Bisimulation and coinduction in higher-order languages

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Bisimulation

- **Behavioural equality**
- **One of the most important contributions of Concurrency Theory to CS (and beyond)**

[Milner, Park, 1980]

Bisimulation: a relation $\mathcal{R}$ on states of an LTS s.t. whenever $M \ \mathcal{R} \ N$:

1. $P \xrightarrow{a} P'$ implies $Q \xrightarrow{a} Q'$ and $P' \ \mathcal{R} \ Q'$
2. the converse.

Bisimilarity ($\sim$): the union of all bisimulations

[in the remainder: converse clauses omitted]

Important

1. The definition gives us a **powerful proof method**:

$$\frac{P\ \mathcal{R}\ Q \qquad \mathcal{R} \text{ is a bisimulation}}{P \sim Q}$$

2. **Coinduction and induction**
   - **Bisimulation**: a coinductive notion
   - **Congruence**: the inductive dual of bisimulation (equivalence)
     [**compatibility** with the constructs of the language]
   - In a language: **we need them both**
     inductive syntax
     coinductive semantics

Higher-order languages

- **Functions and/or processes move and/or used as data**

  Example of higher-order feature:

  $P(x)$ where x can be a program

- **Functional languages, mobile code**

  - What is bisimulation?
  - Compatibility can be hard

The λ-calculus and contextual equivalence

The λ-calculus

The paradigmatical higher-order language

$$M, N ::= x \mid \lambda x.\, M \mid MN$$

Values $\triangleq$ the terms of the form $\lambda x.\, M$

$\Lambda^\bullet \triangleq$ the closed terms

Reduction (call-by-name)

$$\frac{}{(\lambda x.\, M)N \longrightarrow M\{{}^{N}\!/_{x}\}} \qquad \frac{M \longrightarrow M'}{MN \longrightarrow M'N}$$

$\Longrightarrow \triangleq$ the reflexive and transitive closure of $\longrightarrow$.

$M \Downarrow \triangleq M$ terminates

Behavioural equality in sequential languages

for all context C, and for all values V,

$C[\,P\,] \Longrightarrow V$ iff $C[\,Q\,] \Longrightarrow V$

Too strong in higher-order languages:

$$I = \lambda x.\, x \not\simeq \lambda x.\, (I\,I)$$

because $I \Longrightarrow I$ and $\lambda x.\, (I\,I) \Longrightarrow \lambda x.\, (I\,I)$

The observables should be as weak as possible

Contextual equivalence

[Morris,'68]

$M \simeq^{\mathrm{C}} N$ **contextually equivalent** if,

for any context C such that $C[\,M\,]$ and $C[\,N\,]$ are closed,

$C[M] \Downarrow$ iff $C[N] \Downarrow$

No need to check the identity of first-order values returned

Example: if $C[\,P\,] \Longrightarrow 5$ and $C[\,Q\,] \Longrightarrow 7$, wrap C into

`if` $C = 5$ `then` true `else` <diverge>

Problem : definition very hard to use

(utterly useless in higher-order languages)

Proof techniques for contextual equivalence in higher-order languages

Till the 1990s: **denotational techniques**

- hard mathematics
- full abstraction
- scalability in non-purely functional extensions
  (eg, state; worst: concurrency)

After the 1990s: **coinduction (bisimulation)** [Abramsky]

A major factor in the movement towards operationally-based techniques in PL semantics after the 1990s

Still a hot research topic

Applicative bisimulation

Bisimulation in the λ-calculus

[Abramsky, 1990]

Applicative bisimulation: a relation $\mathcal{R} \subseteq \Lambda^\bullet \times \Lambda^\bullet$ s.t. whenever $M \ \mathcal{R} \ N$:

1. $M \Longrightarrow \lambda x.\, M'$ implies $N \Longrightarrow \lambda x.\, N'$ and $M'\{^L/_x\} \ \mathcal{R} \ N'\{^L/_x\}$ for all L;

Applicative bisimilarity ($\sim^{\mathrm{A}}$) : the union of all bisimulations

Questions:

1. $\sim^{\mathrm{A}}$ **vs** $\simeq^{\mathrm{C}}$ **? (contextual equivalence)**
2. does the definition scale to extensions of the λ-calculus?

