Solving Constraint Problems in Constraint Programming

Zeynep KIZILTAN
Department of Computer Science
University of Bologna

Email: zeynep@cs.unibo.it

What is it about?

- 10 hour lectures about the core of constraint solving in CP
  - Part I: Overview of constraint programming
  - Part II: Local consistency & constraint propagation
  - Part III: Search algorithms
  - Part IV: Advanced topics, useful pointers
- Aim:
  - Teach the basics of constraint programming.
  - Emphasize the importance of local consistency & constraint propagation & search.
  - Point out the advanced topics.
  - Inform about the literature.

Warning

- We will see how constraint programming works.
- No programming examples.

PART I: Overview of Constraint Programming

Outline

- Constraint Satisfaction Problems (CSPs)
- Constraint Programming (CP)
  - Modelling
  - Backtracking Tree Search
  - Local Consistency and Constraint Propagation

Constraints are everywhere!

- No meetings before 9am.
- No registration of marks before May 15.
- The lecture rooms have a capacity.
- Two lectures of a student cannot overlap.
- No two trains on the same track at the same time.
- Salary > 45k Euros ☺

…

Constraint Satisfaction Problems

- A constraint is a restriction.
- There are many real-life problems that require to give a decision in the presence of constraints:
  - flight / train scheduling;
  - scheduling of events in an operating system;
  - staff rostering at a company;
  - course time tabling at a university …
- Such problems are called Constraint Satisfaction Problems (CSPs).

Sudoku: An everyday-life example

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

CSPs: More formally

- A CSP is a triple **<X,D,C>** where:
  - **X** is a set of decision variables **$\{X_1,\ldots,X_n\}$**.
  - **D** is a set of domains **$\{D_1,\ldots,D_n\}$** for **X**:
    - D_i is a set of possible values for X_i.
    - usually assume finite domain.
  - **C** is a set of constraints **$\{C_1,\ldots,C_m\}$**:
    - C_i is a relation over $X_j,\ldots,X_k$, giving the set of combination of allowed values.
    - $C_i \subseteq D(X_j) \times \ldots \times D(X_k)$
- A solution to a CSP is an assignment of values to the variables which satisfies all the constraints simultaneously.

CSPs: A simple example

- Variables

  $X = \{X_1, X_2, X_3\}$
- Domains

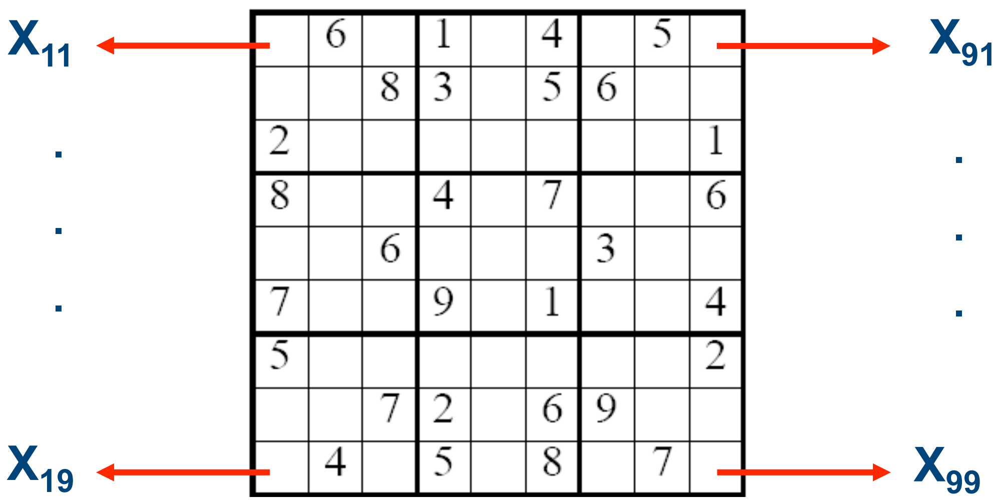
  $D(X_1) = \{1,2\}$, $D(X_2) = \{0,1,2,3\}$, $D(X_3) = \{2,3\}$
- Constraints

  $X_1 > X_2$ and $X_1 + X_2 = X_3$ and $X_1 \neq X_2 \neq X_3 \neq X_1$
- Solution

  $X_1 = 2$, $X_2 = 1$, $X_3 = 3$

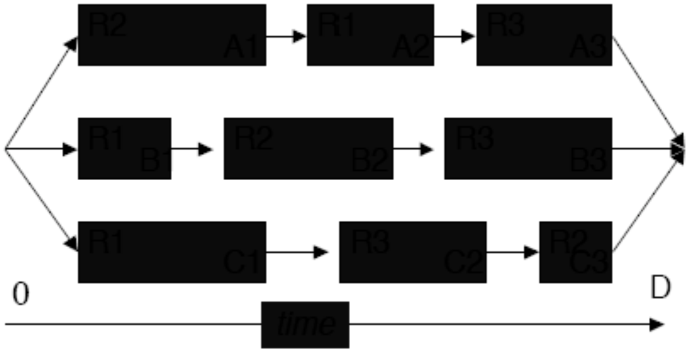
  alldifferent($[X_1, X_2, X_3]$)

Sudoku: An everyday-life example



- A simple CSP
  - 9x9 variables (X_{ij}) with domains {1,...,9}
  - Not-equals constraints on the rows, columns, and 3x3 boxes. E.g.,
    alldifferent([X_{11}, X_{21}, X_{31}, …, X_{91}])
    alldifferent([X_{11}, X_{12}, X_{13}, …, X_{19}])
    alldifferent([X_{11}, X_{21}, X_{31}, X_{12}, X_{22}, X_{32}, X_{13}, X_{23}, X_{33}])

Job-Shop Scheduling: A real-life example



- Schedule jobs, each using a resource for a period, in time D by obeying the precedence and capacity constraints
- A very common industrial problem.
- CSP:
  - variables represent the jobs;
  - domains represent the start times;
  - constraints specify precedence and exclusivity.

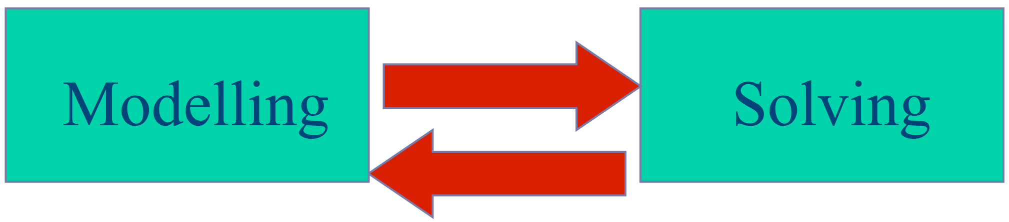
CSPs

- Search space: $D(X_1) \times D(X_2) \times \ldots \times D(X_n)$
  - very large!
- Constraint satisfaction is NP-complete:
  - no polynomial time algorithm is known to exist!
  - I can get no satisfaction ☹
- We need general and efficient methods to solve CSPs:
  - Integer and Linear Programming (satisfying linear constraints on 0/1 variables and optimising a criterion)
  - SAT (satisfying CNF formulas on 0/1 variables)
  - …
  - Constraint Programming

    How does it exactly work?

CP Machinery

- CP is composed of two phases that are strongly interconnected:



Modelling

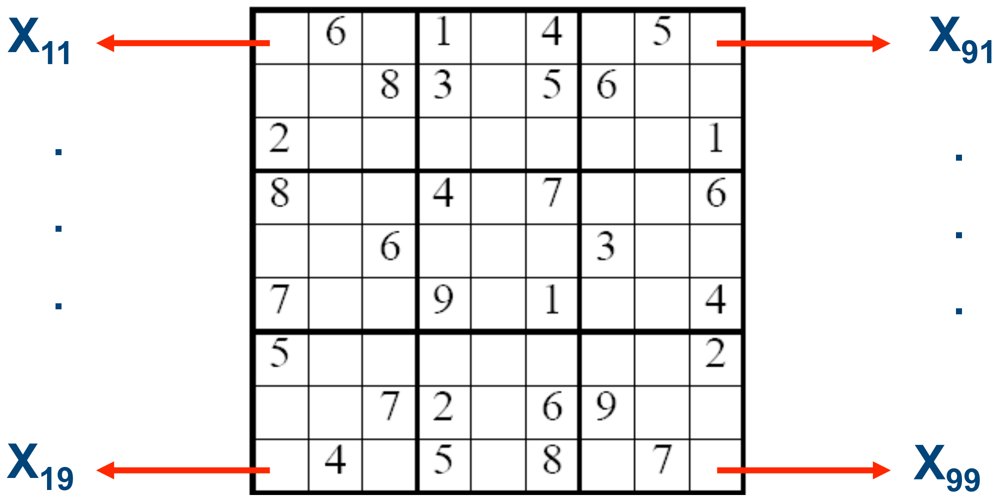
1. The CP user models the problem as a CSP:
   - define the variables and their domains;
   - specify solutions by posting constraints on the variables:
     - off-the-shelf constraints or user-defined constraints.
   - a constraint can be thought of a reusable component with its own propagation algorithm.

WAIT TO UNDERSTAND WHAT I MEAN ☺

Modelling

- Modelling is a critical aspect.
- Given the human understanding of a problem, we need to answer questions like:
  - which variables shall we choose?
  - which constraints shall we enforce?
  - shall we use off-the-self constraints, or define and integrate our own?
  - are some constraints redundant, therefore can be avoided?
  - are there any implied constraints?
  - among alternative models, which one shall I prefer?

A problem with a simple model



- A simple CSP
  - 9x9 variables (X_{ij}) with domains {1,...,9}
  - Not-equals constraints on the rows, columns, and 3x3 boxes, eg.,

    alldifferent([X_{11}, X_{21}, X_{31}, …, X_{91}])

    alldifferent([X_{11}, X_{12}, X_{13}, …, X_{19}])

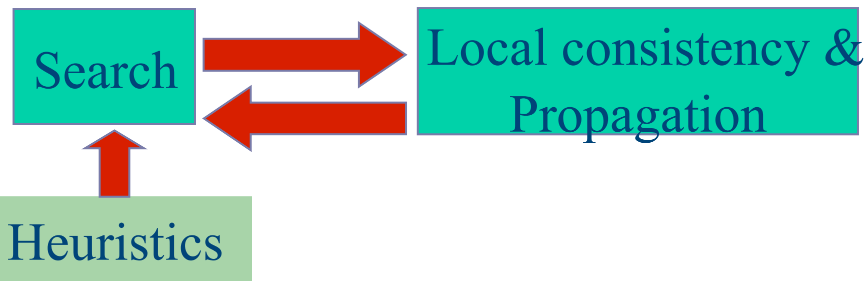
    alldifferent([X_{11}, X_{21}, X_{31}, X_{12}, X_{22}, X_{32}, X_{13}, X_{23}, X_{33}])

A problem with a complex model

- Consider a permutation problem:
  - find a permutation of the numbers {1,...,n} s.t. some constraints are satisfied.
- One model:
  - variables (X_i) for positions, domains for numbers {1,...,n}.
- Dual model:
  - variables (Y_i) for numbers {1,…,n}, domains for positions.
- Often different views allow different expression of the constraints and different implied constraints:
  - can be hard to decide which is better!
- We can use multiple models and combine them via *channelling constraints* to keep consistency between the variables:
  - $X_i = j \leftrightarrow Y_j = i$

Solving

2. The user lets the CP technology solve the CSP:
   - choose a search algorithm:
     - usually backtracking search performing a depth-first traversal of a search tree.
   - integrate local consistency and propagation.
   - choose heuristics for branching:
     - which variable to branch on?
     - which value to branch on?

