A Brief Introduction to Probabilistic and Quantum Programming

Part I

Ugo Dal Lago

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Section 1

Classical and Probabilistic Systems

Notation

- Sets: $A, B, C, \ldots$;
- Cartesian Products: $A \times B$;
- Natural Numbers: 1,2,3,...
- Real Numbers: $\mathbb{R}$;

$$1, \frac{3}{7}, 17463, \pi, \ldots$$

- Complex Numbers: $\mathbb{C}$;

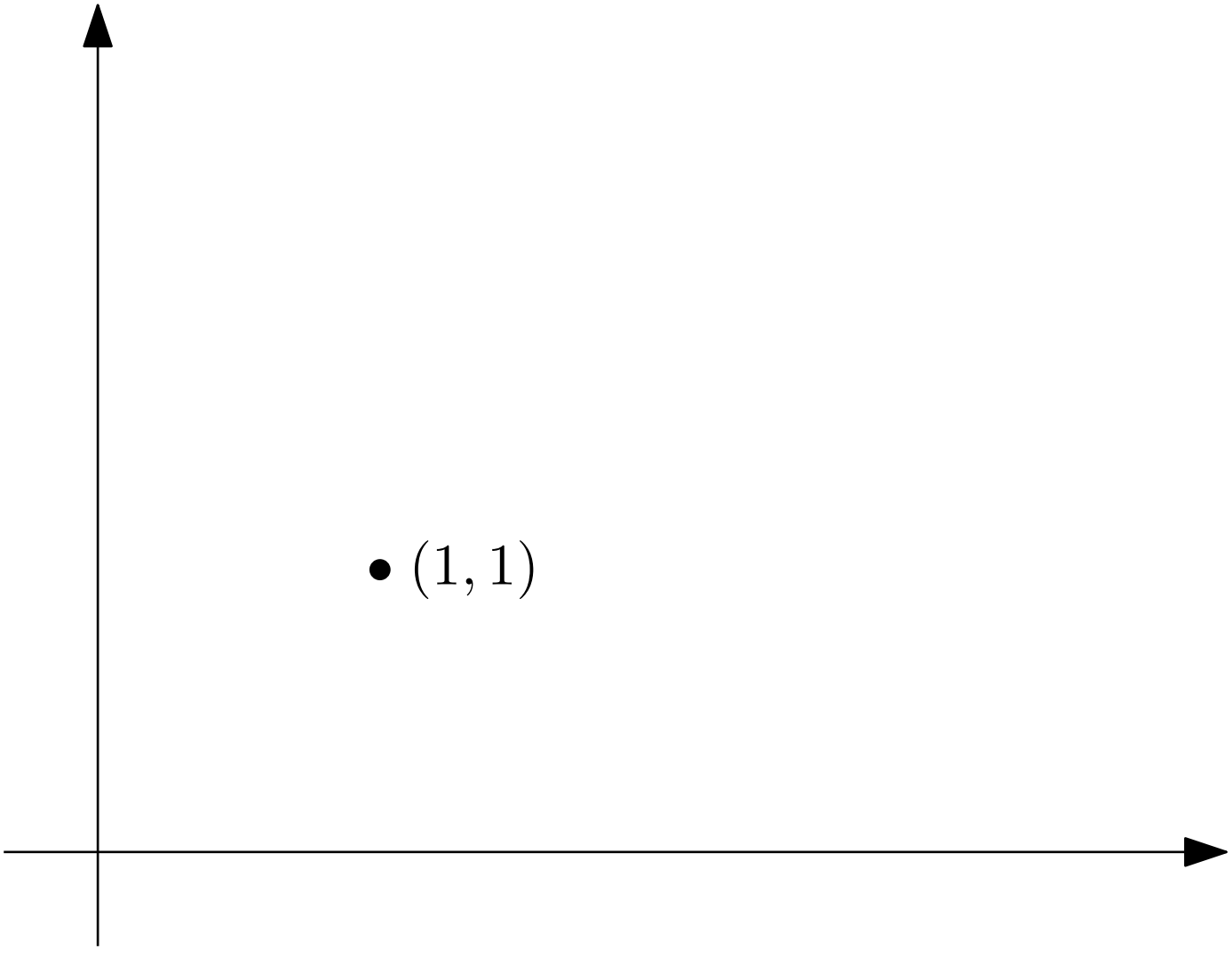
$$1 + 5i, \frac{1}{\sqrt{2}}i, \ldots$$

- Bits: $\mathbb{F} = \{0, 1\}$;
- Vectors: A^n;
- Matrices: $B^{m \times n}$.

State of a Physical System

- **Instantaneous** description of the physical entities forming the system;
- Can change over time;
- Is very often an element of $\mathbb{R}^d$;
- In classical physics, the state of a system is **uniquely determined**.

Example: Particle in $\mathbb{R}^2$ at a Given Time



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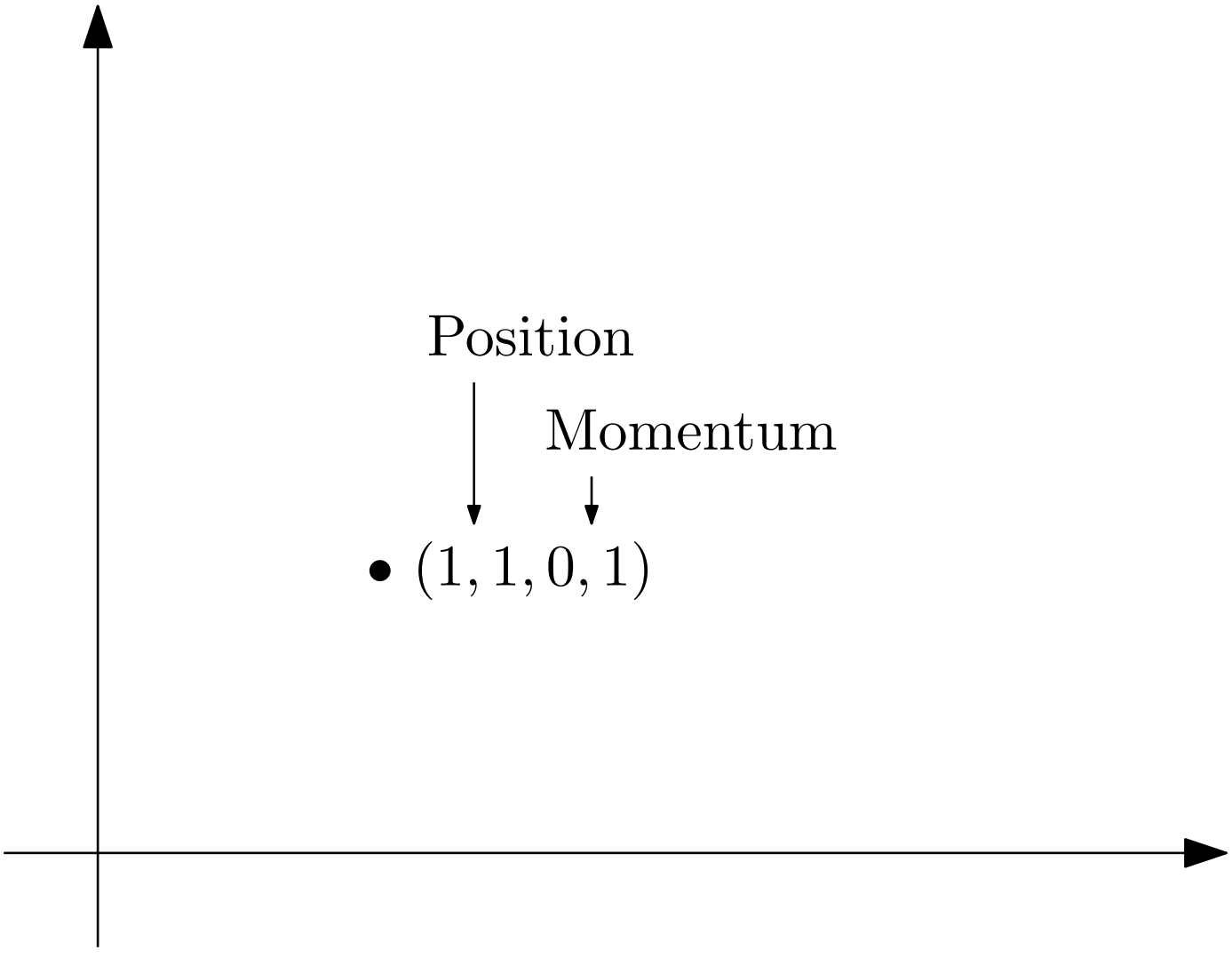