Bisimilarity vs contextual equivalences

– $\sim^{\mathrm{A}} \subseteq \simeq^{\mathrm{C}}$ easy? (cf: bisimilarity implies may testing)

– surprise: what is easy is the converse $\simeq^{\mathrm{C}} \subseteq \sim^{\mathrm{A}}$

$$(\lambda x.\, M)N \simeq^{\mathrm{C}} M\{^{N}/_{x}\} \qquad (\star)$$

$M \simeq^{\mathrm{C}} N$ and $M \Longrightarrow \lambda x.\, M'$ imply $N \Longrightarrow \lambda x.\, N'$

We need: $M'\{^{L}/_{x}\} \simeq^{\mathrm{C}} N'\{^{L}/_{x}\}$, for all L

$$\begin{array}{rcl} M'\{^{L}/_{x}\} & \simeq^{\mathrm{C}} & (\star) \\ ML & \simeq^{\mathrm{C}} & \textbf{(substitutivity)} \\ NL & \simeq^{\mathrm{C}} & (\star) \\ N'\{^{L}/_{x}\} & & \end{array}$$

Conclude from **transitivity** of $\simeq^{\mathrm{C}}$

Congruence?

$\sim^{\mathrm{A}} \subseteq \simeq^{\mathrm{C}}$ would follow from the compatibility of $\sim^{\mathrm{A}}$:

for all M, N, and context C,

if $M \sim^{\mathrm{A}} N$ then $C[\, M \,] \sim^{\mathrm{A}} C[\, N \,]$

A proof attempt :

$$\mathcal{R} \triangleq \{(C[\, \widetilde{M} \,], C[\, \widetilde{N} \,]) \;:\; \widetilde{M} \sim^{\mathrm{A}} \widetilde{N}\}$$

Induction on the structure of C.

Main problematic case: $C = C_1 C_2$

The two congruence problems

Suppose $(\lambda x.\, M_1)M_2 \;\mathcal{R}\; N_1N_2$ with $\begin{cases} \lambda x.\, M_1 \;\mathcal{R}\; N_1 \\ M_2 \;\mathcal{R}\; N_2 \end{cases}$

From the inductive assumption:

$$\begin{array}{ccccc} & & (\lambda x.\, M_1)M_2 & \longrightarrow & M_1\{^{M_2}\!/_x\} \\ & & & & \mathcal{R} \\ N_1N_2 & \Longrightarrow & (\lambda x.\, N_1')N_2 & \longrightarrow & N_1'\{^{N_2}\!/_x\} \end{array}$$

(... only if $M_2 = N_2$!)

But we need more:

(1) if $M_1\{^{M_2}\!/_x\} \Longrightarrow \lambda x.\, M$

then $N_1\{^{L}\!/_x\} \Longrightarrow \lambda x.\, N$ and **for all L ...**

(2) $M_2 \;\mathcal{R}\; N_2$

Techniques for congruence

- Abramsky: via denotational semantics
- Howe's technique: define a relation that is, by definition, a congruence, and then prove that it is the same as $\sim^{\mathrm{A}}$.
  - Difficult to apply
  - Limitations in extensions of the λ-calculus (concurrency)

Bisimulation in the λ-calculus

[Abramsky, 1990]

Applicative bisimulation: a relation $\mathcal{R} \subseteq \Lambda^\bullet \times \Lambda^\bullet$ s.t. whenever $M \ \mathcal{R} \ N$:

1. $M \Longrightarrow \lambda x.\, M'$ implies $N \Longrightarrow \lambda x.\, N'$ and $M'\{^L/_x\} \ \mathcal{R} \ N'\{^L/_x\}$ for all L;

Applicative bisimilarity ($\sim^{\mathrm{A}}$) : the union of all bisimulations

Questions:

1. $\sim^{\mathrm{A}}$ vs $\simeq^{\mathrm{C}}$? (contextual equivalence)
2. **does the definition scale to extensions of the λ-calculus?**

Unsoundness of applicative bisimilarity under language extensions

[example: call-by-value with generation of names]

$$M \triangleq \nu n \,\mathtt{return}\, \lambda f.\, f n$$

$$N \triangleq \mathtt{return}\, \lambda f.\, \nu n\; f n$$

$M \sim^{\mathrm{A}} N$ (the argument supplied for f does not know n)