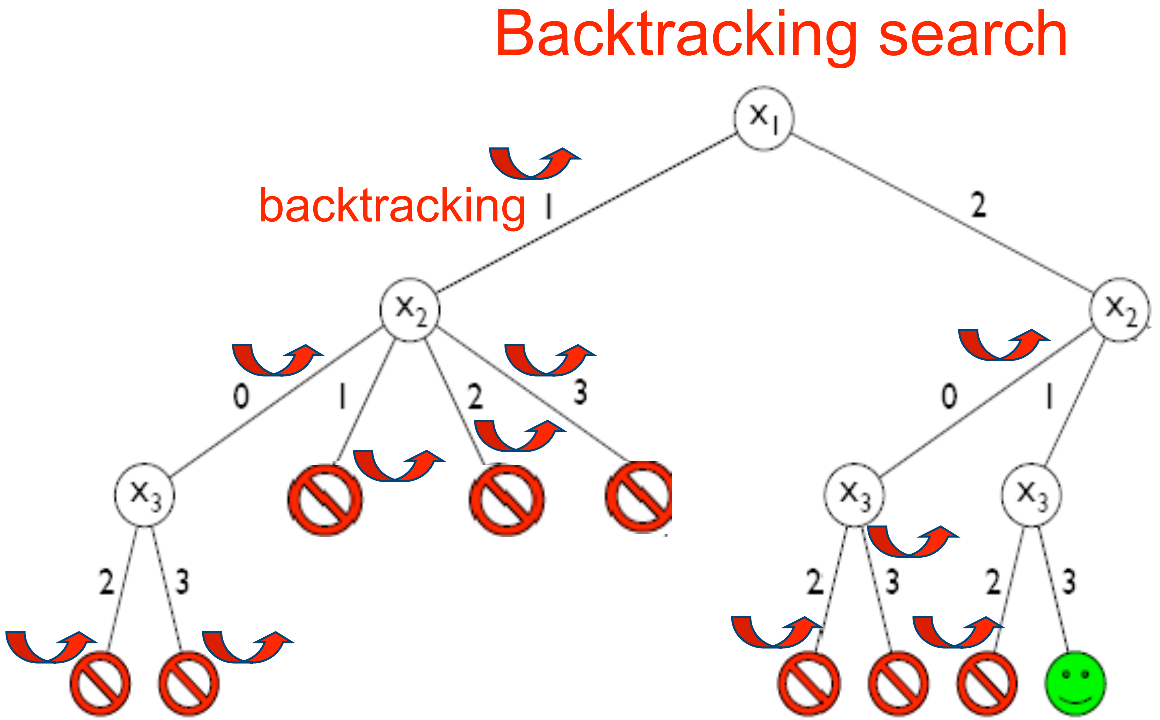


Backtracking Search

- A possible efficient and simple method.
- Variables are instantiated sequentially.
- Whenever all the variables of a constraint is instantiated, the validity of the constraint is checked.
- If a (partial) instantiation violates a constraint, backtracking is performed to the most recently instantiated variable that still has alternative values.
- Backtracking eliminates a subspace from the cartesian product of all variable domains.
- Essentially performs a depth-first search.

Backtracking Search

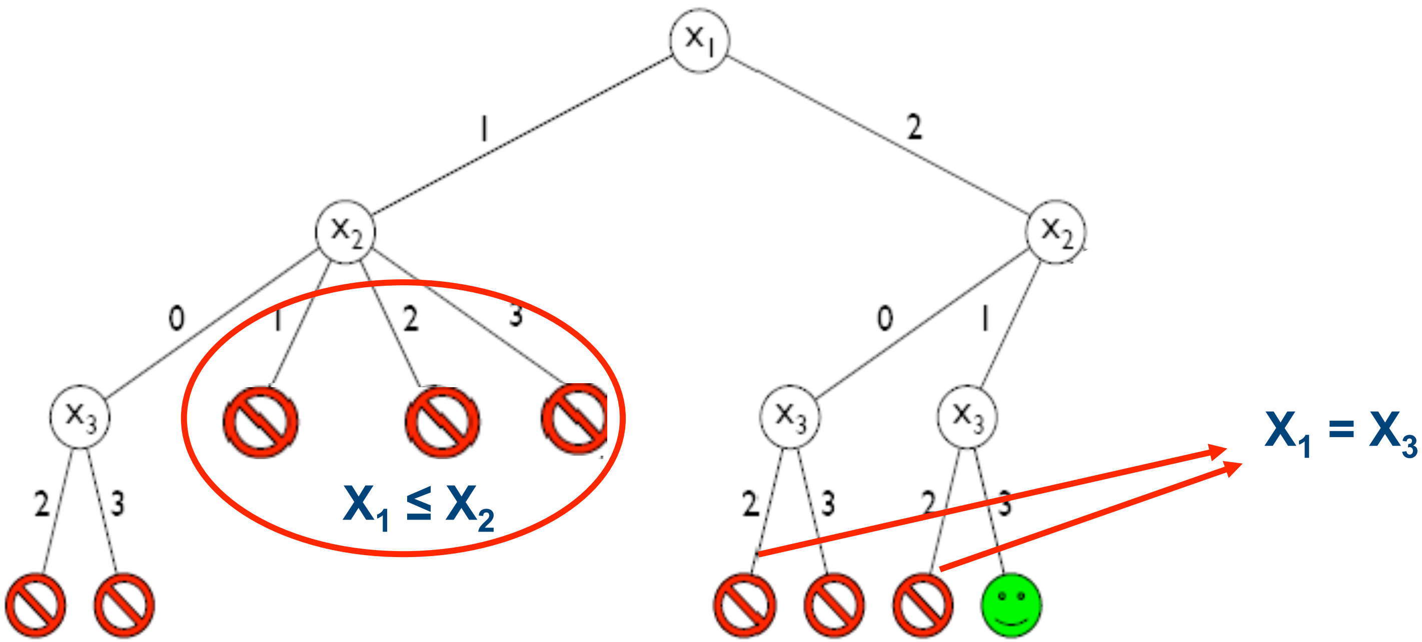
- $X_1 \in \{1,2\}$ $X_2 \in \{0,1,2,3\}$ $X_3 \in \{2,3\}$
- $X_1 > X_2$ and $X_1 + X_2 = X_3$ and alldifferent($[X_1, X_2, X_3]$)



Fails 8 times!

Backtracking Search

- Backtracking suffers from thrashing ☹ :
  - performs checks only with the current and past variables;
  - search keeps failing for the same reasons.



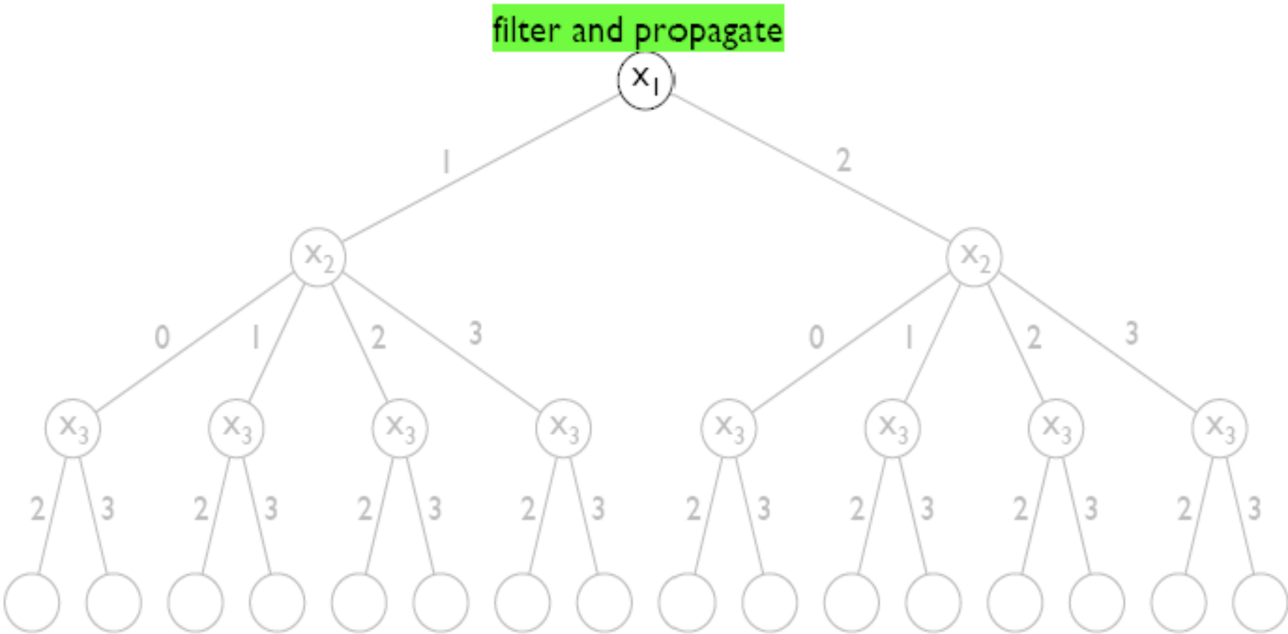
Constraint Programming

- Integrates local consistency and constraint propagation into the search.
- Consequently:
  - we can reason about the properties of constraints and their effect on their variables;
  - some values can be filtered from some domains, reducing the backtracking search space significantly!

Constraint Programming

- $X_1 \in \{1,2\}$ $X_2 \in \{0,1,\cancel{2},\cancel{3}\}$ $X_3 \in \{2,3\}$
- $X_1 > X_2$ and $X_1 + X_2 = X_3$ and alldifferent($[X_1, X_2, X_3]$)

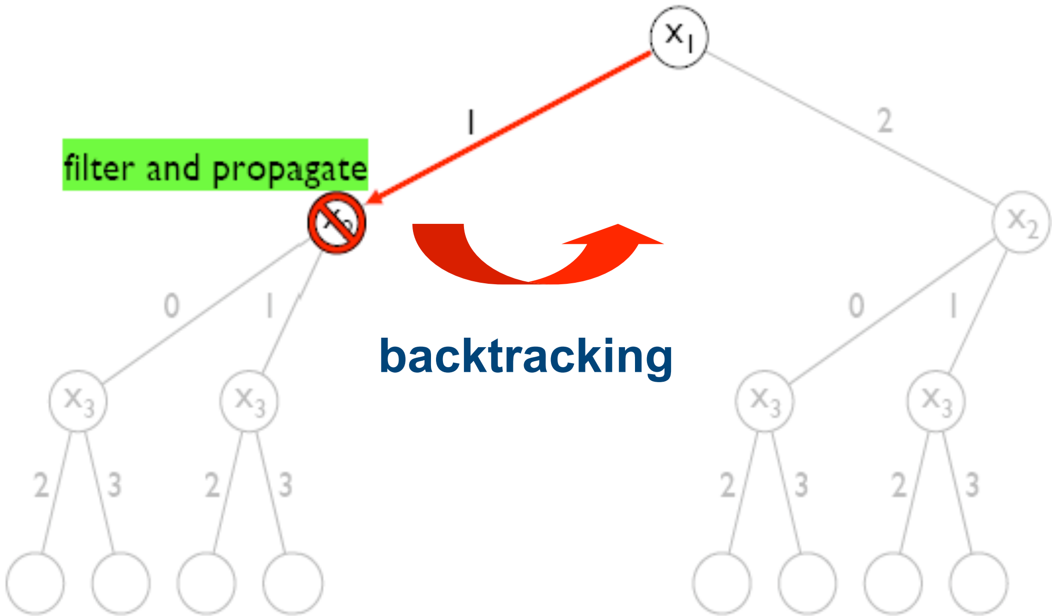
Backtracking search + local consistency/propagation



Constraint Programming

- $X_1 \in \{1, \cancel{2}\}$ $X_2 \in \{0, \cancel{1}\}$ $X_3 \in \{\cancel{2}, \cancel{3}\}$
- $X_1 > X_2$ and $X_1 + X_2 = X_3$ and alldifferent($[X_1, X_2, X_3]$)

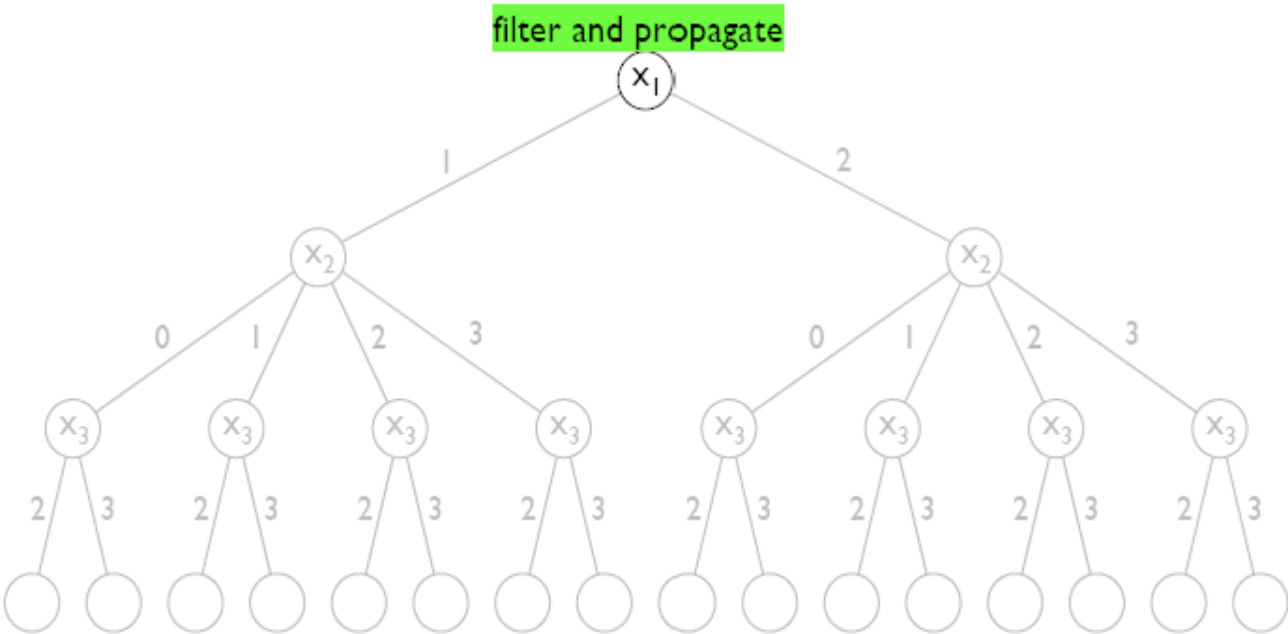
Backtracking search + local consistency/propagation



