

Objectives

**Principles of Constraint
Programming**
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Chapter 5
Local Consistency Notions

- Introduce several local consistency notions:
 - node consistency,
 - arc consistency,
 - hyper-arc consistency,
 - directional arc consistency,
 - path consistency,
 - directional path consistency,
 - k -consistency,
 - strong k -consistency,
 - relational consistency.
- Use the proof theoretic framework to characterize these local consistency notions.

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Node Consistency

- CSP is **node consistent** if for every variable x every unary constraint on x coincides with the domain of x .

- **Examples**

Assume \mathcal{C} contains no unary constraints.

\mathcal{N} – natural numbers,

\mathcal{Z} – integers.

– $\langle \mathcal{C}, x_1 \geq 0, \dots, x_n \geq 0 ; x_1 \in \mathcal{N}, \dots, x_n \in \mathcal{N} \rangle$

is node consistent.

– $\langle \mathcal{C}, x_1 \geq 0, \dots, x_n \geq 0 ;$

$x_1 \in \mathcal{N}, \dots, x_{n-1} \in \mathcal{N}, x_n \in \mathcal{Z} \rangle$

is not node consistent.

Arc Consistency

- A constraint C on the variables x, y with the domains X and Y (so $C \subseteq X \times Y$) is **arc consistent** if

– $\forall a \in X \exists b \in Y (a, b) \in C,$

– $\forall b \in Y \exists a \in X (a, b) \in C.$

- A CSP is **arc consistent** if all its binary constraints are.

- **Examples**

– $\langle x < y ; x \in [2..6], y \in [3..7] \rangle$

is arc consistent.

– $\langle x < y ; x \in [2..7], y \in [3..7] \rangle$

is not arc consistent.

Status of Arc Consistency

- Arc consistency does not imply consistency.

Example Take

$$\langle x = y, x \neq y ; x \in \{a, b\}, y \in \{a, b\} \rangle.$$

- Consistency does not imply arc consistency.

Example Take

$$\langle x = y ; x \in \{a, b\}, y \in \{a\} \rangle.$$

- For some CSP's arc consistency does imply consistency.

(A general result later.)

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Proof Rules for Arc Consistency

ARC CONSISTENCY 1

$$\frac{\langle C ; x \in D_x, y \in D_y \rangle}{\langle C ; x \in D'_x, y \in D_y \rangle}$$

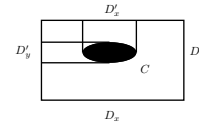
where $D'_x := \{a \in D_x \mid \exists b \in D_y (a, b) \in C\}$

ARC CONSISTENCY 2

$$\frac{\langle C ; x \in D_x, y \in D_y \rangle}{\langle C ; x \in D_x, y \in D'_y \rangle}$$

where $D'_y := \{b \in D_y \mid \exists a \in D_x (a, b) \in C\}$.

Intuition



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Characterization of Arc Consistency

Note A CSP is arc consistent iff it is closed under the applications of the *ARC CONSISTENCY* rules 1 and 2.

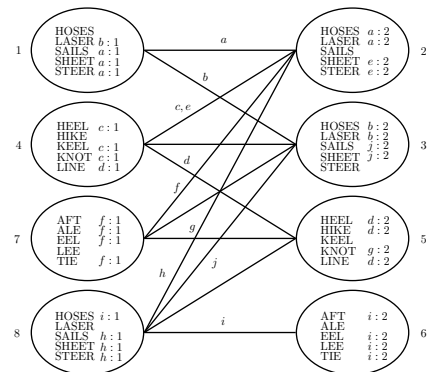
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Derivation: Example



$a : C_{1,2}, b : C_{1,3}, c : C_{4,2}, d : C_{4,5}, e : C_{4,2},$

$f : C_{7,2}, g : C_{7,5}, h : C_{8,2}, i : C_{8,6}, j : C_{8,3}.$



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Hyper-arc Consistency

- A constraint C on the variables x_1, \dots, x_n with the domains D_1, \dots, D_n is **hyper-arc consistent** if

$$\forall i \in [1..n] \forall a \in D_i \exists d \in C \ a = d[x_i].$$

- CSP is **hyper-arc consistent** if all its constraints are.