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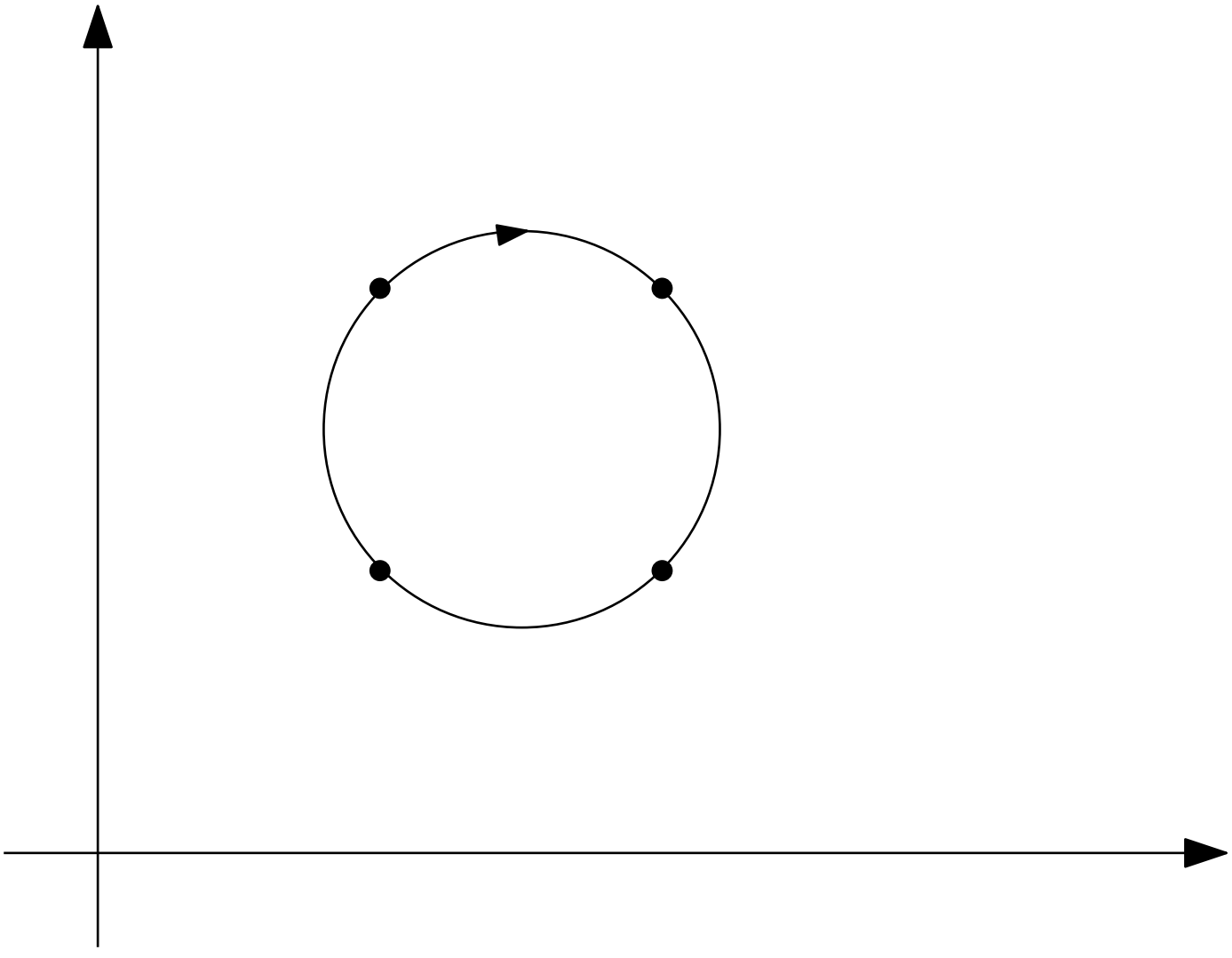