Constraint Programming

- $X_1 \in \{1,2\}$ $X_2 \in \{0,1,\cancel{2},\cancel{3}\}$ $X_3 \in \{2,3\}$
- $X_1 > X_2$ and $X_1 + X_2 = X_3$ and alldifferent($[X_1, X_2, X_3]$)

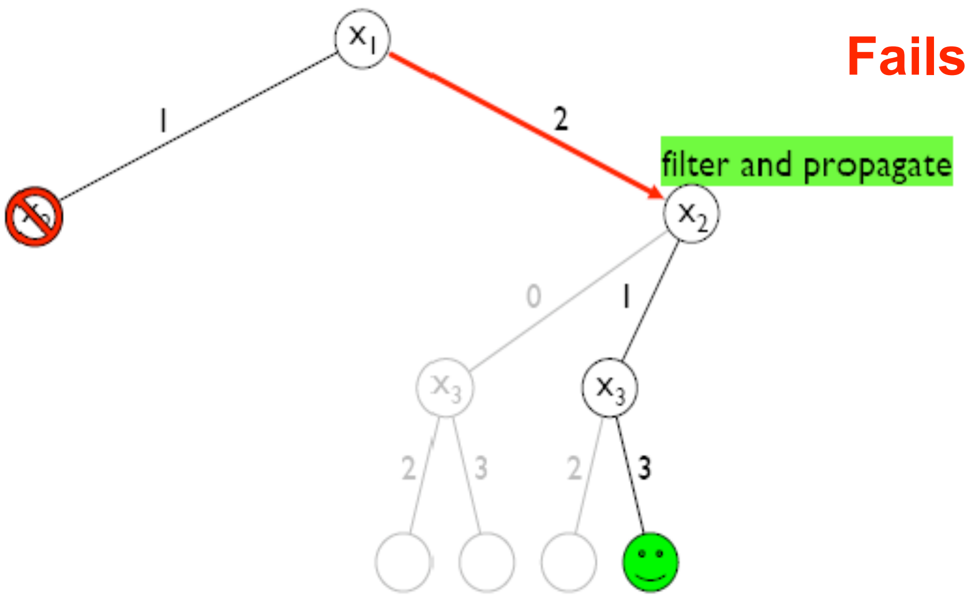
  Backtracking search + local consistency/propagation



Constraint Programming

- $X_1 \in \{\cancel{1},2\}$ $X_2 \in \{\cancel{0},1\}$ $X_3 \in \{\cancel{2},3\}$
- $X_1 > X_2$ and $X_1 + X_2 = X_3$ and alldifferent($[X_1, X_2, X_3]$)

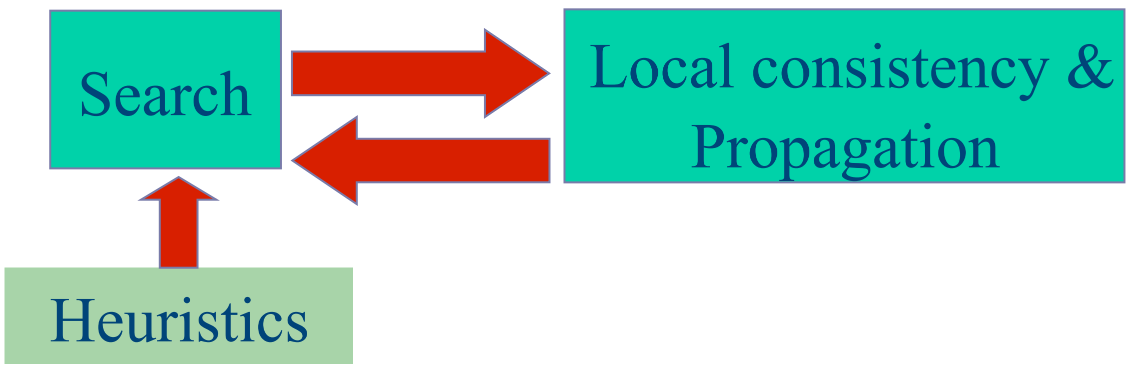
Backtracking search + local consistency/propagation



Fails only once!

Local consistency & Propagation

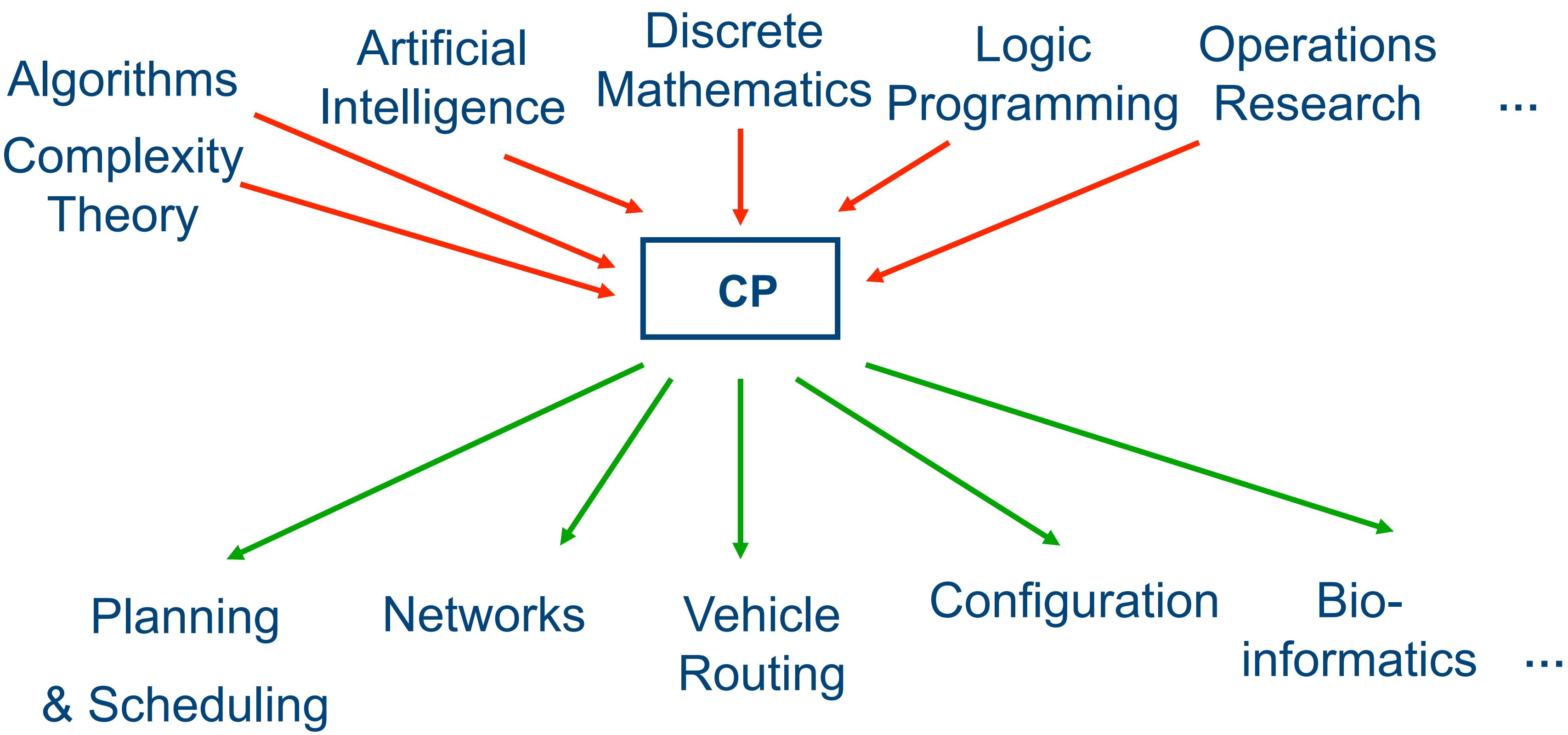
- Central to the process of solving CSPs which are inherently intractable.



CP

- Programming, in the sense of mathematical programming:
  - the user states declaratively the constraints on a set of decision variables.
  - an underlying solver solves the constraints and returns a solution.
- Programming, in the sense of computer programming:
  - the user needs to program a strategy to search for a solution
    - search algorithm, heuristics, …
  - otherwise, solving process can be inefficient.

CP


Algorithms
Complexity Theory
Artificial Intelligence
Discrete Mathematics
Logic Programming
Operations Research
…
CP
Planning & Scheduling
Networks
Vehicle Routing
Configuration
Bio-informatics
…

CP

- Solve SUDOKU using CP!

  http://www.cs.cornell.edu/gomes/SUDOKU/Sudoku.html

  - very easy, not worth spending minutes ☺
  - you can decide which newspaper provides the toughest Sudoku instances ☺

CP

- Constraints can be embedded into:
  - logic programming (constraint logic programming)
    - Prolog III, CLP(R), SICStus Prolog, ECLiPSe, CHIP, …
  - functional programming
    - Oz
  - imperative programming
    - often via a separate library
    - IBM CP Solver, Gecode, Choco, Minion, …

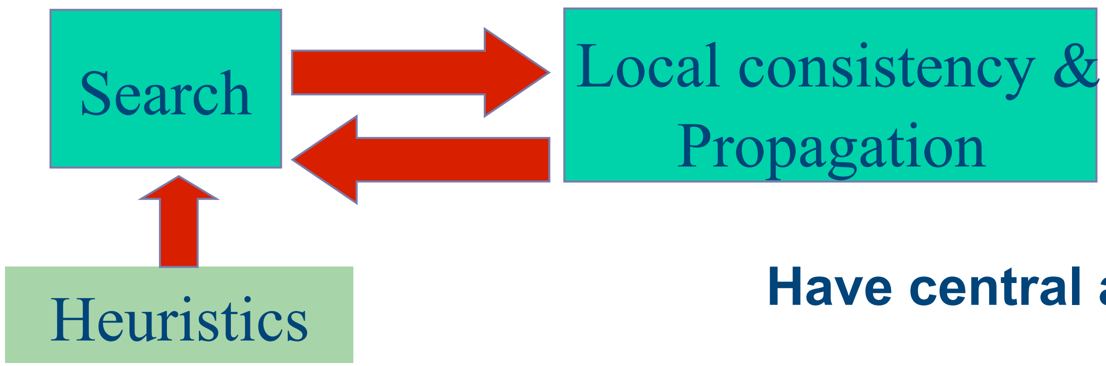
NOTE: We will not commit to any CP language/library, rather use a mathematical and/or natural notation.

PART II: Local Consistency & Constraint Propagation

Local Consistency & Constraint Propagation

PART I: The user lets the CP technology solve the CSP:

- choose a search algorithm;
- design heuristics for branching;
- integrate local consistency and propagation.



Have central affect

What exactly are they?
How do they work?

Outline

- Local Consistency
  - Arc Consistency (AC)
  - Generalised Arc Consistency (GAC)
  - Bounds Consistency (BC)
  - Higher Levels of Consistency
- Constraint Propagation
  - Propagation Algorithms
- Specialised Propagation Algorithms
  - Global Constraints
- Generalised Propagation Algorithms
  - AC algorithms

Local Consistency

- Backtrack tree search aims to extend a partial instantiation of variables to a complete and consistent one.
  - The search space is too large!
- Some inconsistent partial assignments obviously cannot be completed.
- Local consistency is a form of inference which detects inconsistent partial assignments.
  - Consequently, the backtrack search commits into less inconsistent instantiations.
- Local, because we examine individual constraints.
  - Remember that global consistency is NP-complete!

Local Consistency: An example

- $D(X_1) = \{1,2\}$, $D(X_2) = \{3,4\}$, C_1: $X_1 = X_2$, C_2: $X_1 + X_2 \geq 1$
- $X_1 = 1$
- $X_1 = 2$
- $X_2 = 3$
- $X_4 = 4$

all inconsistent partial assignments wrt the constraint $X_1 = X_2$

  - no need to check the individual assignments.
  - no need to check the other constraint.
  - unsatisfiability of the CSP can be inferred without having to search!

Several Local Consistencies

- Most popular local consistencies:
  - Arc Consistency (AC)
  - Generalised Arc Consistency (GAC)
  - Bounds Consistency (BC)
- They detect inconsistent partial assignments of the form $X_i = j$, hence:
  - j can be removed from $D(X_i)$ via propagation;
  - propagation can be implemented easily.

Arc Consistency (AC)

- Defined for binary constraints.
- A binary constraint **C** is a relation on two variables $\mathbf{X_i}$ and $\mathbf{X_j}$, giving the set of allowed combinations of values (i.e. tuples):
  - $\mathbf{C \subseteq D(X_i) \times D(X_j)}$
- C is AC iff:
  - forall $v \in D(X_i)$, exists $w \in D(X_j)$ s.t. $(v,w) \in C$.
    - $v \in D(X_i)$ is said to have a *support* wrt the constraint C.
  - forall $w \in D(X_j)$, exists $v \in D(X_i)$ s.t. $(v,w) \in C$.
    - $w \in D(X_j)$ is said to have a *support* wrt the constraint C.
- A CSP is AC iff all its binary constraints are AC.

