Row Subshifts and Topological Entropy of Cellular Automata

PIETRO DI LENA1*, LUCIANO MARGARA1†

Department of Computer Science, University of Bologna, Mura Anteo Zamboni 7, 40127 Bologna, Italy.

We attach to dynamics of cellular automata a sequence of shift spaces we call row subshifts sequence and we use row subshifts properties to investigate the topological entropy of cellular automata.

1 INTRODUCTION

Cellular Automata (CA) are symbolic dynamical systems often used in computer science as models of complex systems [9].

The topological entropy is often referred to as a measure of the complexity of a dynamical system [1]. The topological entropy of a CA \((A^Z, F)\) is defined in terms of the entropy of its column subshifts \(\Sigma_k\) [4], \(H(F) = \lim_{k \to \infty} \frac{H(\Sigma_k)}{k}\).

Here we address the following question: is it true that for every cellular automaton \((A^Z, F)\) there exists a number \(k > 0\) such that \(H(F) = H(\Sigma_k)\)?

This property holds for some well known classes of CA and, in particular, it is always true for one-sided CA [2].

In order to investigate this problem, we attach to the dynamics of a cellular automaton \((A^Z, F)\) a sequence of shift spaces \((\Omega_t)_{t>0}\), we call row subshifts in contrast to column subshifts. We provide a strong characterization of the shift space sequence \((\Omega_t)_{t>0}\) and we explore how it is related to the growth rate of different blocks of column subshifts.

*email: dilena@cs.unibo.it
†email: margara@cs.unibo.it
Row subshifts allows us to identify some classes of CA for which the topological entropy is concentrated on $\Sigma_{2r+1}$ (where $r$ is the radius of the CA).

The paper is organized as follows. In section 2 we introduce the notation and the minimal background in the context of symbolic dynamics and CA. In section 3 we use symbolic dynamics tools to develop row subshifts theory. In section 4 we use our theory to investigate the topological entropy of CA.

2 NOTATION AND DEFINITIONS

2.1 Symbolic Spaces

For an introduction in symbolic dynamics refer to [7].

Let $A = \{a_1, \ldots, a_n\}$ be a finite alphabet. For $k > 0$, $A^k$ is the set of finite sequences $w = w_1w_2\ldots w_k$ where $w_i \in A, i = 1, \ldots, k$. $A^Z$ is the set of all bi-infinite sequences $x = (x_i)_{i \in \mathbb{Z}}$ of elements of $A$. If $[i,j]$ is an integer interval, we denote with $x_{[i,j]} = x_ix_{i+1}\ldots x_j \in A^{j-i+1}$ the finite subword of $x \in A^Z$ on $[i,j]$.

The shift map $\sigma : A^Z \to A^Z$ is defined by $\sigma(x)_i = x_{i+1}$. A subshift (or shift space) is a closed, $\sigma$-invariant subset $X$ of $A^Z$, i.e. $X \subseteq A^Z$ and $\sigma(X) \subseteq X$. The subshift $X$ is one-sided if $X \subseteq \mathbb{N}$. Let $X,Y$ be shift spaces and let $F : X \to Y$ be a continuous, $\sigma$-commuting map. If $F$ is bijective, it is a conjugacy and $X,Y$ are conjugated. Any subshift $X \subseteq A^Z$ is conjugated to its $k$-block presentation $X^k$, which is the subshift $X$ recoded with alphabet $A^k$.

Definition 2.1 Let $X$ be a shift space and let $k > 0$. Let

$$B_k(X) = \{x \in A^k \mid \exists y \in X, \exists i \in \mathbb{Z}, y_{[i+1,i+k]} = x\}$$

denote the set of the allowed $k$-blocks of $X$.

A subshift $X$ is a shift of finite type (SFT) if $\exists K > 0$ such that $x \in X$ if and only if $\forall i \in \mathbb{Z}, x_{[i+1,i+K]} \in B_K(X)$. We refer to $K$ as the order of $X$.

The entropy of a subshift $X$ is defined as the growth rate of the number of different blocks of $X$, $H(X) = \lim_{k \to \infty} \frac{\log|B_k(X)|}{k}$.