• Examples

– $\langle x \wedge y = z ; x = 1, y \in \{0, 1\}, z \in \{0, 1\} \rangle$
is hyper-arc consistent.

– $\langle x \wedge y = z ; x \in \{0, 1\}, y \in \{0, 1\}, z = 1 \rangle$
is not hyper-arc consistent.

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Characterization of Hyper-arc Consistency

HYPER-ARC CONSISTENCY

$$\frac{\langle C ; x_1 \in D_1, \dots, x_n \in D_n \rangle}{\langle C ; \dots, x_i \in D'_i, \dots \rangle}$$

C a constraint on the variables x_1, \dots, x_n ,
 $i \in [1..n]$,

$$D'_i := \{a \in D_i \mid \exists d \in C \ a = d[x_i]\}.$$

Note A CSP is hyper-arc consistent iff it is closed under the applications of the *HYPER-ARC CONSISTENCY* rule.

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Directional Arc Consistency

Assume a linear ordering \prec on the variables.

- A constraint C on x, y with the domains D_x and D_y is **directionally arc consistent w.r.t. \prec** if

– $\forall a \in D_x \exists b \in D_y \ (a, b) \in C$
provided $x \prec y$,

– $\forall b \in D_y \exists a \in D_x \ (a, b) \in C$
provided $y \prec x$.

- A CSP is **directionally arc consistent w.r.t. \prec** if all its binary constraints are.

Example

$$\langle x < y ; x \in [2..7], y \in [3..7] \rangle$$

is

- not arc consistent,
- directionally arc consistent w.r.t. $y \prec x$.
- not directionally arc consistent w.r.t. $x \prec y$.

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Characterization of Directional Arc Consistency

Define \mathcal{P}_{\prec} :

\mathcal{P} with the variables reordered w.r.t. \prec .

Example

Take $\mathcal{P} :=$

$$\langle x < y, y \neq z ; x \in [2..10], y \in [3..7], z \in [3..6] \rangle$$

and

$$y \prec x \prec z.$$

Then $\mathcal{P}_{\prec} :=$

$$\langle y > x, y \neq z ; y \in [3..7], x \in [2..10], z \in [3..6] \rangle.$$

Note A CSP \mathcal{P} is directionally arc consistent w.r.t. \prec iff the CSP \mathcal{P}_{\prec} is closed under the applications of the *ARC CONSISTENCY* rule 1.

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Limitations of Arc Consistency

Note

$\langle x < y, y < z, z < x; x, y, z \in \{1..100000\} \rangle$.

is inconsistent.

Proof using arc consistency rules.

Applying *ARC CONSISTENCY* rule 1 we get

$\langle x < y, y < z, z < x; x \in \{1..99999\}, y, z \in \{1..100000\} \rangle$,

etc.

Disadvantages:

- Large number of steps.
- Length depends on the size of the domains.

Direct proof: use transitivity of $<$.

Path consistency: a generalizes this form of reasoning to arbitrary binary relations.

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Path Consistency

A normalized CSP is **path consistent** if for each subset $\{x, y, z\}$ of its variables

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z}.$$

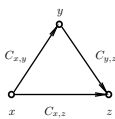
Note A normalized CSP is path consistent iff for each subsequence x, y, z of its variables

$$C_{x,y} \subseteq C_{x,z} \cdot C_{y,z}^T,$$

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z},$$

$$C_{y,z} \subseteq C_{x,y}^T \cdot C_{x,z}.$$

Intuition



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Normalized CSP's

A CSP \mathcal{P} is **normalized** if for each pair x, y of its variables at most one constraint on x, y exists.