AC: An example

- $D(X_1) = \{1,2,3\}$, $D(X_2) = \{2,3,4\}$, C: $X_1 = X_2$
- AC(C)?
  - $1 \in D(X_1)$ does not have a support.
  - $2 \in D(X_1)$ has $2 \in D(X_2)$ as support.
  - $3 \in D(X_1)$ has $3 \in D(X_2)$ as support.
  - $2 \in D(X_2)$ has $2 \in D(X_1)$ as support.
  - $3 \in D(X_2)$ has $3 \in D(X_1)$ as support.
  - $4 \in D(X_2)$ does not have a support.

Propagation!

- $X_1 = 1$ and $X_2 = 4$ are inconsistent partial assignments.
- $1 \in D(X_1)$ and $4 \in D(X_2)$ must be *removed* to achieve AC.
- $D(X_1) = \{2,3\}$, $D(X_2) = \{2,3\}$, C: $X_1 = X_2$.
  - AC(C)

Generalised Arc Consistency

- Generalisation of AC to n-ary constraints.
- A constraint **C** is a relation on k variables $\mathbf{X_1,\ldots, X_k}$:
  - $\mathbf{C \subseteq D(X_1) \times \ldots \times D(X_k)}$
- A *support* is a tuple $<d_1,\ldots,d_k> \in C$ where $d_i \in D(X_i)$.
- C is GAC iff:
  - forall X_i in $\{\mathbf{X_1,\ldots, X_k}\}$, forall $v \in D(X_i)$, v belongs to a support.
- AC is a special case of GAC.
- A CSP is GAC iff all its constraints are GAC.

GAC: An example

- $D(X_1) = \{1,2,3\}$, $D(X_2) = \{1,2\}$, $D(X_3) = \{1,2\}$
  C: alldifferent($[X_1, X_2, X_3]$)

- GAC(C)?
  - $X_1 = 1$ and $X_1 = 2$ are not supported!
- $D(X_1) = \{3\}$, $D(X_2) = \{1,2\}$, $D(X_3) = \{1,2\}$
  C: $X_1 \neq X_2 \neq X_3$
  - GAC(C)

Bounds Consistency (BC)

- Defined for totally ordered (e.g. integer) domains.
- Relaxes the domain of X_i from $D(X_i)$ to $[min(X_i)..max(X_i)]$.
- Advantages:
  - it might be easier to look for a support in a range than in a domain;
  - achieving BC is often cheaper than achieving GAC;
  - achieving BC is enough to achieve GAC for monotonic constraints.
- Disadvantage:
  - BC might not detect all GAC inconsistencies in general.

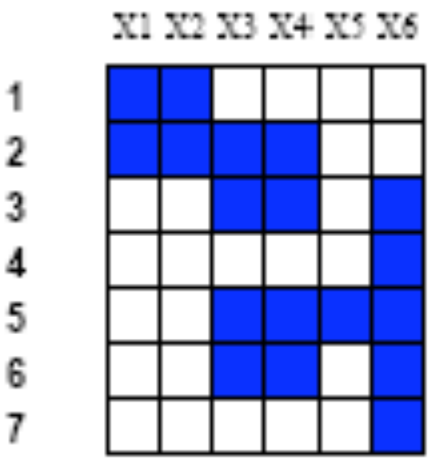
Bounds Consistency (BC)

- A constraint **C** is a relation on k variables $\mathbf{X_1,\ldots, X_k}$:
  - $\mathbf{C \subseteq D(X_1) \times \ldots \times D(X_k)}$
- A *bound support* is a tuple $<d_1,\ldots,d_k> \in C$ where $d_i \in [\min(X_i)..\max(Xi)]$.
- C is BC iff:
  - forall X_i in $\{\mathbf{X_1,\ldots, X_k}\}$, $\min(X_i)$ and $\max(X_i)$ belong to a bound support.

GAC > BC: An example

- $D(X_1) = D(X_2) = \{1,2\}$, $D(X_3) = D(X_4) = \{2,3,5,6\}$, $D(X_5) = \{5\}$, $D(X_6) = \{3,4,5,6,7\}$

  C: alldifferent($[X_1, X_2, X_3, X_4, X_5, X_6]$)
- BC(C): $2 \in D(X_3)$ and $2 \in D(X_4)$ have no support.



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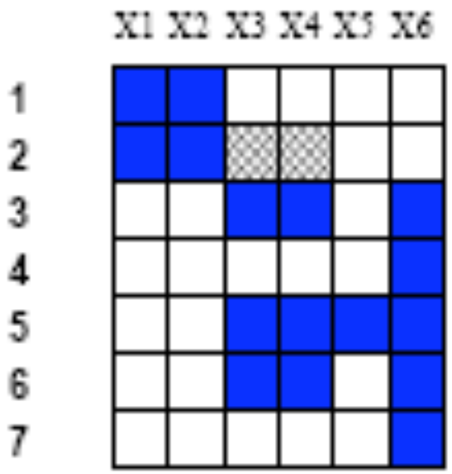


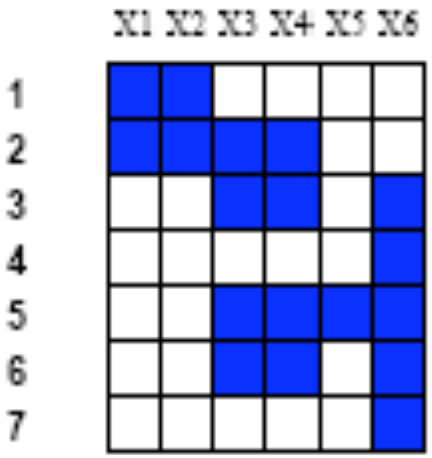
BC

GAC > BC: An example

- $D(X_1) = D(X_2) = \{1,2\}$, $D(X_3) = D(X_4) = \{2,3,5,6\}$, $D(X_5) = \{5\}$, $D(X_6) = \{3,4,5,6,7\}$

  C: alldifferent($[X_1, X_2, X_3, X_4, X_5, X_6]$)

- GAC(C): $\{2,5\} \in D(X_3)$, $\{2,5\} \in D(X_4)$, $\{3,5,6\} \in D(X_6)$ have no support.



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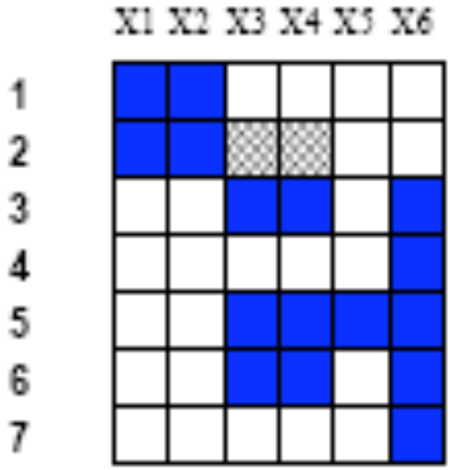
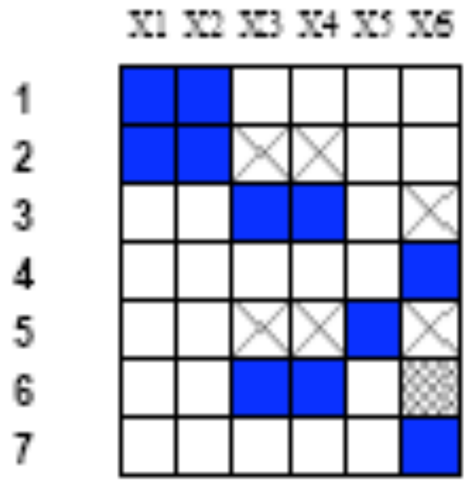


BC



GAC

GAC = BC: An example

- $D(X_1) = \{1,2,3\}$, $D(X_2) = \{1,2,3\}$, C: $X_1 < X_2$
- BC(C):
  - $D(X_1) = \{1,2\}$, $D(X_2) = \{2,3\}$
- BC(C) = GAC(C):
  - a support for $\min(X_2)$ supports all the values in $D(X_2)$.
  - a support for max(X1) supports all the values in D(X1).

Higher Levels of Consistencies

- Path consistency, k-consistencies, (i,j) consistencies, …
- Not much used in practice:
  - detect inconsistent partial assignments with more than one <variable,value> pair.
  - cannot be enforced by removing single values from domains.
- Domain based consistencies stronger than (G)AC.
  - Singleton consistencies, triangle-based consistencies, …
  - Becoming popular:
    - shaving in scheduling.

Outline

- Local Consistency
  - Arc Consistency (AC)
  - Generalised Arc Consistency (GAC)
  - Bounds Consistency (BC)
  - Higher Levels of Consistency
- Constraint Propagation
  - Constraint Propagation Algorithms
- Specialised Propagation Algorithms
  - Global Constraints
- Generalised Propagation Algorithms
  - AC Algorithms

Constraint Propagation

- Can appear under different names:
  - constraint relaxation
  - filtering algorithm
  - local consistency enforcing, …
- Similar concepts in other fields:
  - unit propagation in SAT.
- Local consistencies define properties that a CSP must satisfy after constraint propagation:
  - the operational behaviour is completely left open;
  - the only requirement is to achieve the required property on the CSP.

Constraint Propagation: A simple example

Input CSP:$D(X_1) = \{1,2\}$, $D(X_2) = \{1,2\}$, C: $X_1 < X_2$

$\downarrow$

$$\boxed{\text{A constraint propagation algorithm for enforcing AC}}$$

$\downarrow$

Output CSP:$D(X_1) = \{1\}$, $D(X_2) = \{2\}$, C: $X_1 < X_2$

We can write different algorithms with different complexities to achieve the same effect.

Constraint Propagation Algorithms

- A constraint propagation algorithm propagates a constraint C.
  - It removes the inconsistent values from the domains of the variables of C.
  - It makes C locally consistent.
  - The level of consistency depends on C:
    - GAC might be NP-complete, BC might not be possible, …

Constraint Propagation Algorithms

- When solving a CSP with multiple constraints:
  - propagation algorithms interact;
  - a propagation algorithm can wake up an already propagated constraint to be propagated again!
  - in the end, propagation reaches a fixed-point and all constraints reach a level of consistency;
  - the whole process is referred as constraint propagation.

Constraint Propagation: An example

- $D(X_1) = D(X_2) = D(X_3) = \{1,2,3\}$
  C_1: alldifferent($[X_1, X_2, X_3]$) C_2: $X_2 < 3$ C_3: $X_3 < 3$
- Let's assume:
  - the order of propagation is C_1, C_2, C_3;
  - each algorithm maintains (G)AC.
- Propagation of C_1:
  - nothing happens, C_1 is GAC.
- Propagation of C_2:
  - 3 is removed from $D(X_2)$, C_2 is now AC.
- Propagation of C_3:
  - 3 is removed from $D(X_3)$, C_3 is now AC.
- C_1 is not GAC anymore, because the supports of $\{1,2\} \in D(X_1)$ in $D(X_2)$ and $D(X_3)$ are removed by the propagation of C_2 and C_3.
- Re-propagation of C_1:
  - 1 and 2 are removed from $D(X_1)$, C_1 is now AC.

Properties of Constraint Propagation Algorithms

- It is not enough to be able to remove inconsistent values from domains.
- A constraint propagation algorithm must *wake up* when necessary, otherwise may not achieve the desired local consistency property.
- Events that trigger a constraint propagation:
  - when the domain of a variable changes;
  - when a variable is assigned a value;
  - when the minimum or the maximum values of a domain changes.

Outline

- Local Consistency
  - Arc Consistency (AC)
  - Generalised Arc Consistency (GAC)
  - Bounds Consistency (BC)
  - Higher Levels of Consistency
- Constraint Propagation
  - Propagation Algorithms
- Specialised Propagation Algorithms
  - Global Constraints
    - Decompositions
    - Ad-hoc algorithms
- Generalised Propagation Algorithms
  - AC Algorithms

Specialised Propagation Algorithms

- A constraint propagation algorithm can be general or specialised:
  - general, if it is applicable to any constraint;
  - specialised, if it is specific to a constraint.
- Specialised algorithms:
  - Disadvantage:
    - has limited use;
    - is not always easy to develop one.
  - Advantages:
    - exploits the constraint semantics;
    - is potentially more efficient than a general algorithm.
- Worth developing specialised algorithms for recurring constraints with a reasonable semantics.

Specialised Propagation Algorithms

- C: $X_1 \leq X_2$
- Observation:
  - a support of $\min(X_2)$ supports all the values in $D(X_2)$;
  - a support of $\max(X_1)$ supports all the values in $D(X_1)$.
- Propagation algorithm:
  - filter $D(X_1)$ s.t. $\max(X_1) \leq \max(X_2)$;
  - filter $D(X_2)$ s.t. $\min(X_1) \leq \min(X_2)$.
- The result is GAC (and thus BC).