2.2 Cellular Automata

A cellular automaton (CA) is a pair $(A^Z, F)$ where $A$ is a finite alphabet and $F : A^Z \to A^Z$ is a $\sigma$-commuting, continuous function. The cellular automaton map $F$ is defined by a local rule $f : A^{2r+1} \to A, F(x)_i =$
f(x_{i-r}...x_0...x_{i+r}), i \in \mathbb{Z}. The number r is usually denoted as the radius of \((A^Z, F)\).

A CA is right permutive (resp. left permutive) if the local rule f satisfies the following property: \(\forall a, b \in A, a \neq b, \forall x \in A^{2r}, f(xa) \neq f(xb)\) (resp. \(f(ax) \neq f(bx)\)). A CA is permutive if it is left or right permutive.

A CA is right one-sided (resp. left one-sided) if the local rule f satisfies the following property: \(\forall a \in A^{r+1}, \forall x, y \in A^r, f(xa) = f(ya)\) (resp. \(f(ax) = f(ay)\)). A CA is one-sided if it is either right or left one-sided.

To every cellular automaton one associates the shift spaces generated by the evolution of the CA on an interval of \(A^Z\).

**Definition 2.2** Let \((A^Z, F)\) be a CA. For \(k > 0\) let

\[ \Sigma_k = \{x \in (A^k)^\mathbb{N} \mid \exists y \in A^Z : F^i(y)_{[1,k]} = x_i, i \in \mathbb{N}\} \]

denote the column subshift of width \(k\) associated to \((A^Z, F)\).

Column subshifts play a key role in the study of the dynamics of cellular automata [5], [6], [3]. The topological entropy of \((A^Z, F)\), denoted with \(H(F)\), can be defined in terms of column subshift entropies.

**Proposition 2.3** [4] Let \((A^Z, F)\) be a CA. Then \(H(F) = \lim_{k \to \infty} \frac{H(\Sigma_k)}{k}\).

### 3 THE ROW SUBSHIFT SEQUENCE

We attach to the dynamics of a cellular automaton \((A^Z, F)\) a sequence of shift spaces, \((\Omega_t)_{t>0}\), we call row subshifts in contrast to column subshift. In this section we characterize the sequence \((\Omega_t)_{t>0}\) while in section 4 we use \((\Omega_t)_{t>0}\) to investigate the topological entropy of cellular automata.

**Definition 3.1** Let \((A^Z, F)\) be a CA. For \(t > 0\), let

\[ \Omega_t = \{(x_1, ..., x_t) \in A^Z \times ... \times A^Z \mid F^i(x_1) = x_{i+1}, 0 \leq i < t\} \]

denote the row subshift generated by \(F\) on \(A^Z\) at evolution time \(t\).

**Example 3.2** Let consider Wolfram’s elementary CA (ECA) rule 0 [8], whose local rule \(f: \{0, 1\}^3 \to \{0, 1\}\) is defined by \(f(a, b, c) = 0, \forall a, b, c \in \{0, 1\}\). Let \(q \in A^Z\) denote the biinfinite sequence of all zeros, i.e. \(\forall i \in \mathbb{Z}, q_i = 0\). Then,

\[ \Omega_1 = A^Z, \Omega_2 = A^Z \times \{q\}, \Omega_3 = A^Z \times \{q\} \times \{q\}, ... \]
Remark 3.3 Note that $\forall k, t > 0, B_{k}(\Omega_{t}) = B_{t}(\Sigma_{k})$. Then, according to the context, we denote $b \in B_{k}(\Omega_{t}) = B_{t}(\Sigma_{k})$ either as $b = b_{1}'...b_{t}'$ where $b_{1}', ..., b_{t}' \in A^{k}$ or as $b = b_{1}''...b_{k}''$ where $b_{1}'', ..., b_{k}'' \in A^{t}$.

The following proposition provides a strong characterization of the row subshift sequence $(\Omega_{t})_{t>0}$.