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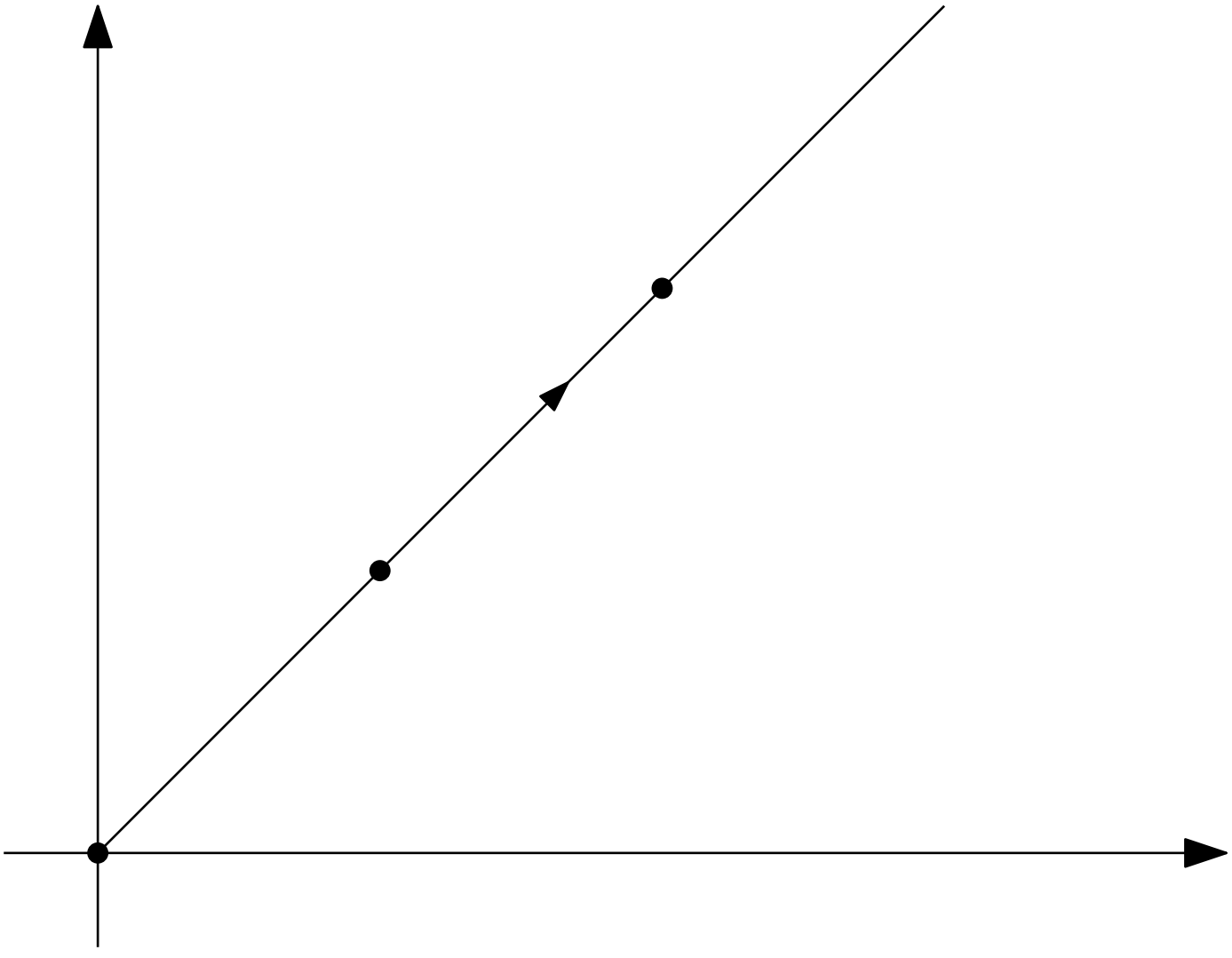

Composing Physical Systems

- If the states of two physical systems live in $\mathbb{R}^n$ e $\mathbb{R}^m$ (respectively), how should we represent the physical system obtained by **composing** the two systems?
- We should keep track of the first **and** the second state.
- Solution: $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$.

How to Discretize Time (and the State Space)

- Traditionally, time is just a number from $\mathbb{R}$.
- But sometimes, it suffices to consider the state of the system only at **certain** instants.

$$0, \alpha, 2\alpha, 3\alpha, 4\alpha, \ldots$$

- Time becomes an element of $\mathbb{N}$, i.e., it is **discrete**.
- Why?
  - Abstraction;
  - Simulation.
- Sometimes also states become **discrete**...
  - E.g., in Turing Machines.
- Often one can also assume that there are **finitely many** states.
  - The system is in one of the n states

$$\mathcal{S} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subseteq \mathbb{R}^n.$$

  - For the sake of simplicity, let us focus our attention on **finite state** systems.

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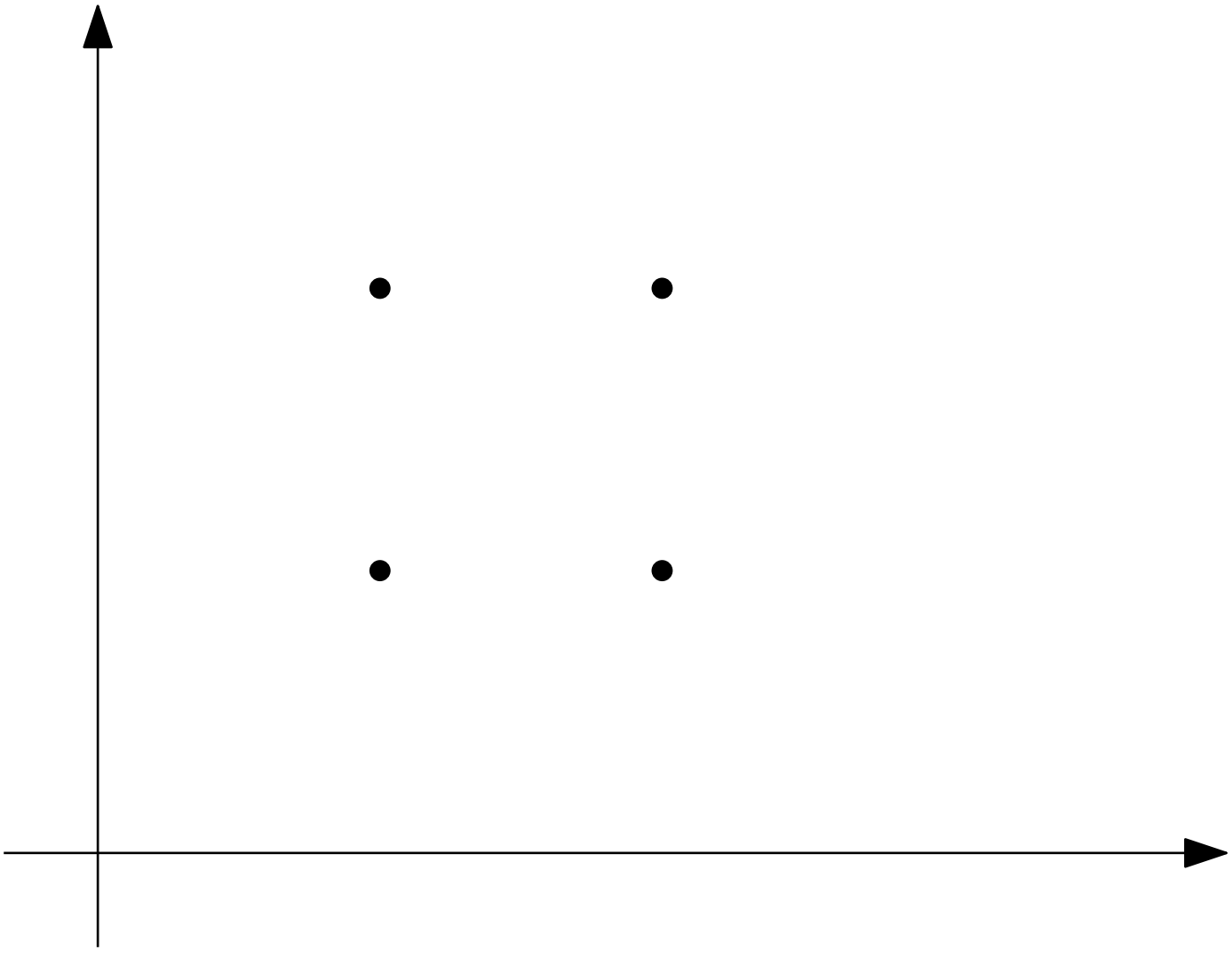