$M \not\sim^{\mathrm{C}} N$,

as $C[\, M \,] \Downarrow \mathtt{true}$ but $C[\, N \,] \Downarrow \mathtt{false}$

for $C \triangleq \mathtt{let}\ [\cdot] = g\ \mathtt{in}\ g(\lambda n.\, g(\lambda m.\, m = n))$

[Koutavas, Levy, Sumii, 2011]

Logical bisimulation, revisited

- **simple congruence proof**
- **separate enhancements of the bisimulation**
- **Basis: logical bisimulation**

  [Kobayashi, Sangiorgi, Sumii, 2008 and 2010]

  cf: logical relations

First congruence problem

From the inductive assumption:

$$(\lambda x.\, M_1)M_2 \longrightarrow M_1\{^{M_2}/_x\}$$

$$\mathcal{R} \qquad \text{(if } M_2 = N_2 \text{ !)}$$

$$N_1N_2 \Longrightarrow (\lambda x.\, N_1')N_2 \longrightarrow N_1'\{^{N_2}/_x\}$$

But we need more:

(1) if $M_1\{^{M_2}/_x\} \Longrightarrow \lambda x.\, M$

then $N_1\{^{L}/_x\} \Longrightarrow \lambda x.\, N$ and **for all L ...**

$\Rightarrow$ **introduce a clause for internal moves**

(cf: concurrency)

First change

... whenever $M \mathcal{R} N$:

1. $M \longrightarrow M'$ **implies** $N \Longrightarrow N'$ **and** $M' \mathcal{R} N'$**;**
2. $M = \lambda x.\, M'$ implies $N \Longrightarrow \lambda x.\, N'$ and $M'\{^{L}/_{x}\} \mathcal{R} N'\{^{L}/_{x}\}$ for all L

Problem: the new definition heavier to use in proofs

The second congruence problem

From the inductive assumption:

$$(\lambda x.\, M_1)M_2 \longrightarrow M_1\{^{M_2}\!/_x\}$$

$$N_1 N_2 \Longrightarrow (\lambda x.\, N_1')N_2 \longrightarrow N_1'\{^{N_2}\!/_x\}$$

$\mathcal{R}$ (if $M_2 = N_2$!)

But we need more:

(2) $M_2 \;\mathcal{R}\; N_2$

first-order substitutivity:

$\forall Q.\ P(x) \simeq P'(x)$ implies $P(Q) \simeq P'(Q)$

higher-order substitutivity:

$\forall Q, Q'.\ P(x) \simeq P'(x)$ and $Q \simeq Q'$ imply $P(Q) \simeq P'(Q')$

Second change

.... whenever $M \mathcal{R} N$:

1. .....
2. $M = \lambda x.\, M'$ implies $N \Longrightarrow \lambda x.\, N'$

   and $M'\{^P/_x\} \mathcal{R} N'\{^Q/_x\}$ for all $P \mathcal{R} Q$

Now: problematic case ok

Problem: definition unsound

$\lambda x.\, x \stackrel{\triangle}{=} I \sim^{\mathrm{A}} K \stackrel{\triangle}{=} \lambda x.\, I$, for $\mathcal{R} = \{(I, I), (I, K)\}$

- **$\mathcal{R}$ and the identity relation are bisimulations, but not their union**
- **a non-monotone functional**

Second change

.... whenever $M \mathcal{R} N$:

1. .....
2. $M = \lambda x.\, M'$ implies $N \Longrightarrow \lambda x.\, N'$

   and $M'\{^P/_x\} \mathcal{R} N'\{^Q/_x\}$ for all $P \mathcal{R} Q$

Problem: definition unsound

– **sound if $\mathcal{R}$ is a congruence (or substitutive)**

– directly from the definition: $\sim^{\mathrm{A}} \subseteq \simeq^{\mathrm{C}}$

Third change

A **congruence (or substitutive)** $\mathcal{R}$ s.t. whenever $M \ \mathcal{R} \ N$:

1. $M \longrightarrow M'$ implies $N \Longrightarrow N'$ and $M' \ \mathcal{R} \ N'$;
2. $M = \lambda x.\, M'$ implies $N \Longrightarrow \lambda x.\, N'$

   and $M'\{^{P}/_{x}\} \ \mathcal{R} \ N'\{^{Q}/_{x}\}$ for all $P \ \mathcal{R} \ Q$