Proposition 3.4 Let $(A^{Z}, F)$ be a CA with radius $r$. Then, $\forall t > 0$

1. $\Omega_{t}$ is conjugated to $A^{Z}$
2. $\Omega_{t}$ is a SFT of order $2r + 1$.

Proof.

1. Let $\varphi_{t} : A^{Z} \rightarrow \Omega_{t}$ be defined in the following way:

   $\forall x \in A^{Z}, \varphi_{t}(x) = (F^{0}(x), F^{1}(x), ..., F^{t-1}(x))$.

   Trivially, the inverse, $\varphi_{t}^{-1}$, is defined by:

   $\forall y = (y_{1}, ..., y_{t}) \in \Omega_{t}, \varphi_{t}^{-1}(y) = y_{1}$.

   Since $F$ is $\sigma$-commuting and continuous it follows that, $\varphi_{t}$ is bijective, shift commuting and continuous then it is a conjugacy.

2. By point 1, for any $t > 0$, $\Omega_{t}$ is conjugated to $A^{Z}$ then it is an SFT. We have just to show that it has order $2r + 1$.

   To prove that $\Omega_{t}$ has order $2r + 1$ it is sufficient to show that for any $x, y \in B_{2r+1}(\Omega_{t})$ such that $x = x_{1}x_{2}...x_{2r+1}, y = y_{1}y_{2}...y_{2r+1}$ and $x_{i} = y_{i-1}, 1 < i \leq 2r + 1$, it follows that $z = x_{1}x_{2}...x_{2r+1}y_{2r+1} \in B_{2r+2}(\Omega_{t})$.

   Since $x, y \in B_{2r+1}(\Omega_{t})$ and $\Omega_{t}$ is shift invariant, there exist $x', y' \in \Omega_{t}$ such that $x'_{[-r,r]} = x$ and $y'_{[-r+1,r+1]} = y$. Let $x'' = \varphi_{t}^{-1}(x')$ and $y'' = \varphi_{t}^{-1}(y')$ and let $z'' \in A^{Z}$ be the configuration defined by $z''_{(-\infty,r]} = x''_{(-\infty,r]}$ and $z''_{[r+1,\infty)} = y''_{[r+1,\infty)}$. Finally, let $z' = \varphi_{t}(z'')$. It is easy to verify that $z'_{[-r,r+1]} = z$, which implies that $z \in B_{2r+2}(\Omega_{t})$. $\square$

Definition 3.5 Let $(A^{Z}, F)$ be a CA with radius $r$. For any $t > 0$ let denote with $M_{t}$ the adjacency matrix related to $\Omega_{t}^{2r+1}$.
By Proposition 3.4, for every \( t > 0 \) the \((2r+1)\)-block presentation \( \Omega_t^{2r+1} \) of the row subshift \( \Omega_t \) is a SFT of order 2 then it can be represented by an adjacency matrix \( M_t \). If \( B_{2r+1}(\Omega_t) = \{ b_1, ..., b_{n_t} \} \) then \( M_t \) is the \( n_t \times n_t \) matrix such that the \((i, j)\)-th entry of \( M_t \) is 1 if and only if \( b_i b_j \in B_2(\Omega_t^{2r+1}) \).

Let consider the \( k \)-th power of \( M_t \), \( M_k^t \), \( k > 0 \). The \((i, j)\)-th entry of \( M_k^t \) is \( l > 0 \) if and only if exist exactly \( l \) blocks in \( B_{k+1}(\Omega_t^{2r+1}) \) starting with block \( b_i \) and ending with block \( b_j \).

We define a quantity we call multiplicity of a block \( b \in B_{2r+1}(\Omega_t) \) and we provide a characterization of \( M_k^t \), for sufficiently large \( k \), in terms of the multiplicities of blocks in \( B_{2r+1}(\Omega_t) \).

**Definition 3.6** Let \( (A^Z, F) \) be a CA with radius \( r \). Fort \( k, t > 0 \), let

\[
\theta_{k,t} : A^{2r(t-1)+k} \rightarrow B_k(\Omega_t)
\]

be the onto mapping defined in the following way: \( \forall a \in A^{2r(t-1)+k}, \theta_{k,t}(a) = b_1...b_t \in B_k(\Omega_t) \) if and only if \( \exists x \in A^Z \) such that \( x_{[-r(t-1),r(t-1)+2]} = a \) and \( F^t(x)_{[0,k)} = b_{i+1}, 0 \leq i < t \).