Denote by $C_{x,y}$ the unique constraint on x, y if it exists and otherwise the “universal” relation on x, y .

R and S : two binary relations.

- **transposition** of R :

$$R^T := \{(b, a) \mid (a, b) \in R\},$$

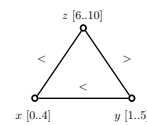
- **composition** of R and S by

$$R \cdot S := \{(a, b) \mid \exists c ((a, c) \in R, (c, b) \in S)\}.$$

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Path Consistency: Example 1

$\langle x < y, y < z, x < z; x \in [0..4], y \in [1..5], z \in [6..10] \rangle$



is path consistent. Indeed

$$C_{x,y} = \{(a, b) \mid a < b, a \in [0..4], b \in [1..5]\},$$

$$C_{x,z} = \{(a, c) \mid a < c, a \in [0..4], c \in [6..10]\},$$

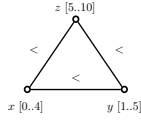
$$C_{y,z} = \{(b, c) \mid b < c, b \in [1..5], c \in [6..10]\},$$

and all 3 conditions are satisfied.

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Path Consistency: Example 2

$\langle x < y, y < z, x < z; \\ x \in [0..4], y \in [1..5], z \in [5..10] \rangle$



is not path consistent. Indeed, now

$C_{x,z} = \{(a, c) \mid a < c, a \in [0..4], c \in [5..10]\}$
and for $4 \in [0..4]$ and $5 \in [5..10]$ no $b \in [1..5]$
exists such that $4 < b$ and $b < 5$.

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Characterization of Path Consistency

PATH CONSISTENCY 1

$$\frac{C_{x,y}, C_{x,z}, C_{y,z}}{C'_{x,y}, C_{x,z}, C_{y,z}}$$

where $C'_{x,y} := C_{x,y} \cap C_{x,z} \cdot C_{y,z}^T$,

PATH CONSISTENCY 2

$$\frac{C_{x,y}, C_{x,z}, C_{y,z}}{C_{x,y}, C'_{x,z}, C_{y,z}}$$

where $C'_{x,z} := C_{x,z} \cap C_{x,y} \cdot C_{y,z}$,

PATH CONSISTENCY 3

$$\frac{C_{x,y}, C_{x,z}, C_{y,z}}{C_{x,y}, C_{x,z}, C'_{y,z}}$$

where $C'_{y,z} := C_{y,z} \cap C_{x,y}^T \cdot C_{x,z}$.

Note A normalized CSP is path consistent iff it is closed under the applications of the *PATH CONSISTENCY* rules 1, 2 and 3.

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m-Path Consistency

A normalized CSP is ***m*-path consistent** ($m \geq 2$) if for each subset $\{x_1, \dots, x_{m+1}\}$ of its variables

$$C_{x_1, x_{m+1}} \subseteq C_{x_1, x_2} \cdot C_{x_2, x_3} \cdot \dots \cdot C_{x_m, x_{m+1}}.$$

Note A normalized CSP is *m*-path consistent if for each subset $\{x_1, \dots, x_{m+1}\}$ of its variables

if $(a_1, a_{m+1}) \in C_{x_1, x_{m+1}}$, then for some a_2, \dots, a_m for all $i \in [1..m]$
 $(a_i, a_{i+1}) \in C_{x_i, x_{i+1}}$.

a_2, \dots, a_m : **path** connecting a_1 and a_{m+1} .

Theorem Every normalized path consistent CSP is *m*-path consistent for each $m \geq 2$.

Proof. Induction on *m*.