Logical bisimilarity: $\sim^{\mathrm{L}}$

Theorem $\sim^{\mathrm{L}} = \simeq^{\mathrm{C}}$

- **Problem: not a good proof technique from the definition**
- **No need to kill two birds with one stone!**
- **Enhancements of the proof method, separately**

  cf: up-to techniques **[cf: Pous/Bonchi talk]**

  bisimilarity results using relations smaller a bisimulation

Example enhancement: up-to context

[$\mathcal{R}^\star \triangleq$ the context closure of $\mathcal{R}$]

A relation $\mathcal{R}$ is a **bisimulation up-to contexts** if whenever $M \ \mathcal{R} \ N$

1. $M \longrightarrow M'$ implies $N \Longrightarrow N'$ and $M' \ \ \mathcal{R}^\star \ \ N'$;
2. $M = \lambda x.\, M'$ implies $N \Longrightarrow \lambda x.\, N'$

   and $M'\{^P/_x\} \ \mathcal{R}^\star \ N'\{^Q/_x\}$ for all $P \ \mathcal{R}^\star \ Q$

Theorem If $\mathcal{R}$ is a bisimulation up-to contexts then $\mathcal{R}^\star$ is a bisimulation.

Proof: essentially the earlier proof of congruence

Big-step up-to contexts and reductions

A relation $\mathcal{R}$ is a **big-step bisimulation up-to contexts and reductions** if whenever $M \; \mathcal{R} \; N$

1. $M \Longrightarrow \lambda x.\, M'$ implies $N \Longrightarrow \lambda x.\, N'$

   and $M'\{^{P}/_{x}\} \Longrightarrow \mathcal{R}^{\star} \Longleftarrow N'\{^{Q}/_{x}\}$ for all $P \; \mathcal{R}^{\star} \; Q$

Theorem If $\mathcal{R}$ is a big-step bisimulation up-to contexts and reductions then $\Longrightarrow \mathcal{R}^{\star} \Longleftarrow$ is a bisimulation.

An example proof with enhancements

$I_1 \simeq^{\mathrm{C}} I_2$ for $I_1 \triangleq \lambda x.\, x$

$I_2 \triangleq \lambda x.\, (\lambda y.\, y)x$

A plain bisimulation $\mathcal{R}$:

– a congruence

– closed under the rules $\dfrac{}{I_1 \;\mathcal{R}\; I_2}$ $\qquad$ $\dfrac{M \;\mathcal{R}\; N}{M \;\mathcal{R}\; (\lambda y.\, y)N}$

$\mathcal{S} \triangleq \{(I_1, I_2)\}$ is a big-step bisimulation up-to contexts and reductions, as for $M \;\; \mathcal{S}^\star \;\; N$:

$$\begin{array}{ccccc} & & \lambda x.\, x & \xrightarrow{M} & M \\ & & & & \mathcal{S}^\star \\ \lambda x.\, (\lambda y.\, y)x & \xrightarrow{N} & (\lambda y.\, y)N & \Longrightarrow & N \end{array}$$

Fixed-points

- **The functional of the final definition non-monotone**

  (even on congruence relations)

  but it has a greatest fixed point ($\simeq^{\mathrm{C}}$)

  non cocontinuous, but it has the stratification approximation

- **A theory of coinduction for non-monotone functionals?**

- **Another possibility: environmental bisimulations**

  [Kobayashi, Sangiorgi, Sumii, 2010]

  monotone functional, robust

  more complex definition

Extensions and variations

The example with language extension

[call-by-value with generation of names]

$$M \triangleq \nu n\, \mathtt{return}\ V \quad \text{for} \quad V \triangleq \lambda f.\, f n$$

$$N \triangleq \mathtt{return}\ W \quad \text{for} \quad W \triangleq \lambda f.\, \nu n\, f n$$

Distinguished in

$$C \triangleq \mathtt{let}\ [\cdot] = g\ \mathtt{in}\ g(\lambda n.\, g(\lambda m.\, m = n))$$

Now, also $M \not\sim^{\mathrm{L}} N$:

$$M \Longrightarrow V \xrightarrow{\lambda n'.V(\lambda m.m=n')} \Longrightarrow \mathtt{true}$$

$$N \Longrightarrow W \xrightarrow{\lambda n'.W(\lambda m.m=n')} \Longrightarrow \mathtt{false}$$