**Remark 3.7** Note that, the block \( b \in B_k(\Omega_t) \) is completely determined by the set of blocks \( \theta_{k,t}^{-1}(b) \). This means that if \( b_1, b_2 \in B_k(\Omega_t) \) and \( b_1 \neq b_2 \) then \( \theta_{k,t}^{-1}(b_1) \cap \theta_{k,t}^{-1}(b_2) = \emptyset \).

**Definition 3.8** Let \( (A^Z, F) \) be a CA with radius \( r \). Let \( t > 0, b \in B_{2r+1}(\Omega_t) \).

- Let \( \omega^+(b) = |\{ x \in A^r \mid \exists y \in \theta_{2r+1,t}^{-1}(b), y_{[r(t-1)+2,2r+t+1]} = x \}| \) denote the right multiplicity of block \( b \).
- Let \( \omega^-(b) = |\{ x \in A^r \mid \exists y \in \theta_{2r+1,t}^{-1}(b), y_{[1,r(t-1)]} = x \}| \) denote the left multiplicity of block \( b \).
- Let \( \omega(b) = \omega^-(b) \cdot \omega^+(b) = |\theta_{2r+1,t}^{-1}(b)| \) denote the multiplicity of block \( b \).

**Example 3.9** Let consider the CA of example 3.2. Then \( \forall t > 0, \forall b \in B_{2r+1}(\Omega_t) \),

\[
\omega^-(b) = \omega^+(b) = 2^{t-1} \text{ and } \omega(b) = 2^{2(t-1)}.
\]

The relation between powers of \( M_t \) and the multiplicities of blocks in \( B_{2r+1}(\Omega_t) \) is expressed in the following proposition.

**Proposition 3.10** Let \( (A^Z, F) \) be a CA with radius \( r \), \( |A| = n \). Let \( t > 0 \), let \( B_{2r+1}(\Omega_t) = \{ b_1, ..., b_{n_t} \} \) and let define the vectors

\[
5
\]
\[ \mathbf{1}_t^T = [\omega^-(b_1) \ldots \omega^-(b_{n_t})], \mathbf{r}_t^T = [\omega^+(b_1) \ldots \omega^+(b_{n_t})]. \]

Then \( \forall k > 0, \)
\[ M_t^k \cdot M_t^{2rt+1} = n^k \cdot M_t^{2rt+1} = n^k \cdot (\mathbf{r}_t \cdot \mathbf{1}_t^T). \]

**Proof.** For any \( 1 \leq i, j \leq n_t, \) the \((i,j)\)-th entry of matrix \( M_t^k \cdot M_t^{2rt+1} = M_t^{2rt+1+k}, \) say \( m_{i,j}^k, \) represents the number of blocks in \( \Omega_t^{2rt+1} \) of length exactly \( 2rt + 2 + k \) starting with block \( b_i \) and ending with block \( b_j. \) We show that the number \( m_{i,j}^k \) equals \( \omega^+(b_i) \cdot n^k \cdot \omega^-(b_j). \)

For any \( b_i, b_j \in \mathcal{B}_{2rt+1}(\Omega_t), k > 0 \) let define the set of blocks (see fig. 1)
\[ X_{i,j}^k = \{ xay \in A^{4rt+2+k} | x \in \varrho_{2r+1,t}(b_i), y \in \varrho_{2r+1,t}(b_j), a \in A^k \}. \]