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Directional Path Consistency

Assume a linear ordering \prec on the variables. A normalized CSP is **directionally path consistent w.r.t. \prec** if for each subset $\{x, y, z\}$ of its variables

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z} \text{ provided } x, z \prec y.$$

Note A normalized CSP is directionally path consistent w.r.t. \prec iff for each subsequence x, y, z of its variables

$$C_{x,y} \subseteq C_{x,z} \cdot C_{y,z}^T \text{ provided } x, y \prec z,$$

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z} \text{ provided } x, z \prec y,$$

$$C_{y,z} \subseteq C_{x,y}^T \cdot C_{x,z} \text{ provided } y, z \prec x.$$

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Examples

Reconsider

$$\langle x < y, y < z, x < z; \\ x \in [0..4], y \in [1..5], z \in [5..10] \rangle$$

Then

$$C_{x,y} = \{(a, b) \mid a < b, a \in [0..4], b \in [1..5]\},$$

$$C_{x,z} = \{(a, c) \mid a < c, a \in [0..4], c \in [5..10]\},$$

$$C_{y,z} = \{(b, c) \mid b < c, b \in [1..5], c \in [5..10]\}.$$

- It is directionally path consistent w.r.t. the ordering \prec in which $x, y \prec z$.
Indeed, for every pair $(a, b) \in C_{x,y}$ there exists $c \in [5..10]$ such that $a < c$ and $b < c$.
- It is directionally path consistent w.r.t. the ordering \prec in which $y, z \prec x$.
Indeed, for every pair $(b, c) \in C_{y,z}$ there exists $a \in [0..4]$ such that $a < b$ and $a < c$.

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Characterization of Directional Path Consistency

Note A normalized CSP \mathcal{P} is directionally path consistent w.r.t. \prec iff \mathcal{P}_{\prec} is closed under the applications of the *PATH CONSISTENCY* rule 1.

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Instantiations

Fix a CSP \mathcal{P} .

- **Instantiation:** function on a subset of the variables of \mathcal{P} . It assigns to each variable a value from its domain.

Notation:

$$\{(x_1, d_1), \dots, (x_k, d_k)\}.$$

- C : a constraint on x_1, \dots, x_k .
Instantiation $\{(x_1, d_1), \dots, (x_k, d_k)\}$
satisfies C if $(d_1, \dots, d_k) \in C$.
- I : instantiation with a domain $X, Y \subseteq X$.
 $I \upharpoonright Y$: **restriction of I to Y** .
- Instantiation I with domain X is **consistent** if for every constraint C of \mathcal{P} on some Y with $Y \subseteq X$ $I \upharpoonright Y$ satisfies C .
- Consistent instantiation is **k -consistent** if its domain consists of k variables.
- An instantiation is a **solution** to \mathcal{P} if it is consistent and defined on all variables of \mathcal{P} .

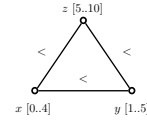
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Example

Consider

$$\langle x < y, y < z, x < z; x \in [0..4], y \in [1..5], z \in [5..10] \rangle.$$

$$\text{Let } I := \{(x, 0), (y, 5), (z, 6)\}.$$



- $I \upharpoonright \{x, y\} = \{(x, 0), (y, 5)\}$.
It satisfies $x < y$.
- $I \upharpoonright \{x, z\} = \{(x, 0), (z, 6)\}$.
It satisfies $x < z$.
- $I \upharpoonright \{y, z\} = \{(y, 5), (z, 6)\}$.
It satisfies $y < z$.
- So I is a 3-consistent instantiation. It is a solution to this CSP.

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k -Consistency

- CSP is **1-consistent** if for every variable x with a domain D each unary constraint on x equals D .
- CSP is **k -consistent**, $k > 1$, if every $(k - 1)$ -consistent instantiation can be extended to a k -consistent instantiation **no matter** which new variable is chosen.

k -consistency aka **node consistency**

Note

- A node consistent CSP is arc consistent iff it is 2-consistent.
- A node consistent normalized binary CSP is path consistent iff it is 3-consistent.