Evaluation contexts

Sometimes useful to separate evaluation contexts

[example: call-by-value λ-calculus with references]

$$M \triangleq \texttt{if}\ !l = 0\ \texttt{then}\ l := 1\ \texttt{else}\ \Omega$$
$$N \triangleq l := 1$$

$\simeq^{\mathrm{EC}} \triangleq$ contextual equivalence, under only evaluation contexts

$\langle [l = 0]\,;\, M\rangle \simeq^{\mathrm{EC}} \langle [l = 0]\,;\, N\rangle$

$\langle [l = 0]\,;\, M\rangle \not\simeq^{\mathrm{C}} \langle [l = 0]\,;\, N\rangle$ for $C = [\cdot];[\cdot]$

Coupled logical bisimulation

$(\mathcal{E}, \mathcal{G})$ with $\mathcal{E}$ closed under contexts, and $\mathcal{E} \subseteq \mathcal{G}$,
$\mathcal{G}$ closed under evaluation contexts

[call-by-value λ-calculus]

... whenever $M \; \mathcal{G} \; N$

1. $M \longrightarrow M'$ implies $N \Longrightarrow N'$ and $M' \; \mathcal{G} \; N'$
2. $M = \lambda x.\, M'$ implies $N \Longrightarrow \lambda x.\, N'$ with
   - $\lambda x.\, M' \; \mathcal{E} \; \lambda x.\, N'$,
   - $M\{^{P}/_{x}\} \; \mathcal{G} \; N\{^{Q}/_{x}\}$ for all $P \; \mathcal{E} \; Q$

$M \; \mathcal{E} \; N$ implies $M \simeq^{\mathrm{C}} N$, and $M \; \mathcal{G} \; N$ implies $M \simeq^{\mathrm{EC}} N$

Non-determinism and probabilities

Non-determinism

$$M, N ::= \ldots \mid M \oplus N$$

Now the easy proof $\simeq^{\mathrm{C}} \subseteq \sim^{\mathrm{A/L}}$ breaks (as $\longrightarrow \not\subseteq \simeq^{\mathrm{C}}$)

- **Convergence: may, must**
- **Variants of $\simeq^{\mathrm{C}}$: may, must, may & must**
- **Bisimulation: different from any of them**

$$\lambda x.\, I \oplus \lambda x.\, \Omega \;=\; \lambda x.\, (I \oplus \Omega)$$

cf: the CCS-like law $\mu.\, P \oplus \mu.\, Q = \mu.\, (P \oplus Q)$

To regain coincidence, two possibilities :

1. stengthen $\simeq^{\mathrm{C}}$
2. weaken $\sim^{\mathrm{A/L}}$

(1) is easy: replace contextual equivalence with barbed congruence (congruence induced by barbed bisimulation)

Barbed bisimulation

whenever $M \; \mathcal{R} \; N$

1. $M \longrightarrow M'$ implies $N \Longrightarrow N'$ and $M' \; \mathcal{R} \; N'$
2. $M \Downarrow$ iff $N \Downarrow$

(2) is be more delicate (cf: proof of $\simeq^{\mathrm{C}} \subseteq \sim^{\mathrm{A/L}}$; first congruence problem)