Example

- $D(X_1) = \{3, 4, 7, 8\}$, $D(X_2) = \{1, 2, 3, 5\}$, C: $X_1 \leq X_2$

Example

- $D(X_1) = \{3, 4, 7, 8\}$, $D(X_2) = \{1, 2, 3, 5\}$, C: $X_1 \leq X_2$
- Propagation:
  - filter $D(X_1)$ s.t. $\max(X_1) \leq \max(X_2)$;

Example

- $D(X_1) = \{3, 4, \cancel{7}, \cancel{8}\}$, $D(X_2) = \{1, 2, 3, 5\}$, C: $X_1 \leq X_2$
- Propagation:
  - filter $D(X_1)$ s.t. $\max(X_1) \leq \max(X_2)$;

Example

- $D(X_1) = \{3, 4, \cancel{7}, \cancel{8}\}$, $D(X_2) = \{1, 2, 3, 5\}$, C: $X_1 \leq X_2$
- Propagation:
  - filter $D(X_1)$ s.t. $\max(X_1) \leq \max(X_2)$;
  - filter $D(X_2)$ s.t. $\min(X_1) \leq \min(X_2)$;

Example

- $D(X_1) = \{3, 4, \cancel{7}, \cancel{8}\}$, $D(X_2) = \{\cancel{1}, \cancel{2}, 3, 5\}$, C: $X_1 \leq X_2$
- Propagation:
  - filter $D(X_1)$ s.t. $\max(X_1) \leq \max(X_2)$;
  - filter $D(X_2)$ s.t. $\min(X_1) \leq \min(X_2)$;

Global Constraints

- Many real-life constraints are complex and not binary.
  - Specialised algorithms are often developed for such constraints!
- A complex and n-ary constraint which encapsulates a specialised propagation algorithm is called a global constraint.

Examples of Global Constraints

- Alldifferent constraint:
  - alldifferent($[X_1, X_2, \ldots, X_n]$) holds iff
    $X_i \neq X_j$ for $i < j \in \{1,\ldots,n\}$
  - useful in a variety of context
    - Timetabling (e.g. exams with common students must occur at different times)
    - Tournament scheduling (e.g. a team can play at most once in a week)
    - Configuration (e.g. a particular product cannot have repeating components)
    - …

Beyond Alldifferent

- NValue constraint:
  - one generalisation of alldifferent
  - nvalue([X_1, X_2, …, X_n], N) holds iff $N = |\{X_i \mid 1 \leq i \leq n\}|$
  - nvalue([1, 2, 2, 1, 3], 3)
  - alldifferent when N = n
  - Useful when values represent resources and we want to limit the usage of resources. E.g.,
    - Minimise the total number of resources used;
    - The total number of resources used must be between a specific interval;
    - …

Beyond Alldifferent

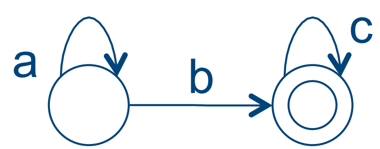
- Global cardinality constraint:
  - another generalisation of alldifferent
  - gcc([X_1, X_2, …, X_n], [v_1, …, v_m], [O_1, …, O_m]) iff
    forall $j \in \{1,\ldots, m\}$ $O_j = |\{X_i \mid X_i = v_j, 1 \leq i \leq n\}|$
  - gcc([1, 1, 3, 2, 3], [1, 2, 3, 4], [2, 1, 2, 0])
  - Useful again when values represent resources
  - We can now limit the usage of each resource individually. E.g.,
    - Resource 1 can be used at most three times
    - Resource 2 can be used min 2 max 5 times
    - …

Symmetry Breaking Constraints

- Consider the following scenario:
  - $[X_1, X_2, \ldots, X_n]$ and $[Y_1, Y_2, \ldots, Y_n]$ represent the 2 day event assignments of a conference
  - Each day has n slots and the days are indistinguishable
  - Need to avoid symmetric assignments
- Global constraints developed for this purpose are called symmetry breaking constraints.
- Lexicographic ordering constraint:
  - $\text{lex}([X_1, X_2, \ldots, X_n], [Y_1, Y_2, \ldots, Y_n])$ holds iff:
    $X_1 < Y_1$ OR $(X_1 = Y_1$ AND $X_2 < Y_2)$ OR …
    $(X_1 = Y_1$ AND $X_2 = Y_2$ AND …. AND $X_n \leq Y_n)$
  - lex ([1, 2, 4],[1, 3, 3])

Grammar Constraints

- We might sometimes want a sequence of variables obey certain patterns. E.g.,
  - regulations in scheduling
- A promising direction in CP is the ability of modelling problems via automata/grammar.
- Global constraints developed for this purpose are called grammar constraints.
- Regular constraint:
  - regular([X_1, X_2, …, X_n], A) holds iff <X_1, X_2, …, X_n> forms a string accepted by the DFA A (which accepts a regular language).
  - regular([a, a, b], A), regular([b], A), regular([b, c, c, c, c, c], A) with A



Specialised Algorithms for Global Constraints

- How do we develop specialised algorithms for global constraints?
- Two main approaches:
  - constraint decomposition
  - ad-hoc algorithm

Constraint Decomposition

- A global constraint is decomposed into smaller and simpler constraints each which has a known propagation algorithm.
- Propagating each of the constraints gives a propagation algorithm for the original global constraint.
  - A very effective and efficient method for some global constraints

Decomposition of Among

- among([X_1, X_2, …, X_n], [d_1, d_2, …, d_m], N) holds iff
  $N = |\{X_i \mid X_i \in \{d_1, d_2, \ldots, d_m\}\ 1 \leq i \leq n\}|$
- Decomposition:
  - B_i with $D(B_i) = \{0, 1\}$ for $1 \leq i \leq n$
  - C_i: $B_i = 1 \leftrightarrow X_i \in \{d_1, d_2, \ldots, d_m\}$ for $1 \leq i \leq n$
  - $\sum_i B_i = N$
- AC(C_i) for $1 \leq i \leq n$ and BC($\sum_i B_i = N$) ensures GAC on among.

Decomposition of Lex

- $lex([X_1, X_2, \ldots, X_n], [Y_1, Y_2, \ldots, Y_n])$
- Decomposition:
  - B_i with $D(B_i) = \{0, 1\}$ for $1 \le i \le n+1$ to indicate the vectors have been ordered by position i-1
  - $B_1 = 0$
  - C_i: ($B_i = B_{i+1} = 0$ AND $X_i = Y_i$) OR ($B_i = 0$ AND $B_{i+1} = 1$ AND $X_i < Y_i$) OR ($B_i = B_{i+1} = 1$) for $1 \le i \le n$
- $GAC(C_i)$ ensures GAC on lex.

Constraint Decompositions

- May not always provide an effective propagation.
- Often GAC on the original constraint is stronger than (G)AC on the constraints in the decomposition.
- E.g., C: alldifferent($[X_1, X_2, \ldots, X_n]$)
- Decomposition following the definition:
  - C_{ij}: $X_i \neq X_j$ for $i < j \in \{1,\ldots,n\}$
  - AC on the decomposition is weaker than GAC on alldifferent.
  - E.g., $D(X_1) = D(X_2) = D(X_3) = \{1,2\}$, C: alldifferent($[X_1, X_2, X_3]$)
  - C_{12}, C_{13}, C_{23} are all AC, but C is not GAC.

Constraint Decompositions

- E.g., C: lex($[X_1, X_2, \ldots, X_n], [Y_1, Y_2, \ldots, Y_n]$)
- OR decomposition:
  - $X_1 < Y_1$ OR $(X_1 = Y_1$ AND $X_2 < Y_2)$ OR …
    $(X_1 = Y_1$ AND $X_2 = Y_2$ AND …. AND $X_n \leq Y_n)$
  - AC on the decomposition is weaker than GAC on lex.
  - E.g., $D(X_1) = \{0, 1, 2\}$, $D(X_2) = \{0, 1\}$, $D(Y_1) = \{0, 1\}$, $D(Y_2) = \{0, 1\}$
    C: Lex($[X_1, X_2], [Y_1, Y_2]$)
  - C is not GAC but the decomposition does not prune anything.

Constraint Decompositions

- AND decomposition of lex($[X_1, X_2, \ldots, X_n]$, $[Y_1, Y_2, \ldots, Y_n]$):
  - $X_1 \leq Y_1$ AND $(X_1 = Y_1 \rightarrow X_2 \leq Y_2)$ AND …
    $(X_1 = Y_1$ AND $X_2 = Y_2$ AND …. $X_{n-1} = Y_{n-1} \rightarrow X_n \leq Y_n)$
  - AC on the decomposition is weaker than GAC on lex.
  - E.g., $D(X_1) = \{0, 1\}$, $D(X_2) = \{0, 1\}$, $D(Y_1) = \{1\}$, $D(Y_2) = \{0\}$
    C: Lex($[X_1, X_2]$, $[Y_1, Y_2]$)
  - C is not GAC but the decomposition does not prune anything.

Constraint Decompositions

- Different decompositions of a constraint may be incomparable.
  - Difficult to know which one gives a better propagation for a given instance of a constraint.
- C: $Lex([X_1, X_2], [Y_1, Y_2])$
  - $D(X_1) = \{0, 1\}$, $D(X_2) = \{0, 1\}$, $D(Y_1) = \{1\}$, $D(Y_2) = \{0\}$
    - AND decomposition is weaker than GAC on lex, whereas OR decomposition maintains GAC.
  - $D(X_1) = \{0, 1, 2\}$, $D(X_2) = \{0, 1\}$, $D(Y_1) = \{0, 1\}$, $D(Y_2) = \{0, 1\}$
    - OR decomposition is weaker than GAC on lex, whereas OR decomposition maintains GAC.

Constraint Decompositions

- Even if effective, may not always provide an efficient propagation.
- Often GAC on a constraint via a specialised algorithm is maintained faster than (G)AC on the constraints in the decomposition.

Constraint Decompositions

- C: Lex([X_1, X_2], [Y_1, Y_2])
  - $D(X_1) = \{0, 1\}$, $D(X_2) = \{0, 1\}$, $D(Y_1) = \{1\}$, $D(Y_2) = \{0\}$
    - AND decomposition is weaker than GAC on lex, whereas OR decomposition maintains GAC
  - $D(X_1) = \{0, 1, 2\}$, $D(X_2) = \{0, 1\}$, $D(Y_1) = \{0, 1\}$, $D(Y_2) = \{0, 1\}$
    - OR decomposition is weaker than GAC on lex, whereas OR decomposition maintains GAC
- AND or OR decompositions have complementary strengths!
- – Combining them gives us a decomposition which maintains GAC on lex.
- Too many constraints to post and propagate!
- A dedicated algorithm runs amortised in O(1).

Dedicated Algorithms

- Dedicated ad-hoc algorithms provide effective and efficient propagation.
- Often:
  - GAC is maintained in polynomial time.
  - Many more inconsistent values are detected compared to the decompositions.

Benefits of Global Constraints

- Modelling benefits
  - Reduce the gap between the problem statement and the model.
  - Capture recurring modelling patterns.
  - May allow the expression of constraints that are otherwise not possible to state using primitive constraints (semantic).
- Solving benefits
  - More inference in propagation (operational).
  - More efficient propagation (algorithmic).

Dedicated Algorithm for Alldifferent

- GAC algorithm based on matching theory.
  - Establishes a relation between the solutions of the constraint and the properties of a graph.
  - Runs in time $O(dn^{1.5})$.
- Value graph: bipartite graph between variables and their possible values.
- Matching: set of edges with no two edges having a node in common.
- Maximal matching: largest possible matching.

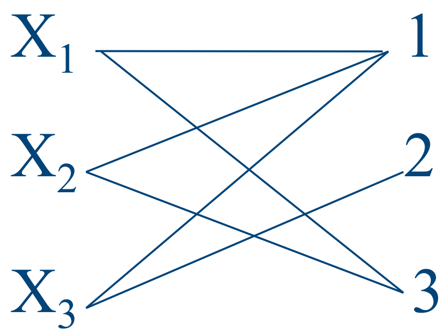
Dedicated Algorithm for Alldifferent

- An assignment of values to the variables $X_1, X_2, \ldots, X_n$ is a solution iff it corresponds to a maximal matching.
  - Edges that do not belong to a maximal matching can be deleted.
- The challenge is to compute such edges efficiently.
  - Exploit concepts like strongly connected components, alternating paths, …

Dedicated Algorithm for Alldifferent

- $D(X_1) = \{1,3\}$, $D(X_2) = \{1,3\}$, $D(X_3)= \{1,2\}$

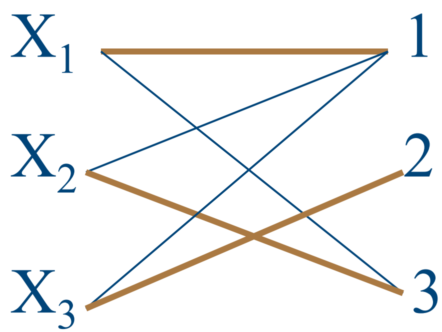
Variable-value graph



Dedicated Algorithm for Alldifferent

- $D(X_1) = \{1,3\}$, $D(X_2) = \{1,3\}$, $D(X_3)= \{1,2\}$

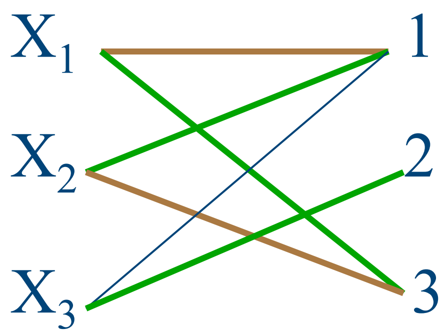
A maximal matching



Dedicated Algorithm for Alldifferent

- $D(X_1) = \{1,3\}$, $D(X_2) = \{1,3\}$, $D(X_3)= \{\cancel{1},2\}$

Another maximal matching



Does not belong to any maximal matching

Dedicated Algorithms

- Is it always easy to develop a dedicated algorithm for a given constraint?
- There's no single recipe!
- A nice semantics often gives us a clue!
  - Graph Theory
  - Flow Theory
  - Combinatorics
  - Complexity Theory, …
- GAC may as well be NP-hard!
  - In that case, algorithms which maintain weaker consistencies (like BC) are of interest.