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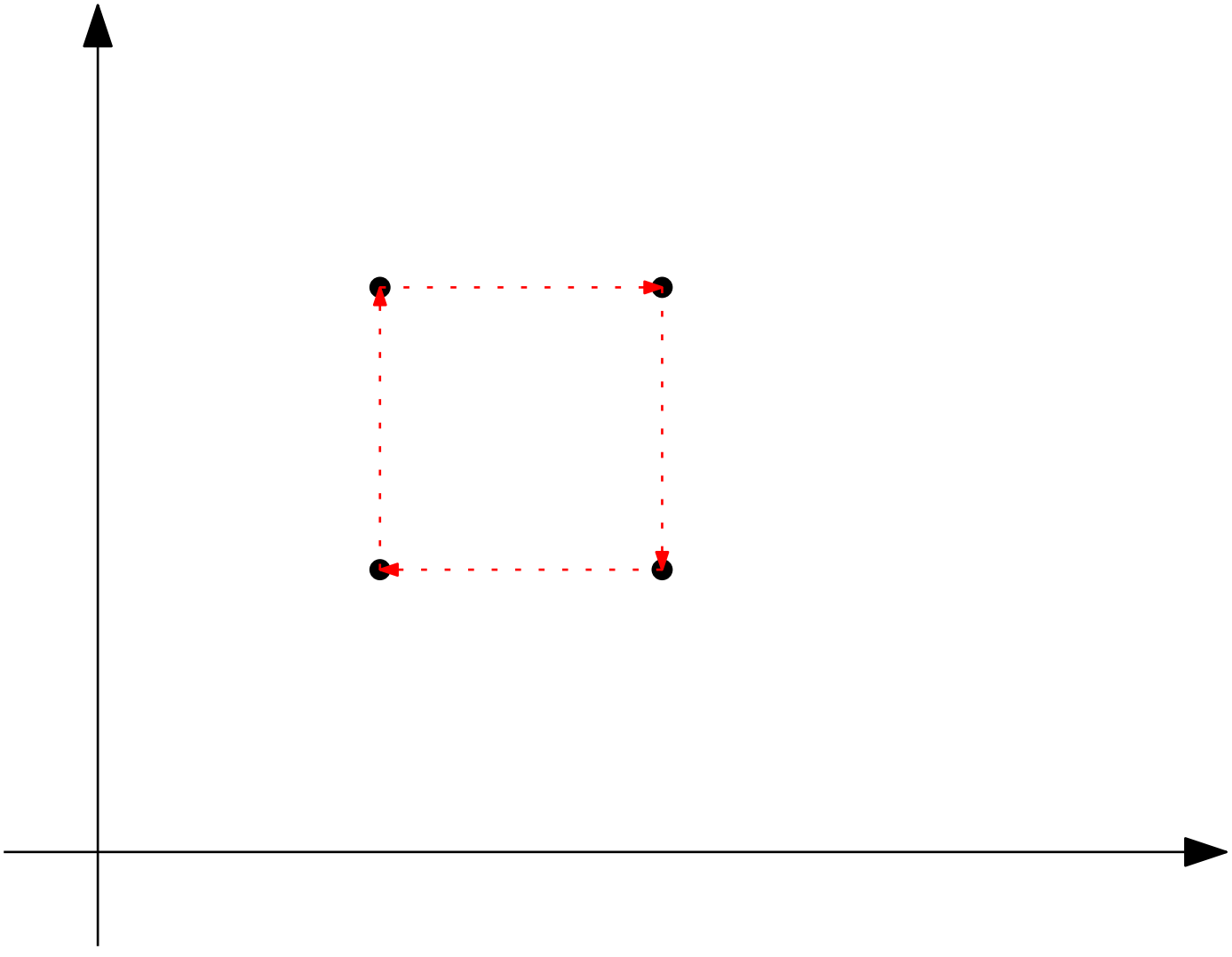


Probabilistic Systems, Statically

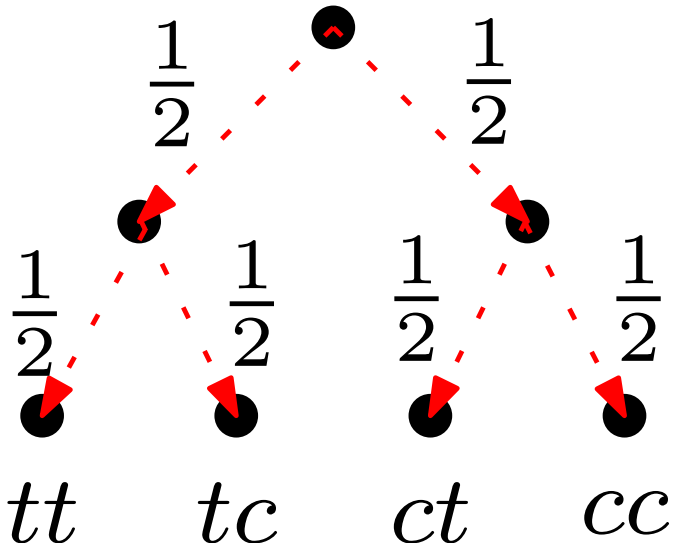
- In classical physics, the state of any system is **uniquely determined**.
- For reasons of abstraction and complexity, one ofter uses a **probabilistic** representation of states.
  - Example: a **dice**.
- If there is no uncertainty about a state, the latter is said to be **pure**.
  - Elements of $\mathcal{S} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ are precisely the pure states.
- In general, however, a state (at any given time) is described as a function $\mathbf{P} : \mathcal{S} \to \mathbb{R}$ such that $\sum_{\mathbf{x} \in \mathcal{S}} \mathbf{P}(\mathbf{x}) = 1$, or by a vector in $\mathbb{R}^{|\mathcal{S}|}$:

$$\begin{pmatrix} \mathbf{P}(\mathbf{x}_1) \\ \vdots \\ \mathbf{P}(\mathbf{x}_n) \end{pmatrix} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

- This is a **mixed state**.