A case study: **the probabilistic λ-calculus**

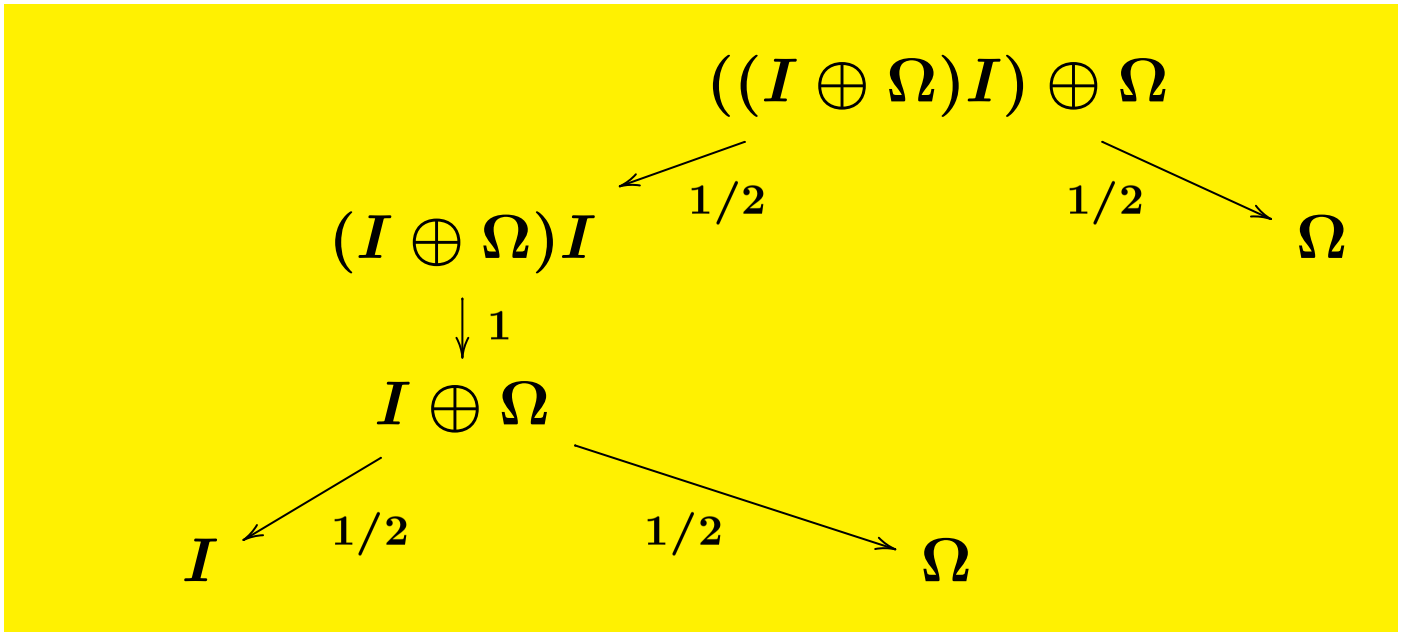
The probabilistic λ-calculus

[Alberti, Dal Lago, Sangiorgi, on-going work]

$M \oplus N$ abbreviates for $M \oplus_{1/2} N$

$$M_1 \oplus M_2 \longrightarrow_{1/2} M_1 \qquad\qquad M_1 \oplus M_2 \longrightarrow_{1/2} M_2$$

Example:

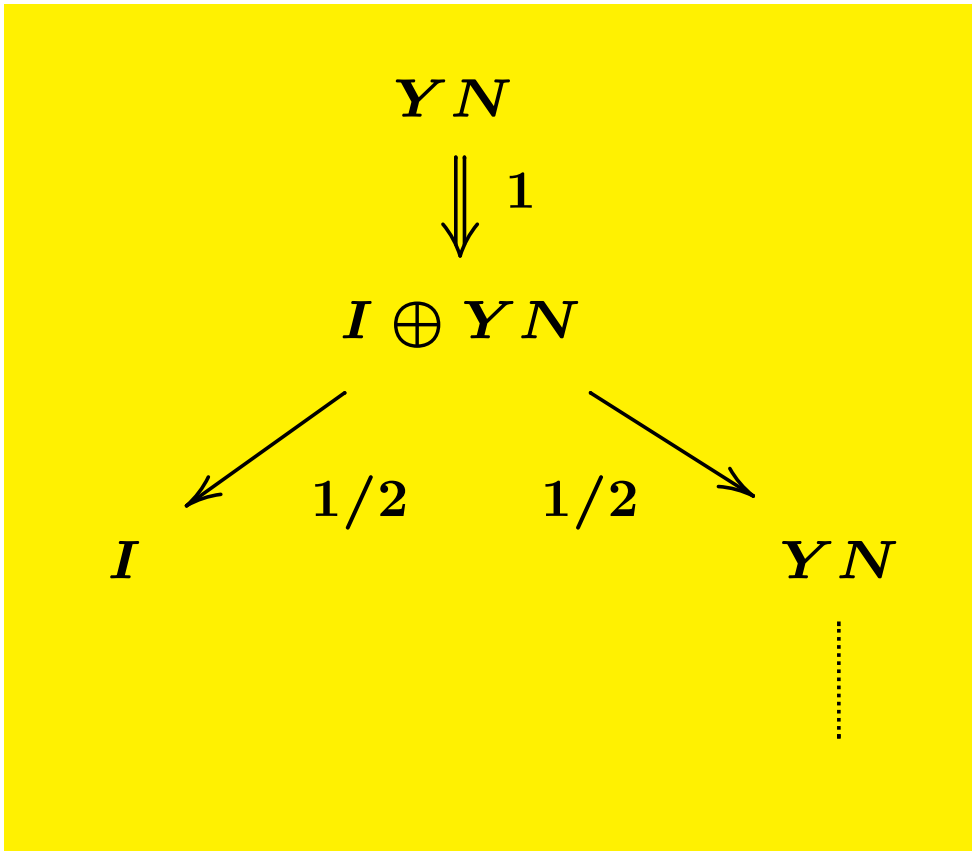


which shows $((I \oplus \Omega)I) \oplus \Omega \Downarrow_{1/4}$

Distributions

$Y \triangleq$ fix point operator

$N \triangleq \lambda f.\,(I \oplus f)$



For all n, $Y\,N \Longrightarrow_{1/2^n} I$ hence $Y\,N \Downarrow_1$ (ie, $Y\,N \Uparrow_0$)

Using (partial) distributions: $Y\,N \Downarrow \Sigma_n\langle I, 1/2^n\rangle$

Probabilistic contextual equivalence, $\simeq_{\mathrm{P}}^{\mathrm{C}}$

$M \simeq^{\mathrm{C}} N$ if, $C[M] \Downarrow_p$ iff $C[N] \Downarrow_p$, for all contexts C.

No issues of may and must convergence

Probabilistic applicative bisimulation, $\sim_{\mathrm{P}}^{\mathrm{A}}$, following Larsen-Skou

An **equivalence** s.t. $M \; \mathcal{R} \; N$ implies,

for all equivalence classes E of $\mathcal{R}$ and for all inputs L:

$$\mathtt{prob}(M \stackrel{L}{\Longrightarrow} E) = \mathtt{prob}(N \stackrel{L}{\Longrightarrow} E)$$

Theorem $\sim_{\mathrm{P}}^{\mathrm{A}}$ is a congruence [Howe's technique]

$\sim_{\mathrm{P}}^{\mathrm{A}} = \simeq_{\mathrm{P}}^{\mathrm{C}}$ **? And how discriminating?**

The effect of probabilities on pure λ-terms

$=_{\mathrm{LL}} \triangleq$ Lévy–Longo Tree equality

The finest equivalence for pure λ-term, under call-by-name

[Dezani,Giovannetti tutorial, 2001]

Theorem In $\Lambda \times \Lambda$: $=_{\mathrm{LL}} \;=\; \sim_{\mathrm{P}}^{\mathrm{A}} \;=\; \simeq_{\mathrm{P}}^{\mathrm{C}}$

Higher-order and probability: maximal discriminating power (on pure λ-terms)

[cf: work in concurrency: eg Deng, Hennessy 2010]

Quite different from (non-probabilistic) non-determinism :

$\lambda x.\, xx$ and $\lambda x.\, x(\lambda y.\, xy)$ contextually equivalent

(both may and must)

[under may (similarly for must): if $L \Downarrow_{\mathrm{may}}$ then $\lambda z.\, Lz = L$;
otherwise $L\widetilde{N} = \Omega$]

Different in $\simeq_{\mathrm{P}}^{\mathrm{C}}$:

$$(\lambda x.\, xx)(I \oplus \Omega) \Downarrow_{1/4} I$$
$$(\lambda x.\, x(\lambda y.\, xy))(I \oplus \Omega) \Downarrow_{1/2} \lambda y.\, (I \oplus \Omega)y$$

Similarly, different under bisimulation, and different LL trees

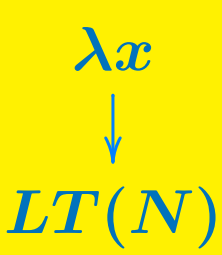
Outside pure λ-terms, usual counterexample:

$$\lambda x.\, I \oplus \lambda x.\, \Omega \qquad \text{vs} \qquad \lambda x.\, (I \oplus \Omega)$$