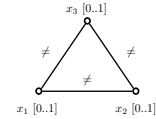
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k -Consistency, ctd

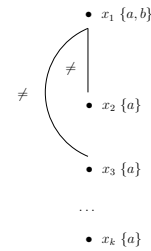
Fix $k > 1$.

- There exists a CSP that is $(k - 1)$ -consistent but not k -consistent.
- There exists a CSP that is not $(k - 1)$ -consistent but is k -consistent.

Proof of (i) for $k = 3$:



Proof of (ii):



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Strong k -Consistency

CSP **strongly k -consistent**, $k \geq 1$, if it is i -consistent for every $i \in [1..k]$.

Theorem Take a CSP with k variables, $k \geq 1$, such that

- at least one domain is non-empty,
- it is strongly k -consistent.

Then it is consistent.

Proof. Construct a solution by induction.

Prove that

- there exists a 1-consistent instantiation,
- for every $i \in [2..k]$ each $(i - 1)$ -consistent instantiation can be extended to an i -consistent instantiation.

Disadvantage Required level of strong consistency = # of variables.

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Graphs and CSP's

Graph is **associated with a CSP \mathcal{P}** .

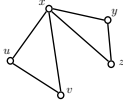
Nodes: variables of \mathcal{P} .

Arcs: connect two variables if they appear jointly in some constraint.

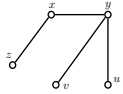
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Examples

- $SEND + MORE = MONEY$ puzzle.
The graph has 8 nodes, S, E, N, D, M, O, R, Y , and is complete.
- $\langle x + y = z, x + u = v ; \mathcal{DE} \rangle$



- $\langle x < z, x < y, y < u, y < v ; \mathcal{DE} \rangle$



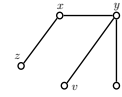
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Width of a Graph

- G : a finite graph G .
- \prec : linear ordering on the nodes of G .
- **\prec -width** of a node of G : number of arcs in G that connect it to \prec -smaller nodes.
 - **\prec -width** of G : maximum of the \prec -widths of its nodes.
 - The **width** of G : minimum of \prec -widths for all linear orderings \prec .

Examples

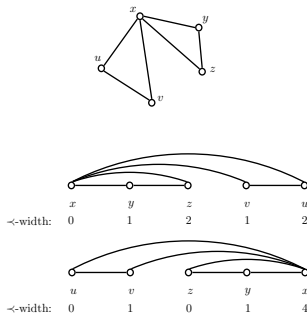
- $SEND + MORE = MONEY$ puzzle.
Complete graph with 8 nodes, so its width = 7.



- It is a tree, so its width = 1.

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Examples, ctd



Two examples of the \prec -widths of the nodes
Here width = 2.

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Consistency via Strong k -Consistency

Theorem Given: a CSP such that

- all domains are non-empty,
- it is strongly k -consistent,
- the graph associated with it has width $k - 1$.

Then this CSP is consistent.

Proof. (Sketch)

Assume n variables.

- Reorder the variables so that the resulting \prec -width is $k - 1$.
- Prove by induction that
 - there exists consistent instantiation with domain $\{x_1\}$,
 - for every $j \in [1..n - 1]$ each consistent instantiation with domain $\{x_1, \dots, x_j\}$ can be extended to a consistent instantiation with domain $\{x_1, \dots, x_{j+1}\}$.

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Useful Corollaries

Corollary 1

Given: \mathcal{P} and a linear ordering \prec such that

- all domains are non-empty,
- \mathcal{P} is
 - node consistent,
 - directionally arc consistent w.r.t. \prec ,
- the \prec -width of the graph associated with \mathcal{P} is 1.

Then \mathcal{P} is consistent.

Corollary 2

Given: \mathcal{P} and a linear ordering \prec such that

- all domains are non-empty,
- \mathcal{P} is
 - directionally arc consistent w.r.t. \prec ,
 - directionally path consistent w.r.t. \prec ,
- the \prec -width of the graph associated with it is 2.