**FIGURE 1**
Blocks in \( \mathcal{B}_{2rt+2+k}(\Omega_t^{2r+1}) \) starting with \( b_i \) and ending with \( b_j. \)

Obviously, \( \varrho_{2(rt+r+1)+k,t}(X_{i,j}^k) \) is in one-to-one correspondence with the set of blocks in \( \Omega_t^{2rt+1} \) of length \( 2rt + 2 + k \) starting with block \( b_i \) and ending with block \( b_j. \) Then \( m_{i,j}^k = |\varrho_{2(rt+r+1)+k,t}(X_{i,j}^k)| \) and, by definition of \( X_{i,j}^k, \) it is easy to see that
\[ |\varrho_{2(rt+r+1)+k,t}(X_{i,j}^k)| = \omega^+(b_i) \cdot n^k \cdot \omega^-(b_j). \]

This implies that \( M_t^k \cdot M_t^{2rt+1} = n^k \cdot (\mathbf{r}_t \cdot \mathbf{1}_t^T) = n^k \cdot M_t^{2rt+1}. \) \( \square \)

**4 TOPOLOGICAL ENTROPY OF CELLULAR AUTOMATA**

In this section we address the following question: is it true that for any cellular automaton \((A^\mathbb{Z}, F)\) there exists an integer \( k > 0 \) such that \( H(F) = H(\Sigma_k)? \)
Rephrasing the question in another way: is it true that the topological entropy of every CA is concentrated on a finite number of columns?

There exist numerous well known examples of cellular automata for which this is true. In the general case, however, the question is open. The theory developed in the previous section provides a way to identify some classes of CA for which the topological entropy is concentrated on a finite number of columns.

First of all note that, since $\forall k, t > 0$, $B_t(\Sigma_{2r+1+k}) = |M^k_t|$ we can define the entropy of $\Sigma_{2r+1+k}$ as

$$H(\Sigma_{2r+1+k}) = \lim_{t \to \infty} \frac{\log |M^k_t|}{t}.$$ 

Our theory permits to exploit a relation between the growth rate of blocks in $\Sigma_{2r+1+k}$ and the growth rate of the multiplicities of blocks in $\Sigma_{2r+1}$. In order to see this we need the following lemma.

**Lemma 4.1** Let $(A^2, F)$ be a CA with radius $r$. Let $k, t > 0$. and let $B_{2r+1}(\Omega_t) = \{b_1, ..., b_{nt}\}$. Let denote with $c^k_i$ and $r^k_i, 1 \leq i \leq nt$, the sum of the entries of the $i$-th column and of the $i$-th row of $M^k_t$, respectively.

Then,

1. $|M^k_t \cdot M^{2r+1}_t| = \sum_{i=1}^{nt} c^k_i \cdot \sum_{j=1}^{nt} \omega^+(b_j)$.

2. $|M^k_t \cdot M^{2r+1}_t| = \sum_{i=1}^{nt} r^k_i \cdot \sum_{j=1}^{nt} \omega^-(b_j)$.

**Proof.** We provide a proof only of point 1 since the proof of point 2 is similar.

By Proposition 3.10,

$$M^{2r+1}_t = \begin{pmatrix} \omega^+(b_1)\omega^-(b_1) & \omega^+(b_1)\omega^-(b_{nt}) \\ \omega^+(b_{nt})\omega^-(b_1) & \omega^+(b_{nt})\omega^-(b_{nt}) \end{pmatrix}.$$ 

Note that the summation of the entries of the $i$-th row of $M^{2r+1}_t$ is equal to $\omega^+(b_i) \cdot \sum_{j=1}^{nt} \omega^-(b_j)$. Now, let consider the matrix product $M^k_t \cdot M^{2r+1}_t$

$$M^k_t \cdot M^{2r+1}_t = \begin{pmatrix} x_{i1} & \cdots & x_{in} \\ \cdots & \cdots & \cdots \\ x_{in} & \cdots & x_{nn} \end{pmatrix} \cdot \begin{pmatrix} \omega^+(b_1)\omega^-(b_1) & \omega^+(b_1)\omega^-(b_{nt}) \\ \omega^+(b_{nt})\omega^-(b_1) & \omega^+(b_{nt})\omega^-(b_{nt}) \end{pmatrix}.$$ 

Note that, in the matrix product $M^k_t \cdot M^{2r+1}_t$, the $(i, j)$-th entry of $M^k_t$, $x_{ij}$, is multiplied only with the entries of the $i$-th row of $M^{2r+1}_t$ which is equal to
\( \omega^+(b_i) \cdot \sum_{j=1}^{n_i} \omega^-(b_j) \). Recall that, by definition, \( c_i = \sum_{j=1}^{n_i} x_{ij}, 1 \leq i \leq n_t \). Then it is easy to check that \( |M_t^k \cdot M_t^{2r+1}| = \sum_{i=1}^{n_t} c_i^k \cdot \omega^+(b_i) \cdot \sum_{j=1}^{n_t} \omega^-(b_j) \).