GAC for Nvalue Constraint

- nvalue($[X_1, X_2, \ldots, X_n]$, N) holds iff $N = |\{X_i \mid 1 \leq i \leq n\}|$
- Reduction from 3 SAT.
  - Given a Boolean fomula in k variables (labelled from 1 to k) and m clauses, we construct an instance of nvalue($[X_1, X_2, \ldots, X_{k+m}]$, N):
    - $D(X_i) = \{i, i'\}$ for $i \in \{1,\ldots, k\}$ where X_i represents the truth assignment of the SAT variables;
    - X_i where $i > k$ represents a SAT clause (disjunction of literals);
    - for a given clause like x V y' V z, $D(X_i) = \{x, y', z\}$.
  - By construction, $X_1, \ldots, X_k$ will consume all the k distinct values.
  - When N = k, nvalue has a solution iff the original SAT problem has a satisfying assignment.
    - Otherwise we will have more than k distinct values.
    - Hence, testing a value for support is NP-complete, and enforcing GAC is NP-hard!

GAC for Nvalue Constraint

- E.g., C_1: (a OR b' OR c) AND
  C_2: (a' OR b OR d) AND
  C_3: (b' OR c' OR d)
- The formula has 4 variables (a, b, c, d) and 3 clauses (C_1, C_2, C_3).
- We construct nvalue([X_1, X_2, …, X_7], 4) where:
  - $D(X_1) = \{a, a'\}$, $D(X_2) = \{b, b'\}$, $D(X_3) = \{c, c'\}$, $D(X_4) = \{d, d'\}$, $D(X_5) = \{a, b', c\}$, $D(X_6) = \{a', b, d\}$, $D(X_7) = \{b', c', d\}$
- An assignment to X_1, …, X_4 will consume 4 distinct values.
- Not to exceed 4 distinct values, the rest of the variables must have intersecting values with X_1, …, X_4.
- Such assignments will make the SAT formula TRUE.

Outline

- Local Consistency
  - Arc Consistency (AC)
  - Generalised Arc Consistency (GAC)
  - Bounds Consistency (BC)
  - Higher Levels of Consistency
- Constraint Propagation
  - Propagation Algorithms
- Specialised Propagation Algorithms
  - Global Constraints
    - Decompositions
    - Ad-hoc algorithms
- Generalised Propagation Algorithms
  - AC Algorithms

Generalised Propagation Algorithms

- Not all constraints have nice semantics we can exploit to devise an efficient specialised propagation algorithm.
- Consider a product configuration problem:
  - compatibility constraints on hardware components:
    - only certain combinations of components work together.
  - compatibility may not be a simple pairwise relationship:
    - video cards supported function of motherboard, CPU, clock speed, O/S, ...

Production Configuration Problem

- 5-ary constraint:
  - Compatible (motherboard345, intelCPU, 2GHz, 1GBRam, 80GBdrive).
  - Compatible (motherboard346, intelCPU, 3GHz, 2GBRam, 100GBdrive).
  - Compatible (motherboard346, amdCPU, 2GHz, 2GBRam, 100GBdrive).
  - …

Crossword Puzzle

- Constraints with different arity:
  - $\text{Word}_1\ ([X_1,X_2,X_3])$
  - $\text{Word}_2\ ([X_1,X_{13},X_{16}])$
  - …
- No simple way to decide acceptable words other than to put them in a table.

GAC Schema

- A generic propagation algorithm.
  - Enforces GAC on an n-ary constraint given by:
    - a set of allowed tuples;
    - a set of disallowed tuples;
    - a predicate answering if a constraint is satisfied or not.
  - Sometimes called the “table” constraint:
    - user supplies table of acceptable values.
- Complexity: $O(ed^n)$ time
- Hence, n cannot be too large!
  - Many solvers limits it to 3 or so.

Arc Consistency Algorithms

- Generic AC algorithms with different complexities and advantages:
  - AC3
  - AC4
  - AC6
  - AC2001
  - …

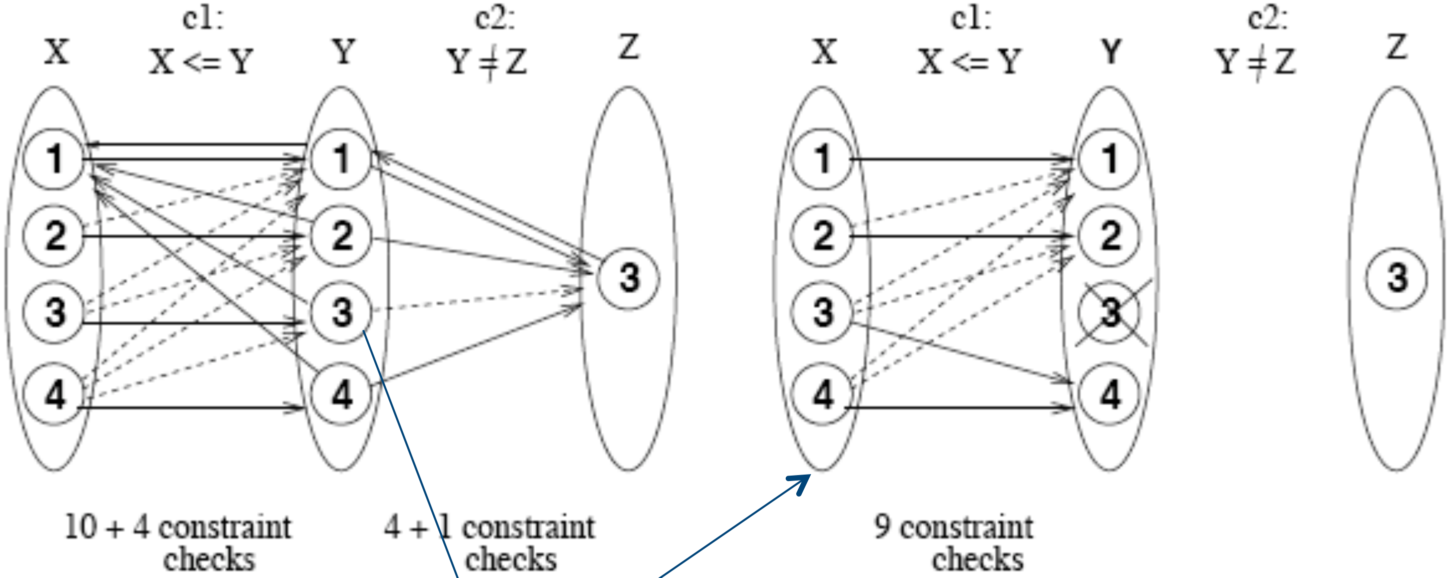
AC-3

- Idea:
  - Revise (X_i, C): removes unsupported values of X_i and returns TRUE.
  - Place each (X_i, C) where X_i participates to C and its domain is potentially not AC, in a queue Q;
  - While Q is not empty:
    - Select and remove (X_i, C) from Q;
    - If revise(X_i, C) then
      - If $D(X_i) = \{ \}$ then return FALSE;
      - else place $\{(X_j, C') \mid X_i, X_j$ participate in some $C'\}$ into Q.

AC-3

- AC-3 achieves AC on binary CSPs in $O(ed^3)$ time and $O(e)$ space.
  - Time complexity is not optimal ☹
  - Revise does not remember anything about past computations and re-does unnecessary work.

AC-3



(a)

(b)

(X, C_1) is put in Q

only check of X ← 3 was necessary!

AC-4

- Stores max. amount of info in a preprocessing step so as to avoid redoing the same constraints checks.
- Idea:
  - Start with an empty queue Q.
  - Maintain counter[X_i, v_j, X_k] where X_i, X_k participate in a constraint C_{ik} and $v_j \in D(X_i)$
    - Stores the number of supports for $X_i \leftarrow v_j$ on C_{ik}.
  - Place all supports of $X_i \leftarrow v_j$ (in all constraints) in a list S[X_i, v_j].

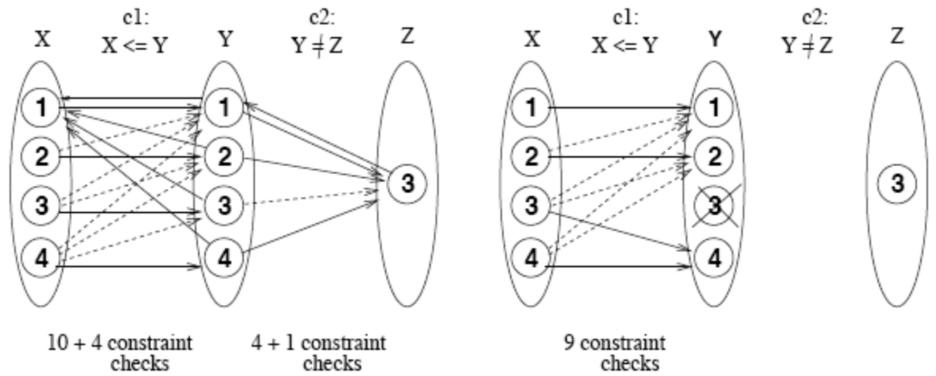
AC-4

- Initialisation:
  - All possible constraint checks are performed.
  - Each time a support for $X_i \leftarrow v_j$ is found, the corresponding counters and lists are updated.
  - Each time a support for $X_i \leftarrow v_j$ is not found, remove v_j from $D(X_i)$ and place (X_i, v_j) in Q for future propagation.
  - If $D(X_i) = \{ \}$ then return FALSE.

AC-4

- Propagation:
  - While Q is not empty:
    - Select and remove (X_i, v_j) from Q;
    - For each (X_k, v_t) in $S[X_i, v_j]$
      - If $v_t \in D(X_k)$ then
        - decrement counter$[X_k, v_t, X_i]$
        - If counter$[X_k, v_t, X_i] = 0$ then
          - Remove v_t from $D(X_k)$; add (X_k, v_t) to Q
          - If $D(X_k) = \{ \}$ then return FALSE.

AC-4



(a) (b)

No additional constraint check!

(y,3) is put in Q

$$
\begin{array}{lll}
\texttt{counter}[x,1,y]=4 & \texttt{counter}[y,1,x]=1 & \texttt{counter}[y,1,z]=1\\
\texttt{counter}[x,2,y]=3 & \texttt{counter}[y,2,x]=2 & \texttt{counter}[y,2,z]=1\\
\texttt{counter}[x,3,y]=2 & \texttt{counter}[y,3,x]=3 & \texttt{counter}[y,3,z]=0\\
\texttt{counter}[x,4,y]=1 & \texttt{counter}[y,4,x]=4 & \texttt{counter}[y,4,z]=1\\
 & & \texttt{counter}[z,3,y]=3
\end{array}
$$

$$
\begin{aligned}
S[x,1] &= \{(y,1),(y,2),(y,3),(y,4)\}\\
S[x,2] &= \{(y,2),(y,3),(y,4)\}\\
S[x,3] &= \{(y,3),(y,4)\}\\
S[x,4] &= \{(y,4)\}
\end{aligned}
$$

$$
\begin{aligned}
S[y,1] &= \{(x,1),(z,3)\}\\
S[y,2] &= \{(x,1),(x,2),(z,3)\}\\
S[y,3] &= \{(x,1),(x,2),(x,3)\}\\
S[y,4] &= \{(x,1),(x,2),(x,3),(x,4),(z,3)\}\\
S[z,3] &= \{(y,1),(y,2),(y,4)\}
\end{aligned}
$$

AC-4

- AC-3 achieves AC on binary CSPs in $O(ed^2)$ time and $O(ed^2)$ space.
  - Time complexity is optimal ☺
  - Space complexity is not optimal ☹
- AC-6 and AC-2001 achieve AC on binary CSPs in $O(ed^2)$ time and $O(ed)$ space.
  - Time complexity is optimal ☺
  - Space complexity is optimal ☺