Then \mathcal{P} is consistent.

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Relational Consistency

“Ultimate” notion of local consistency

- **Given:** \mathcal{P} and a subsequence \mathcal{C} of its constraints.
 - $\mathcal{P} \mid \mathcal{C}$:
 - remove from \mathcal{P} all constraints not in \mathcal{C} ,
 - delete all domain expressions involving variables not present in any constraint in \mathcal{C} .
- \mathcal{P} is **relationally (i, m) -consistent** if for every sequence \mathcal{C} of m constraints and $X \subseteq \text{Var}(\mathcal{C})$ of size i :
 - every consistent instantiation with the domain X can be extended to a solution to $\mathcal{P} \mid \mathcal{C}$.

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Relational Consistency, ctd

Intuition:

For every sequence of m constraints and for every set X of i variables, each present in one of these m constraints:

each consistent instantiation with the domain X can be extended to a solution to all these m constraints.

Some properties

- A node consistent binary CSP is arc consistent iff it is relationally $(1, 1)$ -consistent.
- A node consistent CSP is hyper-arc consistent iff it is relationally $(1, 1)$ -consistent.
- Every node consistent normalized relationally $(2, 3)$ -consistent CSP is path consistent.
- Every strictly binary relationally $(k - 1, k)$ -consistent CSP is k -consistent.
- A CSP with m constraints is consistent iff it is relationally $(0, m)$ -consistent.

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Some Notation

- **Given:** constraint C on variables X , subsequence Y of X .

$$\Pi_Y(C) := \{d[Y] \mid d \in C\}.$$

- X : sequence of variables,
 X_1, \dots, X_n : subsequences of X .
union of X_1, \dots, X_n : shortest subsequence of X containing each X_i as a subsequence.
Example: Take x_1, x_2, x_4, x_5 .
 Union of $(x_2, x_4), (x_4, x_5), (x_2, x_5)$ is x_2, x_4, x_5 .
- **Given:** a sequence of constraints C_1, \dots, C_m on variables X_1, \dots, X_m .

$$C_1 \bowtie \dots \bowtie C_m := \{d \mid d[X_i] \in C_i \text{ for } i \in [1..m]\}.$$

$C_1 \bowtie \dots \bowtie C_m$ is a constraint on the “union” of X_1, \dots, X_m .

- X : a sequence of variables
 $\overline{C_X} := \bowtie \{C_Y \mid Y \text{ is a subsequence of } X\}.$

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Characterization of k -Consistency

Note d is a solution to $\langle C_1, \dots, C_m; \mathcal{DE} \rangle$ iff $d \in C_1 \bowtie \dots \bowtie C_m$.

A CSP \mathcal{P} **regular** if for each sequence X of its variables a unique constraint on X exists. Denote it by C_X .

k -CONSISTENCY

$$\frac{C_X}{C_X \cap \Pi_X(C_{X,y})}$$

Note If a regular CSP is closed under the applications of the k -CONSISTENCY rule for all subsequences X of $k - 1$ variables and all variables y not in X , then it is k -consistent.

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Characterization of Relational Consistency

RELATIONAL (i, m) -CONSISTENCY

$$\frac{C_X}{C_X \cap \Pi_X(C_1 \bowtie \dots \bowtie C_m)}$$

Note If a regular CSP is closed under the applications of

RELATIONAL (i, m) -CONSISTENCY rule for each subsequence of constraints C_1, \dots, C_m and each subsequence X of $\text{Var}(C_1, \dots, C_m)$ of length i , then it is relationally (i, m) -consistent.

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Objectives

- Introduce several local consistency notions:
 - node consistency,
 - arc consistency,
 - hyper-arc consistency,
 - directional arc consistency,
 - path consistency,
 - directional path consistency,
 - k -consistency,
 - strong k -consistency,
 - relational consistency.
- Use the proof theoretic framework to characterize these local consistency notions.

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