Using Lemma 4.1 we are in the position to show that if the growth rate of the multiplicities is (in some sense) exponentially uniform, the topological entropy of \((A^Z, F)\) is the entropy of \(\Sigma_{2r+1} \).

**Definition 4.2** Let \((A^Z, F)\) be a CA and let \(B_{2r+1}(\Omega_t) = \{b_1, \ldots, b_{n_t}, t > 0\) let define

\[
max^+(t) = Max\{\omega^+(b_1), \ldots, \omega^+(b_{n_t})\}, \quad min^+(t) = min\{\omega^+(b_1), \ldots, \omega^+(b_{n_t})\}
\]

and

\[
max^-(t) = Max\{\omega^-(b_1), \ldots, \omega^-(b_{n_t})\}, \quad min^-(t) = min\{\omega^-(b_1), \ldots, \omega^-(b_{n_t})\}.
\]

**Definition 4.3** Let \((A^Z, F)\) be a CA. Let define

\[
R^+ = \lim_{t \to \infty} \frac{\log max^+(t)}{t} \quad \text{and} \quad R^- = \lim_{t \to \infty} \frac{\log min^-(t)}{t}.
\]

**Proposition 4.4** Let \((A^Z, F)\) be a CA. If either \(R^+ = 0\) or \(R^- = 0\), then

\[
H(F) = H(\Sigma_{2r+1}).
\]

**Proof.** Suppose that \(R^+ = \lim_{t \to \infty} \frac{\log max^+(t)}{t} = 0\). The proof for the other case is similar.

It is sufficient to show that \(H(\Sigma_{2r+1}) \geq H(\Sigma_{2r+1+k}), \forall k > 0\) (the converse is trivially true). Recall that \(H(\Sigma_{2r+1+k}) = \lim_{t \to \infty} \log \frac{|M_t^k|}{t}\).

Let \(B_{2r+1}(\Omega_t) = \{b_1, \ldots, b_{n_t}\}\) and let denote with \(c_i^k\) the sum of the entries of \(i\)-th column of \(M_t^k, 1 \leq i \leq n_t\). By Proposition 3.10,

\[
|M_t^k \cdot M_t^{2r+1}| = |A|^k \cdot |M_t^{2r+1}|
\]

and by Lemma 4.1, equation 1 can be rewritten as

\[
\sum_{i=1}^{n_t} c_i^k \cdot \omega^+(b_i) \cdot \sum_{j=1}^{n_t} \omega^-(b_j) = |A|^k \cdot \sum_{i=1}^{n_t} \omega^+(b_i) \cdot \sum_{j=1}^{n_t} \omega^-(b_j).
\]

Substituting every \(\omega^+(b_i)\) with \(min^+(t)\) and \(max^+(t)\) respectively in the left and right part of equation (2) we can derive the following inequality:

\[
|M_t^k| \leq \sum_{i=1}^{n_t} c_i^k \leq |A|^k \cdot n_t \cdot \frac{\max^+(t)}{\min^+(t)}.
\]
Then, taking the logarithm and the limit for \( t \to \infty \) we can conclude:

\[
H(\Sigma_{2r+1+k}) \leq \lim_{t \to \infty} \frac{\log |A|^k}{t} + \lim_{t \to \infty} \frac{\log n_t}{t} + \lim_{t \to \infty} \frac{\log \max_{0 \leq i < t} n(t)}{t} = H(\Sigma_{2r+1}).
\]

The following example shows that the condition imposed by Proposition 4.4 is not necessary to have the topological entropy concentrated on a finite number of columns.