PART IV: Search Algorithms

Outline

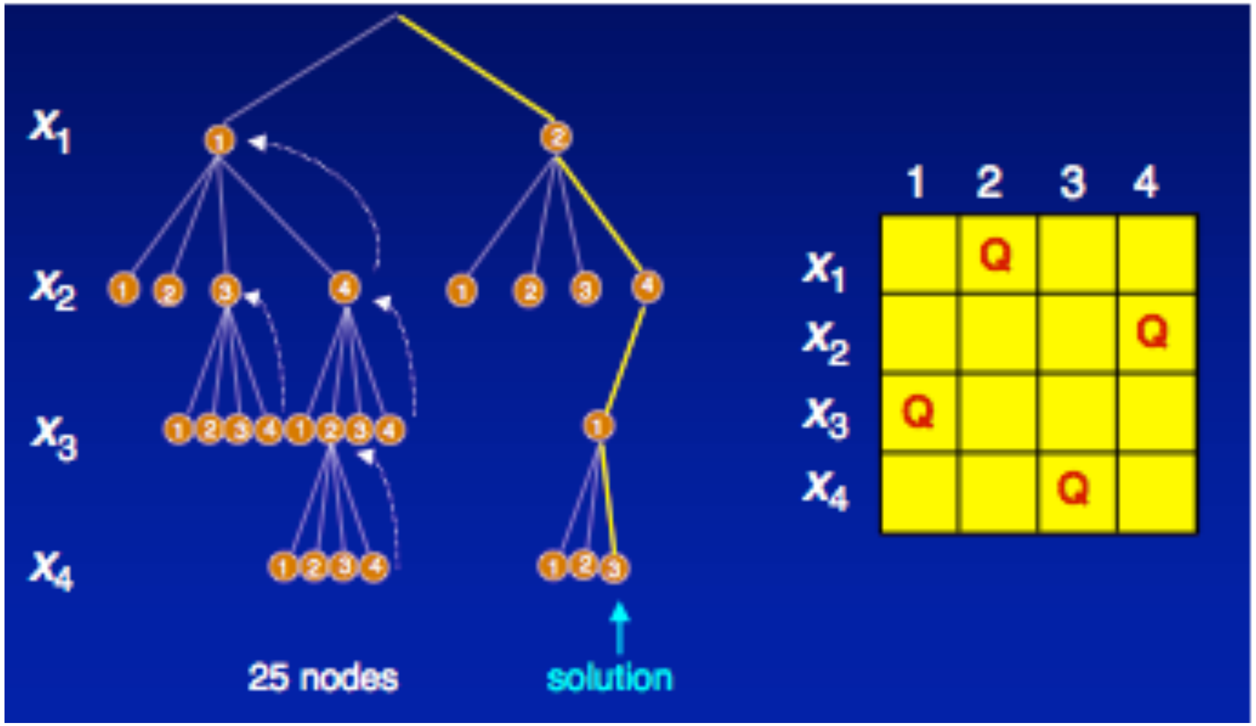
- Depth-first Search Algorithms
  - Chronological Backtracking
  - Conflict Directed Backjumping
  - Dynamic Backtracking
  - Branching Strategies
  - Heuristics
- Best-First Search Algorithms
  - Limited Discrepancy Search

Depth-first Search Algorithms

- Backtracking tree search algorithms essentially perform depth-first traversal of a search tree.
  - Every node represents a decision made on a variable.
  - At each node:
    - check every completely assigned constraint;
    - If consistent continue down in the tree;
    - otherwise prune the underlying subtrees and backtrack to an uninstantiated variable that still has alternative values.

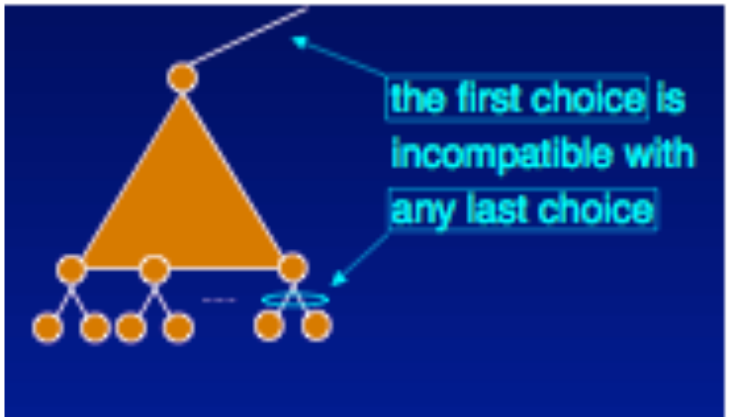
Chronological Backtracking

- Backtracks to the most recent variable.



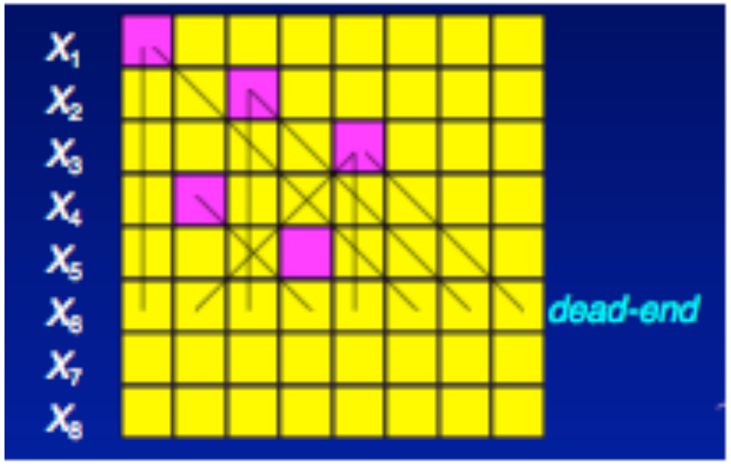
Chronological Backtracking

- Suffers from trashing.
  - The same failure can be remade an exponential number of times.



Non-Chronological Backtracking

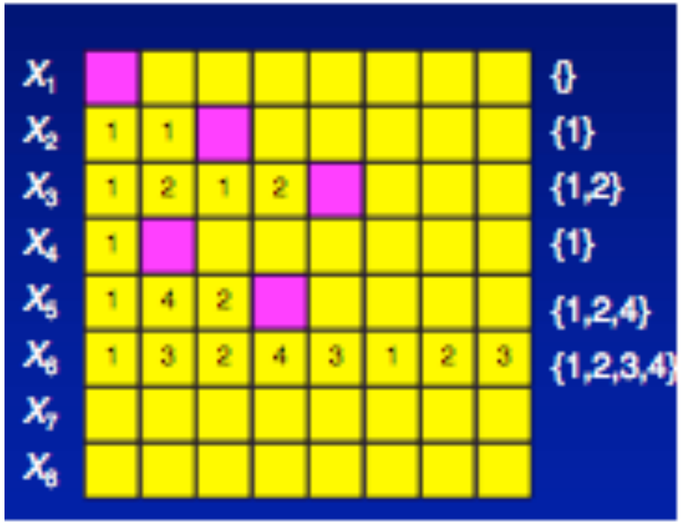
- Backtrack on a culprit variable.
- E.g.,



  - Backtracking to X_5 is pointless.
  - Better to backtrack on X_4.

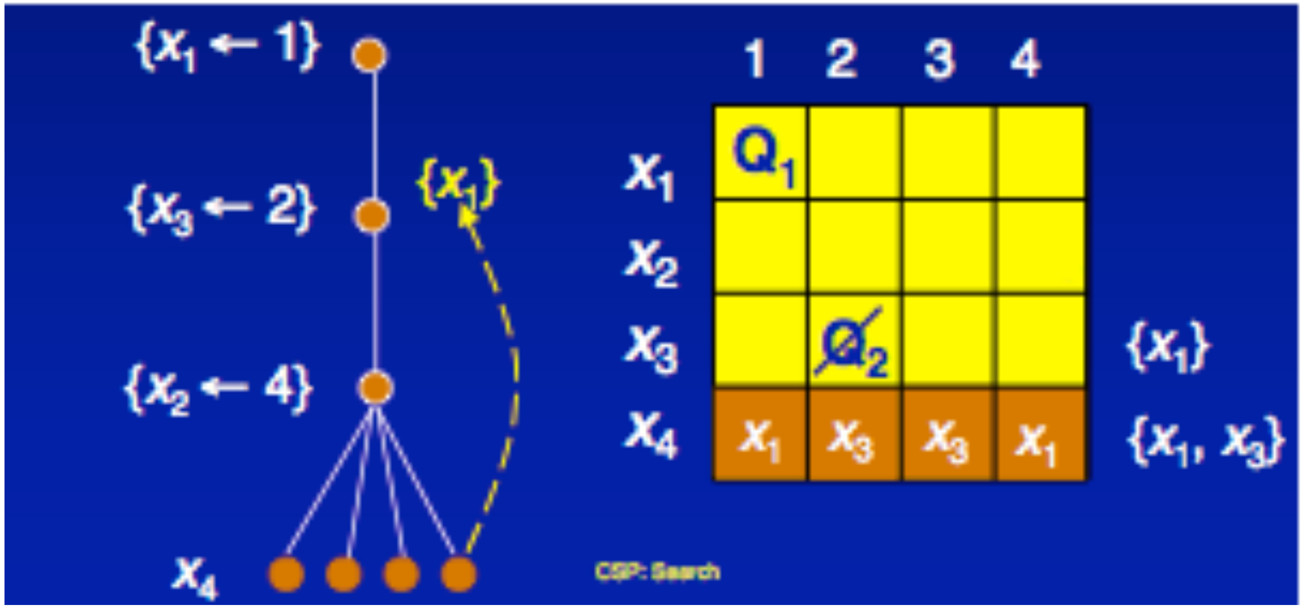
Conflict Sets

- CS(X_k): assigned variables in conflict with some value of X_k.



Conflict Directed Backjumping

- Backtracks to the last variable in the conflict set.
- Intermediate decisions are removed.



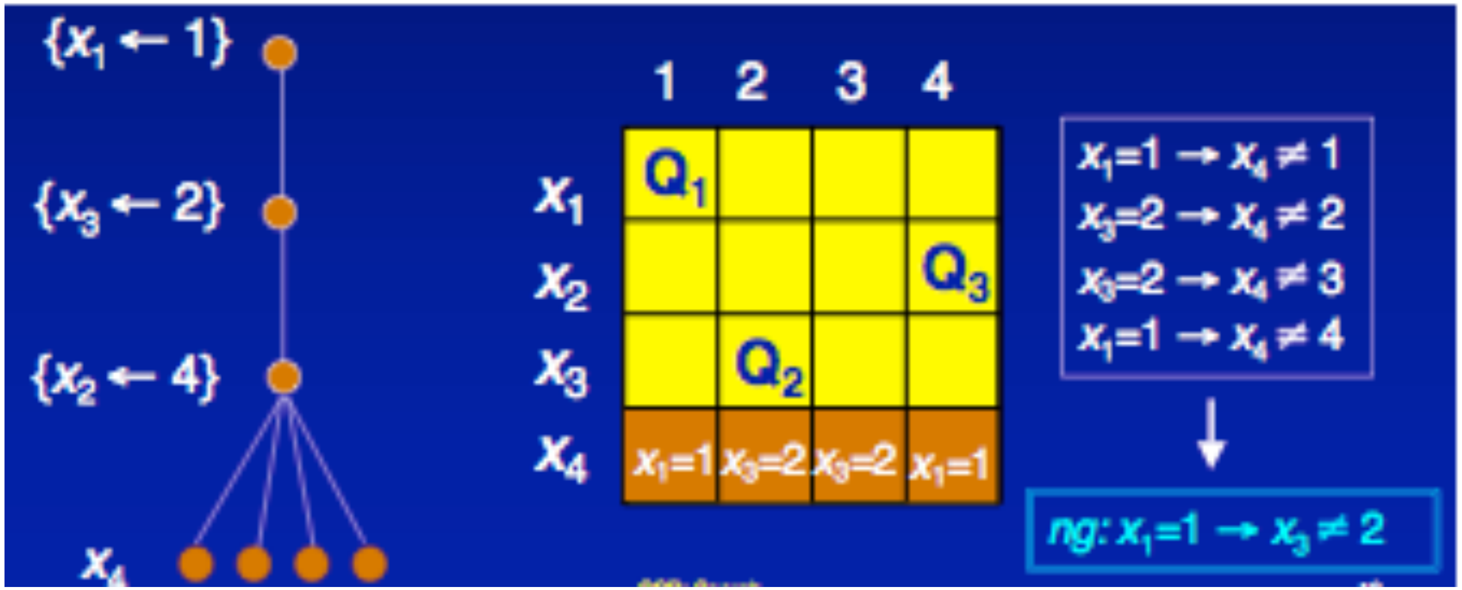
No-goods

- Subset of incompatible assignments.
- E.g., map colouring problem.
  - X_1, X_2, X_3 are adjacent with $D = \{1, 2\}$.
  - ($X_1 = a$ and $X_3 = a$) or equivalently ($X_1 = a \rightarrow X_3 \neq a$) is a no-good.
- No-good resolution:

$$\left.\begin{array}{l} X_1 = a \rightarrow X_3 \neq a \\ X_2 = b \rightarrow X_3 \neq b \end{array}\right\} \quad X_1 = a \rightarrow X_2 \neq b$$