Probabilistic Systems, Dynamically

- In any given pure state $\mathbf{x}_i$, the system can evolve into *any* pure state $\mathbf{x}_j$ with probability p_{ij}.
- As an example:



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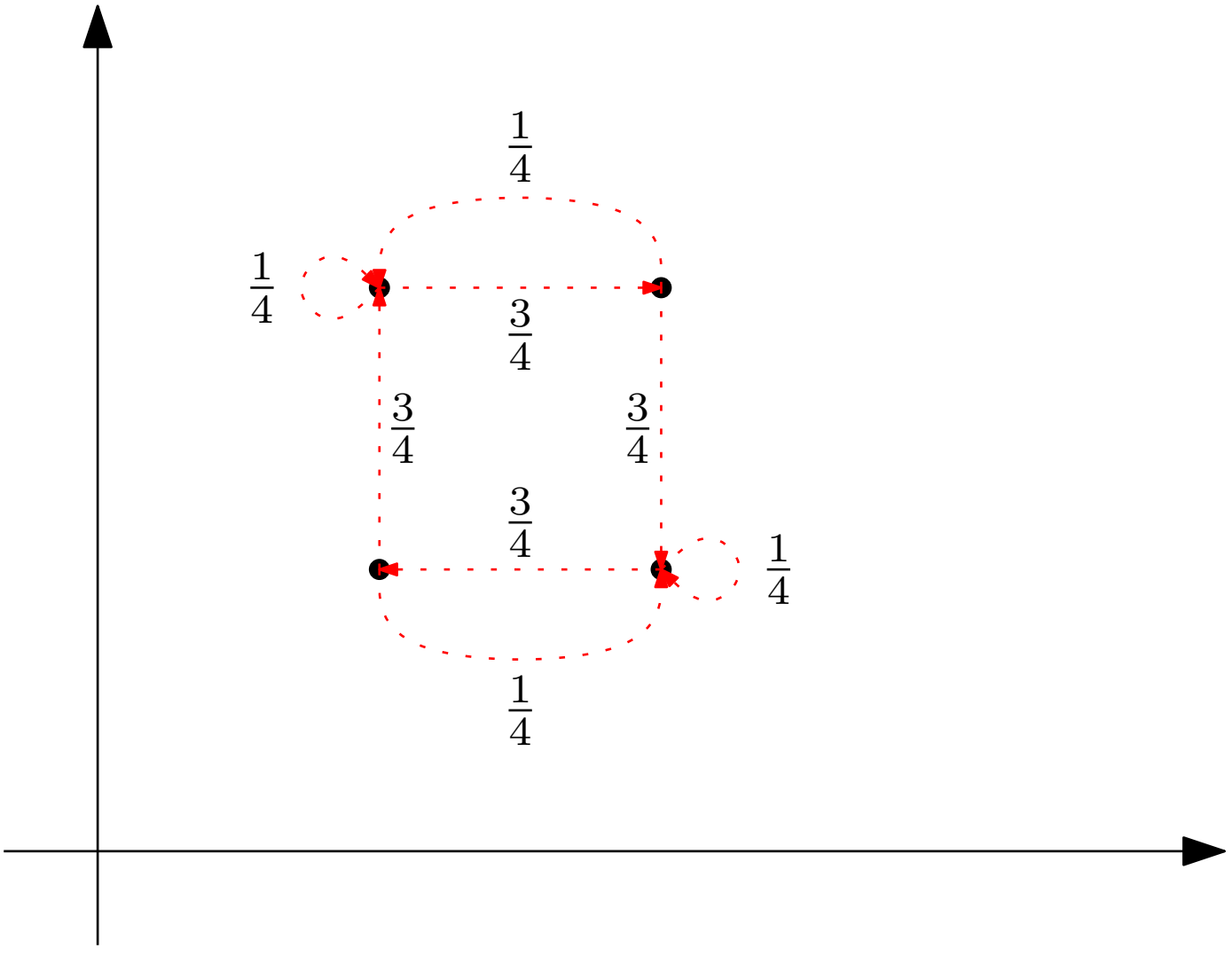

Dynamics: Matrix Representation

- Suppose we work in a system with n pure states:

$$\mathcal{S} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subseteq \mathbb{R}^m.$$

- The dynamic evolution of the system is described, compactly, by the matrix $\mathbf{M}$ in $\mathbb{R}^{n \times n}$:

$$\mathbf{M} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

- A transition step corresponds to **multiplying M** and **P** (the latter seen as a vector).
- In other words, the system can be seen as a so-called **Markov Chain**.

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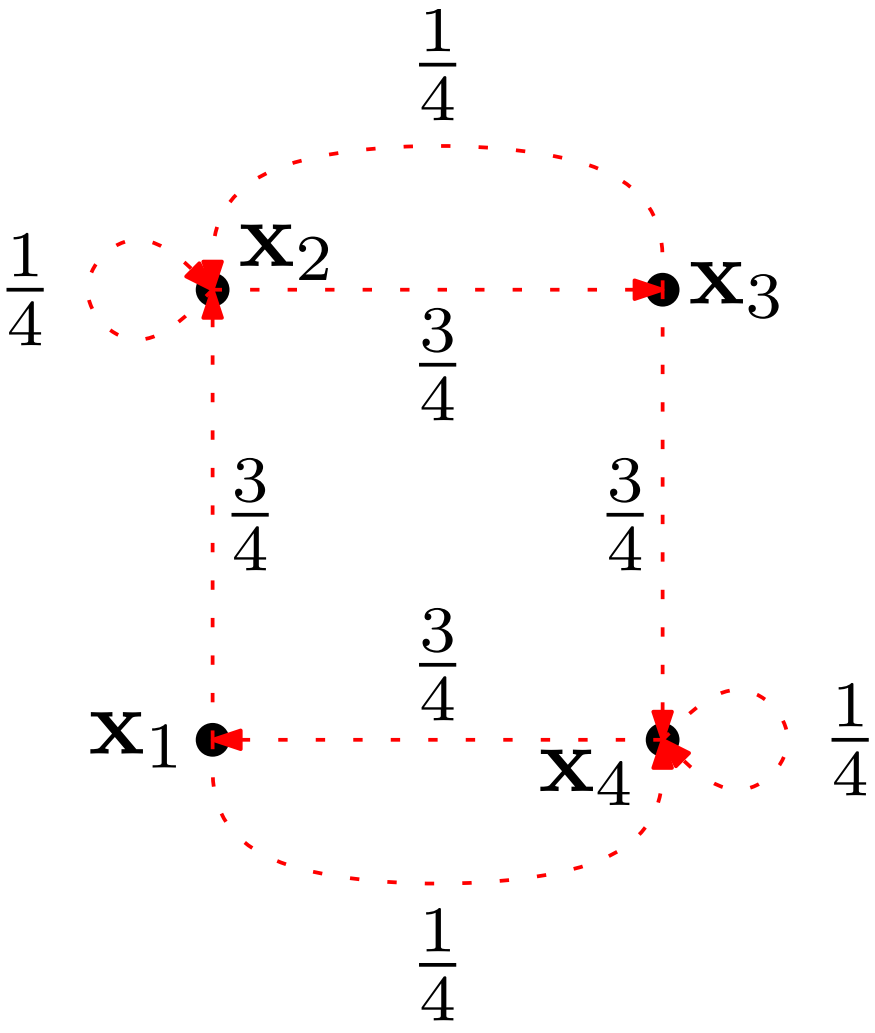