**Example 4.5** Let \((A^Z, F)\) be a CA where \( A = \{0, 1, 2\} \) and where the local rule \( f : A^3 \to A \) is defined by:

1. \( f(a, b, c) = 2 \), if \( a = 2 \) or \( b = 2 \) or \( c = 2 \)
2. \( f(a, b, c) = (a + c) \mod 2 \), otherwise

Let denote with \( b_{3,t}(0) \), \( b_{3,t}(2) \in B_3(\Omega_t) \) the blocks constituted of all 0s and all 2s, respectively. It is easy to see that \( \forall t > 0, \omega^+ (b_{3,t}(0)) = \omega^- (b_{3,t}(0)) = 1 \) and \( \omega^+ (b_{3,t}(2)) = \omega^- (b_{3,t}(2)) = 3^t - 1 \). Thus, Proposition 4.4 doesn’t apply to this CA. However we show that the topological entropy of \((A^Z, F)\) is concentrated on a finite number of columns.

Let \( X = \{0, 1\}^Z \) and \( Y = A^Z \setminus X \). Let consider the dynamical systems \((X, F_X)\) and \((Y, F_Y)\) obtained by restricting \( F \) on \( X \) and \( Y \), respectively. Let denote with \( \Sigma_k(X) \) (resp. \( \Sigma_k(Y) \)) the column subshift of width \( k > 0 \) of \((X, F_X)\) (resp. \( (Y, F_Y) \)). Then, \((X, F_X)\) is the ECA rule 90. It is a positively expansive CA and \( H(F_X) = H(\Sigma_3(X)) = \log 2 \). Let \( b_k(2) \in \Sigma_k(Y) \) be the column of width \( k \) constituted of all 2s. It is easy to see that,

\[
\Sigma_k(Y) = \{x \in (A^k)^N \mid \exists t \in \mathbb{N}, x_{[0,t]} \in B_{t+1}(\Sigma_k(X)), x_{[t+1,\infty]} = b_k(2)\} \cup \{b_k(2)\}.
\]

Thus \( H(\Sigma_k(Y)) = H(\Sigma_k(X)), \forall k > 0 \). Then \( H(F) = H(\Sigma_3) = \log 2 \).

To conclude, as an example of application, we use Proposition 4.4 to identify some classes of CA for which the topological entropy is concentrated on \( \Sigma_{2r+1} \).

**Corollary 4.6** Let \((A^Z, F)\) be a CA with radius \( r \).

1. If \((A^Z, F)\) is one-sided then \( H(F) = H(\Sigma_{2r+1}) \).
2. If \((A^Z, F)\) is permutive then \( H(F) = H(\Sigma_{2r+1}) \).
3. If \((A^Z, F)\) is positively expansive then \(H(F) = H(\Sigma_{2r+1})\).

**Proof.**

1. Suppose that \((A^Z, F)\) is right one-sided, the proof for the left case is similar. It is easy to see that \(\forall t > 0, \forall b \in B_{2r+1}(\Omega_t), \omega^+(b) = |A|^r (t-1)\). Then \(\forall t > 0, min^+(t) = max^+(t) = |A|^r (t-1)\) and the proof follows from Proposition 4.4.

2. Suppose that \((A^Z, F)\) is right permutive, the proof for the left case is similar. It is easy to see that \(\forall t > 0, \forall b \in B_{2r+1}(\Omega_t), \omega^+(b) = 1\). Then \(\forall t > 0, min^+(t) = max^+(t) = 1\) and the proof follows from Proposition 4.4.

3. Since \((A^Z, F)\) is positively expansive, it is conjugated to \(\Sigma_{2r+1}\) [4]. This implies that \(\forall y \in \Sigma_{2r+1}, \exists x \in A^Z\) such that \(F^i(x)_{[0,2r]} = y_i, i \in \mathbb{N}\). Then

\[
\lim_{t \to \infty} max^+(t) = \lim_{t \to \infty} min^+(t) = \lim_{t \to \infty} max^-(t) = \lim_{t \to \infty} min^-(t) = 1
\]

and the proof follows from Proposition 4.4.

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