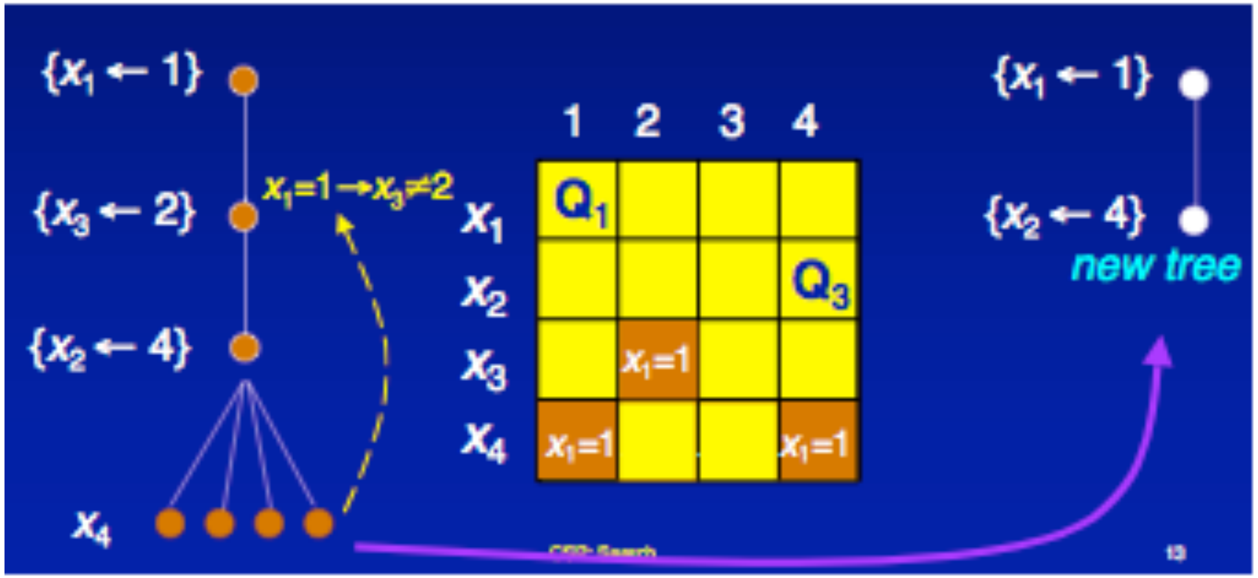
Dynamic Backtracking

- One no-good for each incompatible value is maintained.
  - Empty domain: new no-good by no-good resolution.
  - Backtrack to the variable in the right hand side of the no-good.



Dynamic Backtracking

- Backtracks to the last decision responsible for the dead-end.
- Intermediate decisions are not removed.

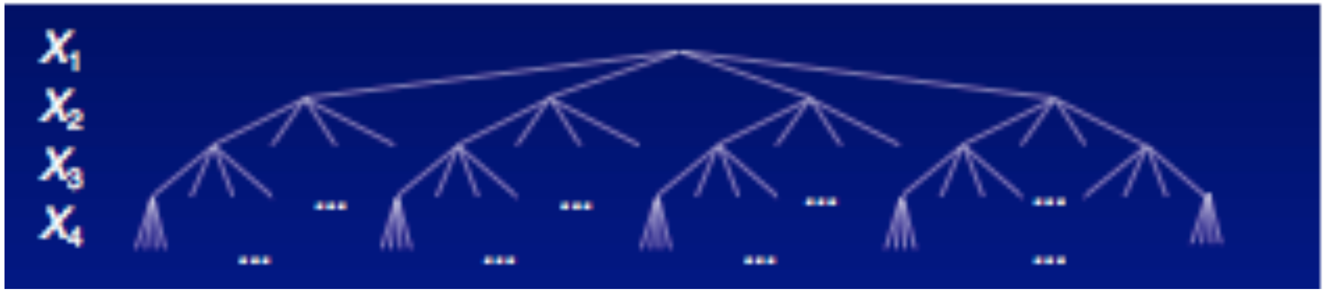


Branching Strategies

- The method of extending a node in the search tree.
  - Usually consists of posting a unary constraint on a chosen variable X_i.
  - X_i & the ordering of the branches are chosen by the heuristics.
- D-way branching:
  - One branch is generated for each $v_j \in D(X_i)$ by $X_i \leftarrow v_j$.
- 2-way branching:
  - 2 branches are generated for each $v_j \in D(X_i)$ by $X_i \leftarrow v_j$ and $X_i \leftarrow\backslash\, v_j$.
- Domain splitting:
  - k branches are generated by $X_i \in D_j$ where $D_1 \ldots D_k$ are partitions of D_i.

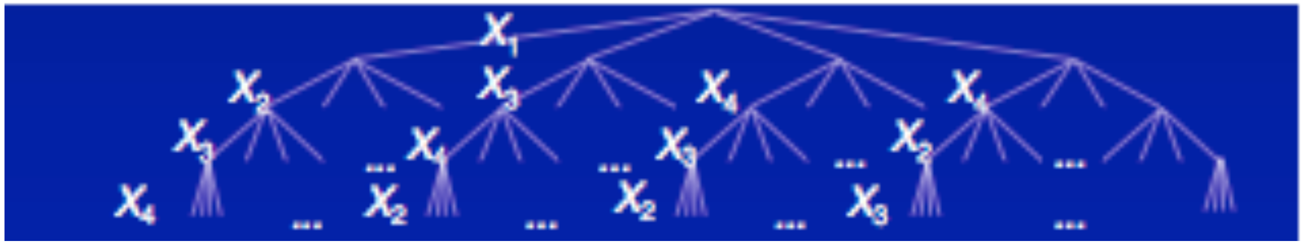
Variable and Value Ordering Heuristics

- Guide the search.
- Problem specific vs generic heuristics.
- Static Heuristics:
  - a variable is associated with each level;
  - branches are generated in the same order all over the tree;
  - calculated once and for all before search starts, hence cheap to evaluate.



Variable and Value Ordering Heuristics

- Dynamic Heuristics:
  - at any node, any variable & branch can be considered;
  - decided dynamically during search, hence costly;
  - takes into account the current state of the search tree.



Variable Ordering Heuristics

- Fail-first principle: to succeed, try first where you are most likely to fail.
- Min domain (dom):
  - choose next the variable with minimum domain.
- Most constrained (deg):
  - choose next the variable involved in most number of constraints.
- Combinations
  - dom + deg; dom / deg

Value Ordering Heuristics

- Succeed-first principle: choose next the value most likely to be part of a solution.
  - Approximating the number of solutions.
  - Looking at the remaining domain sizes when a value is assigned to a variable.

Problems with Depth-first Search

- The branches out of a node, ordered by a value ordering heuristic, are explored in left-to-right order, the left-most branch being the most promising.
- For many problems, heuristics are more accurate at deep nodes.
- Depth-first search:
  - puts tremendous burden on the heuristics early in the search and light burden deep in the search;
  - consequently mistakes made near the root of the tree can be costly to correct.
- Best-first search strategy is of interest.

Limited Discrepancy Search

- A discrepancy is the case where the search does not follow the value ordering heuristic and thus does not take the left-most branch out of a node.
- LDS:
  - Trusts the value ordering heuristic and gives priority to the left branches.
  - Iteratively searches the tree by increasing number of discrepancies, preferring discrepancies that occur near the root of the tree.

Limited Discrepancy Search

- The search recovers from mistakes made early in the search.

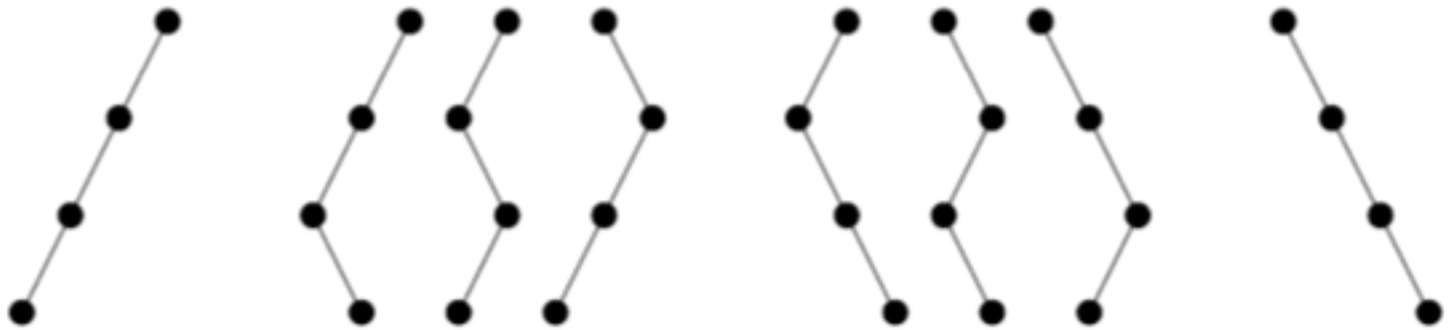


Figure 1: Paths with 0, 1, 2, and 3 Discrepancies in a Depth 3 Binary Tree

PART IV: Some Useful Pointers about CP

(Incomplete) List of Advanced Topics

- Modelling
- Global constraints, propagation algorithms
- Search algorithms
- Heuristics
- Symmetry breaking
- Optimisation
- Local search
- Soft constraints, preferences
- Temporal constraints
- Quantified constraints
- Continuous constraints
- Planning and scheduling
- SAT
- Complexity and tractability
- Uncertainty
- Robustness
- Structured domains
- Randomisation
- Hybrid systems
- Applications
- Constraint systems
- No good learning
- Explanations
- Visualisation

Literature

- Books
  - Handbook of Constraint Programming

    F. Rossi, P. van Beek, T. Walsh (eds), Elsevier Science, 2006.

    Some online chapters:

    Chapter 1 - Introduction

    Chapter 3 - Constraint Propagation

    Chapter 6 - Global Constraints

    Chapter 10 - Symmetry in CP

    Chapter 11 - Modelling

Literature

- Books
  - Constraint Logic Programming Using Eclipse
    K. Apt and M. Wallace, Cambridge University Press, 2006.
  - Principles of Constraint Programming
    K. Apt, Cambridge University Press, 2003.
  - Constraint Processing
    Rina Dechter, Morgan Kaufmann, 2003.
  - Constraint-based Local Search
    Pascal van Hentenryck and Laurent Michel, MIT Presss, 2005.
  - The OPL Optimization Programming Languages
    Pascal Van Hentenryck, MIT Press, 1999.

Literature

- People
  - Barbara Smith
    - Modelling, symmetry breaking, search heuristics
    - Tutorials and book chapter
  - Christian Bessiere
    - Constraint propagation
    - Global constraints
      - Nvalue constraint
    - Book chapter
  - Jean-Charles Regin
    - Global constraints
      - Alldifferent, global cardinality, cardinality matrix
  - Toby Walsh
    - Modelling, symmetry breaking, global constraints
    - Various tutorials

Literature

- **Journals**
  - Constraints
  - Artificial Intelligence
  - Journal of Artificial Intelligence Research
  - Journal of Heuristics
  - Intelligenza Artificiale (AI*IA)
  - Informs Journal on Computing
  - Annals of Mathematics and Artificial Intelligence

Literature

- **Conferences**
  - Principles and Practice of Constraint Programming (CP)
    http://www.cs.ualberta.ca/~ai/cp/
  - Integration of AI and OR Techniques in CP (CP-AI-OR)
    http://www.cs.cornell.edu/~vanhoeve/cpaior/
  - National Conference on AI (AAAI)
    http://www.aaai.org
  - International Joint Conference on Artificial Intelligence (IJCAI)
    http://www.ijcai.org
  - European Conference on Artificial Intelligence (ECAI)
    http://www.eccai.org
  - International Symposium on Practical Aspects of Declarative Languages (PADL)
    http://www.informatik.uni-trier.de/~ley/db/conf/padl/index.html

Literature

- **Schools and Tutorials**
  - ACP summer schools:

    2005: http://www.math.unipd.it/~frossi/cp-school/

    2006: http://www.cse.unsw.edu.au/~tw/school.html

    2007: http://www.iiia.csic.es/summerschools/sscp2007/

    2008: http://www-circa.mcs.st-and.ac.uk/cpss2008/

    2009: http://www.cs.ucc.ie/~osullb/ACPSS2009/Welcome.html

    2010: http://becool.info.ucl.ac.be/summerschool2010/
  - AI conference tutorials (IJCAI'09, 07, 05, ECAI'04 …).
  - CP conference tutorials.
  - CP-AI-OR master classes.

Literature

- **Solvers & Languages**
  - Choco (http://choco.sourceforge.net/)
  - Comet (http://www.comet-online.org/)
  - Eclipse (http://eclipse.crosscoreop.com/)
  - FaCiLe (http://www.recherche.enac.fr/opti/facile/)
  - Gecode (http://www.gecode.org/)
  - IBM ILOG Solver (http://www-01.ibm.com/software/websphere/products/optimization/)
  - Koalog Constraint Solver (http://www.gecode.org/)
  - Minion (http://minion.sourceforge.net/)
  - OPL (http://www.ilog.com/products/oplstudio/)
  - Sicstus Prolog (http://www.sics.se/isl/sicstuswww/site/index.html)