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

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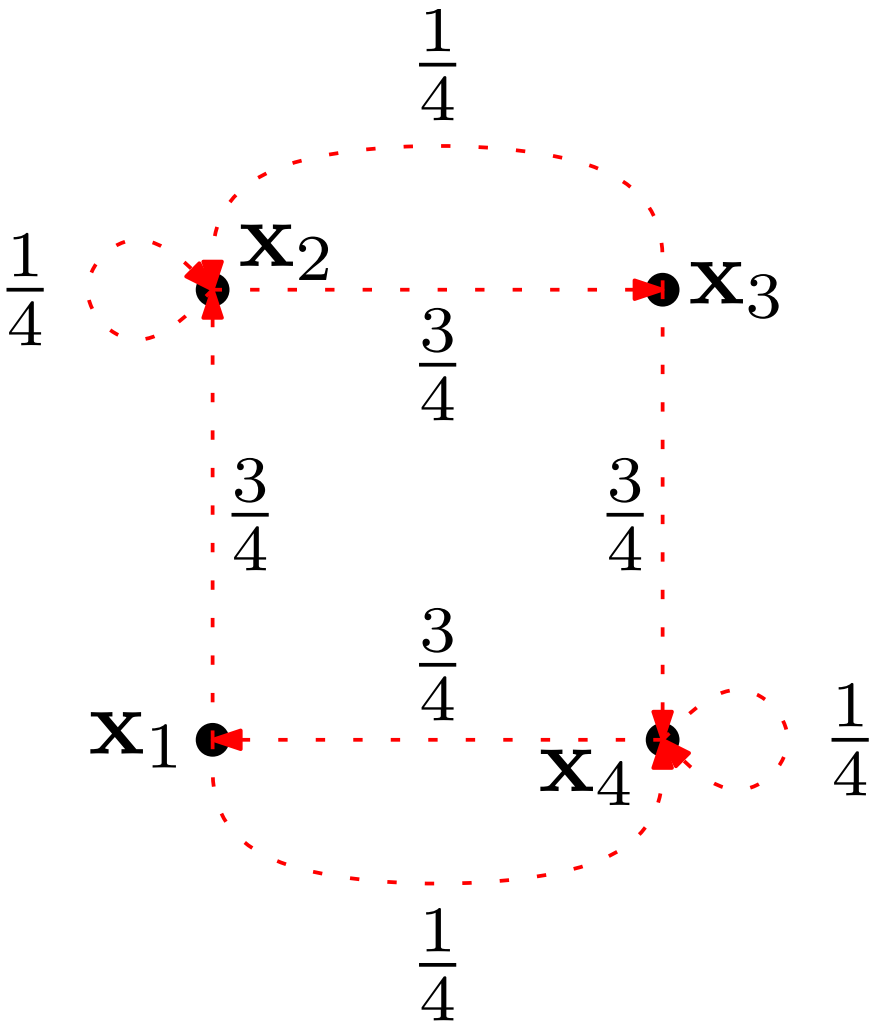

$$(1, 0, 0, 0)$$

$$\Rightarrow \left(0, \frac{3}{4}, 0, \frac{1}{4}\right)$$

$$\Rightarrow \left(\frac{3}{16}, \frac{3}{16}, \frac{9}{16}, \frac{1}{16}\right)$$

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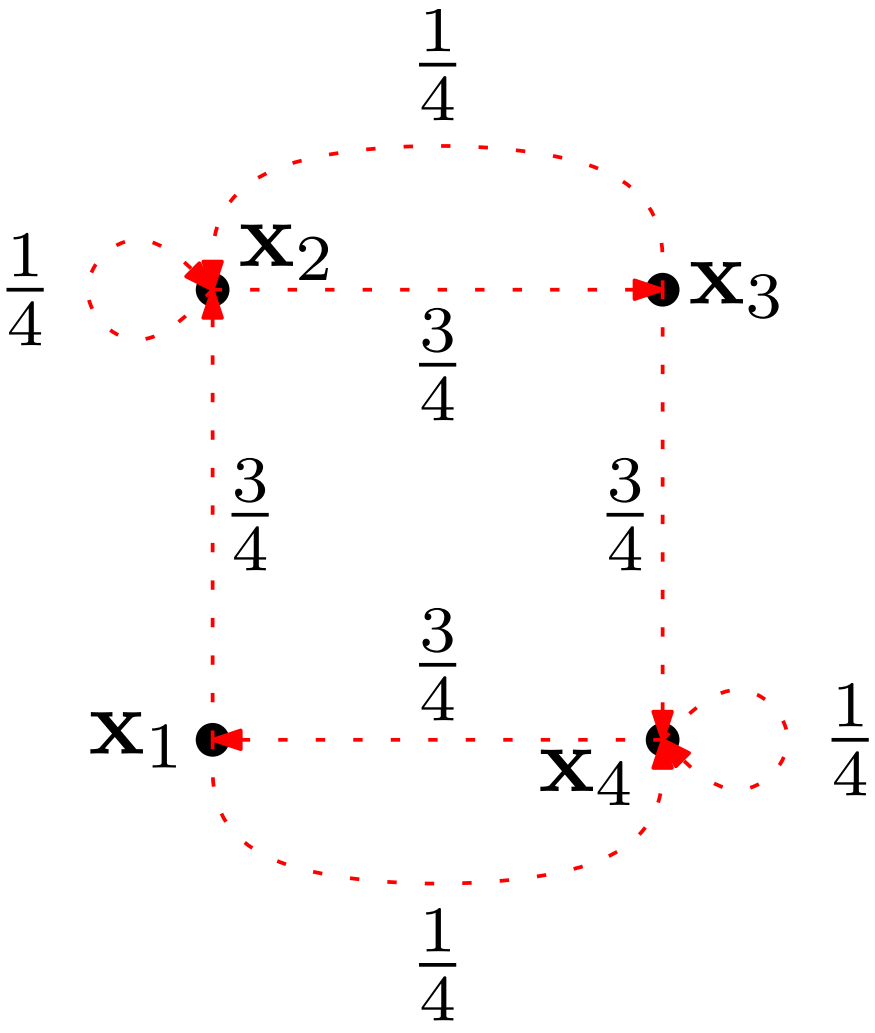

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Intermezzo: Linearity

- Vectors in $\mathbb{R}^n$ support:
  - **Binary Sums**:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

  - **Products by Scalars in** $\mathbb{R}$:

$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

- Multiplication with a matrix gives rise to a **linear map**:

$$\mathbf{M}(\alpha \mathbf{x}) = \alpha(\mathbf{M}\mathbf{x});$$
$$\mathbf{M}(\mathbf{x} + \mathbf{y}) = \mathbf{M}\mathbf{x} + \mathbf{M}\mathbf{y}.$$

Classical Systems as Specific Probabilistic Systems

- A **pure state** can be seen as a function $\mathbf{P} : \mathcal{S} \to \mathbb{R}$ such that $\mathbf{P}(\mathbf{x}_i) = 1$ for a given $\mathbf{x}_i$.
  - The system is in $\mathbf{x}_i$, without any uncertainty.
- If the matrix $\mathbf{M} = (p_{ij})$ is such that for every $i \in \{1, \ldots, n\}$ there is $j \in \{1, \ldots, n\}$ such that $p_{ij} = 1$, then $\mathbf{M}$ is **classical**.
- A classical matrix turns pure states into pure states!
- As an example:

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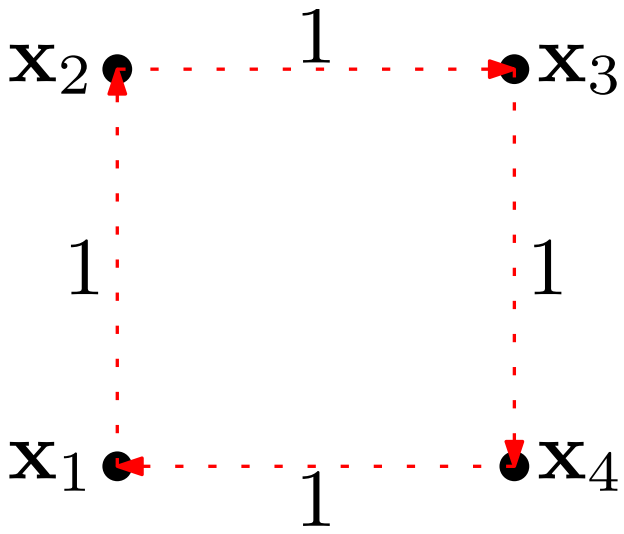


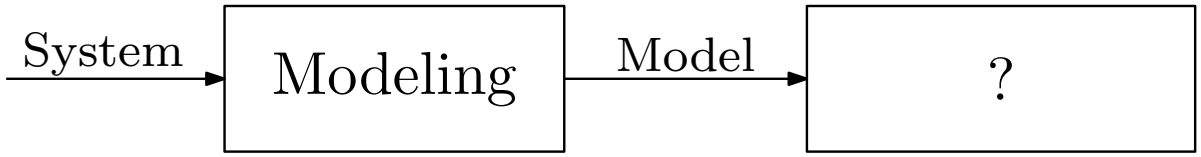
$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Section 2

Probabilistic Computational Models

Incepting Probability into Algorithms

- In what we have seen so far, probability has been seen a way to facilitate **modeling** of existing systems, e.g.



- Probability, however, can also be seen as a very powerful way to **compute**.
  - Algorithms can "flip a coin".
  - This is somehow a paradigm shift: probabilism is not there for modeling uncertainty, but rather for the purpose of **relaxing determinism** as an algorithm design principle.
  - This "new" computation paradigm is called **probabilistic computation**.
  - Your algorithms (then your programs) can be *seen* and *treated* as probabilistic systems.

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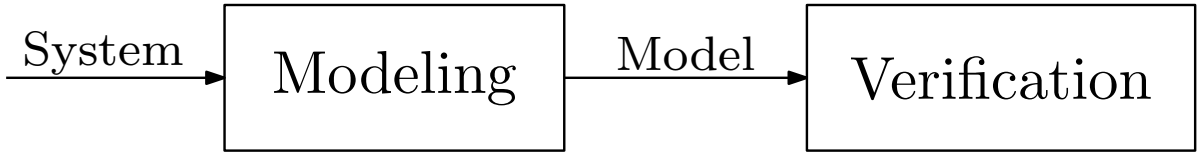

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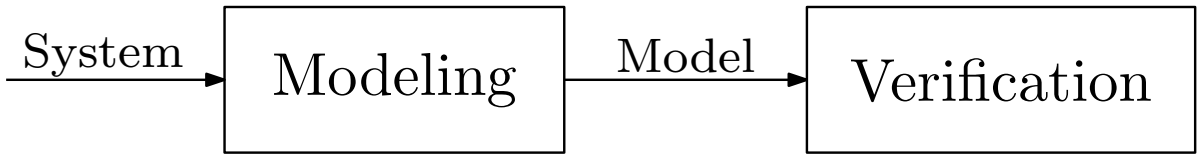

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But Why?

- **Achieving Better Performances**
  - Allowing computation to proceed probabilistically *somehow* allows to exploit different computation paths **at the same time**.
  - Of course, each computation path comes with its own probability.
  - . . . but all this can (sometime) be turned into a way to alleviate exponential blowups.
- **Allowing a Form of "Confusion" in Computation**
  - Deterministic computation has its own shortcomings.
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Example

MillerRabin(n)

If $n > 2$ and n is even, return **composite.**

/* Factor $n-1$ as $2^s t$ where t is odd. */

$s \leftarrow 0$

$t \leftarrow n-1$

while t is even

    $s \leftarrow s+1$

    $t \leftarrow t/2$

end /* Done. $n-1 = 2^s t$. */

Choose $x \in \{1, 2, \ldots, n-1\}$ uniformly at random.

Compute each of the numbers $x^t, x^{2t}, x^{4t}, \ldots, x^{2^s t} = x^{n-1} \bmod n$.

If $x^{n-1} \not\equiv 1 \pmod{n}$, return **composite.**

for $i = 1, 2, \ldots, s$

    If $x^{2^i t} \equiv 1 \pmod{n}$ and $x^{2^{i-1} t} \not\equiv \pm 1 \pmod{n}$, return **composite.**

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Quantum Systems

Quanta and Computation

- Starting from the beginning of the XX century, physical sciences faced a series of **revolutions**.
- The principles of classical physics were proved **not** to hold when the involved magnitudes become as little as those in atoms, or as big as those in astrophysics.
- One of the answers to the subsequent crisis was the introduction of **quantum mechanics**.
  - It is a **complex, non-intuitive, theory**;
  - Mathematically, it is highly **not trivial**;
- In 1982, Richard Feynman suggests, for the first time, that quantum mechanics can form the basis of a **new model of computation**, more efficient than the one introduced by Turing and later generalized to a proper model of computation.

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State of a Quantum System H_n

- It is a vector in $\mathbb{C}^n$:

$$\mathbf{x} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

- The following condition must hold:

$$\sum_{i=1}^{n} ||c_i||^2 = 1$$

- We can write

$$\mathbf{x} = c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n,$$

  where the $\mathbf{x}_i$ are always 0 except in the ith coordinate, which is 1 ($\mathbf{x}_1, \ldots, \mathbf{x}_n$ is the **computational base** of the system).
- What does $\mathbf{x}_i$ represent? It is an **observable** of the system.
  - Every **orthonormal base** for $\mathbb{C}^n$ corresponds to one such observable property.