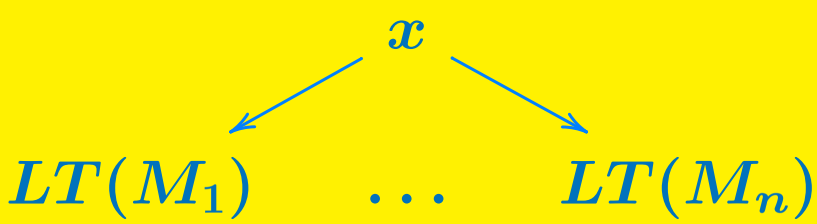
Coinductive characterisation of $\simeq_{\mathrm{P}}^{\mathrm{C}}$?

Lévy–Longo Trees

The **Lévy–Longo Tree** of $M \in \Lambda$ is the labeled tree, LT(M), defined coinductively as follows:

1. 

if $M \Longrightarrow \lambda x.\, N$

2. 

if $M \Longrightarrow x M_1 \ldots M_n$

3. $LT(M) = \bot$ otherwise (ie, $M \Uparrow$)

Probabilistic coupled logical bisimulation

A partial distribution : $\Sigma_i \langle M_i, p_i \rangle$

A distribution value : $\lambda x.\, \Sigma_i \langle M_i, p_i \rangle \stackrel{\triangle}{=} \Sigma_i \langle \lambda x.\, M_i, p_i \rangle$

Allow distributions in redex position

Extended λ-terms (Λ_{D}):

$$E, F ::= EM \;\Big|\; \Sigma_i \langle M_i, p_i \rangle \;\Big|\; M_1 \oplus M_2 \;\Big|\; \lambda x.\, M$$

$(M \in \Lambda)$

A bisimulation: $(\mathcal{E}, \mathcal{G})$ with

$$\mathcal{E} \subseteq \Lambda^{\bullet} \times \Lambda^{\bullet}, \quad \mathcal{G} \subseteq \Lambda^{\bullet}_{\mathrm{D}} \times \Lambda^{\bullet}_{\mathrm{D}}, \quad \mathcal{E} \subseteq \mathcal{G}.$$

$(\mathcal{E}, \mathcal{G})$ is a **bisimulation** if for each $E \; \mathcal{G} \; F$ we have:

1. if $E \longrightarrow E'$ then $F \Longrightarrow F'$ and $E' \; \mathcal{G} \; F'$;
2. if $E = \lambda x.\, E'$ then $F \Longrightarrow \lambda x.\, F'$ with
   $\mathtt{prob}(E') = \mathtt{prob}(F')$, and
   $E'\{^{M}/_{x}\} \; \mathcal{G} \; W\{^{N}/_{x}\}$ for all $M \; \mathcal{E} \; N$

Write $M \sim^{\mathrm{CL}}_{\mathrm{P}} N$ if $M \; \mathcal{E} \; N$ for some bisimulation $(\mathcal{E}, \mathcal{G})$.

Theorem $\sim^{\mathrm{CL}}_{\mathrm{P}} = \simeq^{\mathrm{C}}_{\mathrm{P}}$

Probabilistic λ-calculus: big-step operational semantics, call-by-name

[Dal Lago, Zorzi, 2012]

$$\textsc{Lam}\ \frac{}{\lambda x.\, M \rightsquigarrow \langle \lambda x.\, M, 1\rangle} \qquad \textsc{Emp}\ \frac{}{M \rightsquigarrow \emptyset}$$

$$\textsc{App}\ \frac{M \rightsquigarrow \lambda x.\, \Sigma_i \langle M_i, p_i\rangle \qquad M_i\{^N/_x\} \rightsquigarrow \Sigma_j \langle N_{i,j}, q_j\rangle}{MN \rightsquigarrow \Sigma_{i,j} \langle N_{i,j}, p_i \cdot q_j\rangle}$$

$$\textsc{Plus}\ \frac{M \rightsquigarrow E \qquad M \rightsquigarrow F}{M \oplus N \rightsquigarrow E \cdot 1/2 + F \cdot 1/2}$$

Inductively: $M \Longrightarrow E$ if $E = \sup\{F \ : \ M \rightsquigarrow F\}$

Coinductively, without EMP**:**

$$M \Longrightarrow E \text{ if } E = \inf\{F \ : \ M \rightsquigarrow F\}$$