no. 1, 51–86.
2. M. Cook. Universality in Elementary Cellular Automata. Complex Systems 15, 1-40, 2004.
3. J.C. Delvenne, P. Kůrka, V. Blondel. Decidability and universality in symbolic dynamical systems. Machines, computations, and universality, 104–115, Lecture Notes in Comput. Sci., 3354, Springer, Berlin, 2005.
4. P. Di Lena. Decidable properties for Regular Cellular Automata. In Proceedings of IFIP/TCS conference 22-24 August 2006, Santiago, Chile.
5. E. Formenti, P. Kůrka. Subshift attractors of cellular automata. Nonlinearity 20 (2007), 105–117.
6. R.H. Gilman. Classes of linear automata. Ergod. Th. & Dynam. Sys. 7 (1987), 105-118.
7. Hedlund, G. A. Endormorphisms and automorphisms of the shift dynamical system. Math. Systems Theory 3 1969 320–375.
8. M. Hurley. Attractors in cellular automata. Ergodic Theory Dynam. Systems 10 (1990), no. 1, 131–140.
9. P. Kůrka. Languages, equicontinuity and attractors in cellular automata. Ergodic Theory Dynam. Systems 17 (1997), no. 2, 417–433.
10. P. Kůrka. On the measure attractor of a cellular automaton. Discrete Contin. Dyn. Syst. 2005, suppl., 524–535.
11. P. Kůrka. Topological and symbolic dynamics. Cours Spcialiss [Specialized Courses], 11. Socit Mathmatique de France, Paris, 2003. xii+315 pp. ISBN 2-85629-143-0
12. P. Kůrka. Cellular automata with infinite number of subshift attractors. Preprint.
13. Smith, A. R. III. Simple Computation-Universal Cellular Spaces. J. Assoc. Comput. Mach. 18, 339-353, 1971.
14. S. Wolfram. Universality and complexity in cellular automata, Physica D, 10:1-35, (1984).