Evolution of a Quantum System H_n

- The system evolves **linearly**, thus its dynamics can be described by a matrix:

$$\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

- It must be that $\sum_{i=1}^{n} ||d_i||^2 = 1$.
- A necessary and sufficient condition is that $\mathbf{Q} = (a_{ij})$ must be **unitary**: $\mathbf{Q}^* = \mathbf{Q}^{-1}$.
  - $\mathbf{Q}^*$ is the conjugate transpose of $\mathbf{Q}$. As an example,

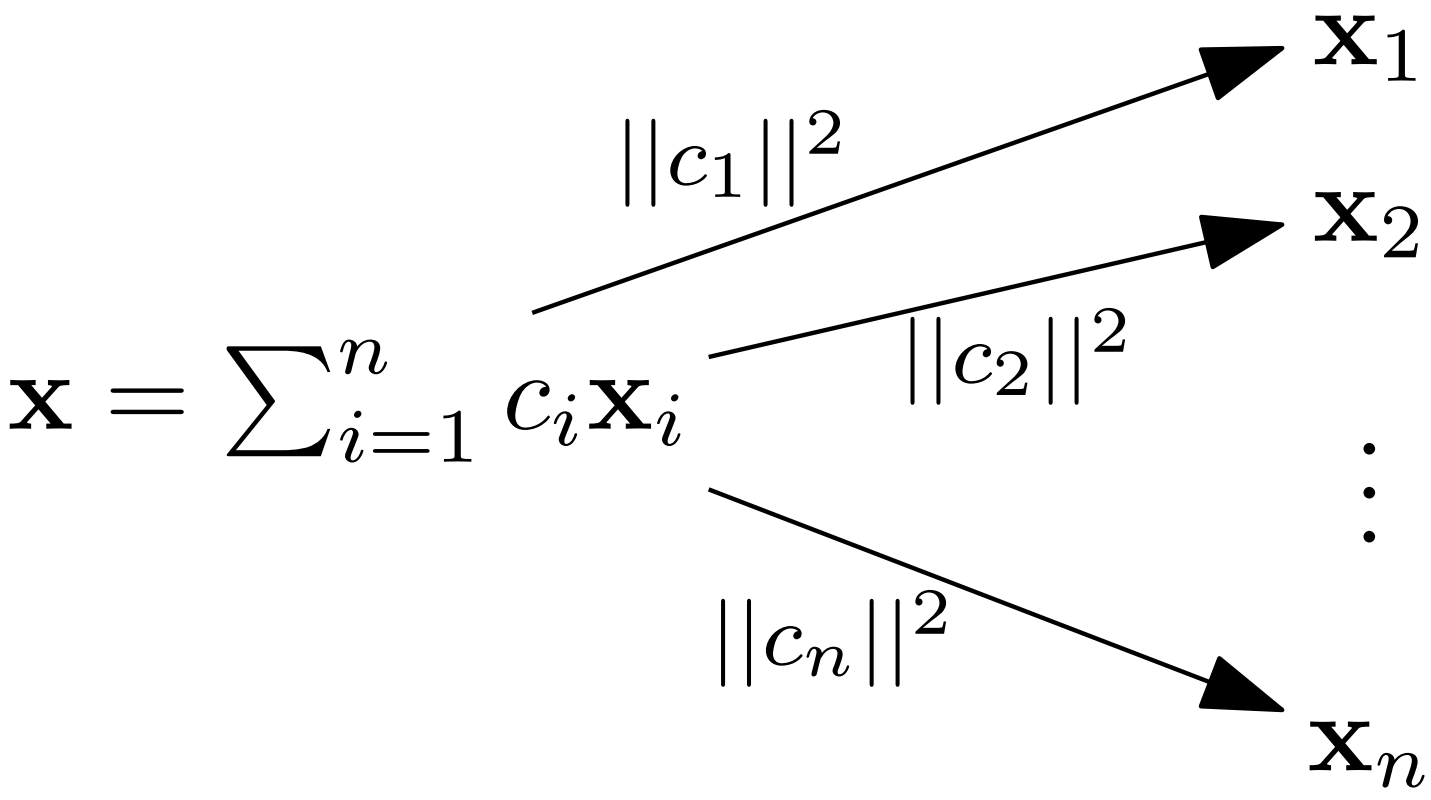
$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}^* = \begin{pmatrix} a_{11}^* & \cdots & a_{n1}^* \\ \vdots & & \vdots \\ a_{1n}^* & \cdots & a_{nn}^* \end{pmatrix}$$

- The product of two unitary matrices is again unitary.

Measurement in a Quantum System H_n

- Given a Quantum System $\mathbf{x} = (c_i)$, the quantity $||c_i||^2$ **should not** be interpreted as the probability of being in state $\mathbf{x}_i$.
- The only way to observe the state of a quantum system is by way of **measurements**:
  - When measured, the state of a system goes from $\mathbf{x} = (c_i)$ to each of the states $\mathbf{x}_j$ with probability equals to $||c_j||^2$.
  - At that point the system is in a **classical state**, i.e., it can be observed without being altered.
  - Everything depends on the chosen base, which corresponds to the desired observable.
- Again: measuring a quantum system **can alter** its state!

Measurement in a Quantum System H_n



Composing two Quantum Systems H_n and H_m

- Given two quantum systems with computational bases

$$\mathbf{x}_1, \ldots, \mathbf{x}_n;$$
$$\mathbf{y}_1, \ldots, \mathbf{y}_m;$$

  their **composition** has computational basis

$$(\mathbf{x}_1, \mathbf{y}_1), \ldots, (\mathbf{x}_n, \mathbf{y}_m).$$

- Thus, we get the system

$$H_{nm} = H_n \otimes H_m.$$

- Let us indicate the pair $(\mathbf{x}_i, \mathbf{x}_j)$ by $\mathbf{x}_i \otimes \mathbf{x}_j$;

Composing two Quantum Systems H_n and H_m

- We can extend this operation to all states of H_n and H_m:

$$\left(\sum_{i=1}^{n} c_i \mathbf{x}_i\right) \otimes \left(\sum_{j=1}^{m} d_j \mathbf{y}_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} c_i d_j \mathbf{x}_i \otimes \mathbf{y}_j.$$

- A quantum state $\mathbf{x}$ is sometimes indicated as $|\mathbf{x}\rangle$.
- Another notational abuse:

$$|\mathbf{x}\rangle \otimes |\mathbf{y}\rangle = |\mathbf{x}\rangle|\mathbf{y}\rangle = |\mathbf{xy}\rangle.$$

- If a state $\mathbf{x}$ of $H_n \otimes H_m$ is such that $\mathbf{x} = \mathbf{y} \otimes \mathbf{z}$, then $\mathbf{x}$ is said to be **decomposable**, and otherwise **entangled**.
- Given two unitary transforms $\mathbf{Q}_n, \mathbf{Q}_m$ on H_n and H_m (respectively), we can build $\mathbf{Q}_n \otimes \mathbf{Q}_m \ldots$

Example: H_2

- The states of the system are the unitary vectors in $\mathbb{C}^2$.
  - The states of the orthonormal base $|\mathbf{x}_1\rangle, |\mathbf{x}_2\rangle$ are sometime denoted with $|\mathbf{0}\rangle$ e $|\mathbf{1}\rangle$.
- Unitary transforms: examples:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathbf{Z} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- States of H_2 are said to be **qubits**.

Example: $H_2 \otimes H_2$

- States of the systems are the unitary vectors in $\mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$.
- Let us consider two example states in $H_2 \otimes H_2$.
  - The state $|\mathbf{01}\rangle$ is decomposable: $|\mathbf{01}\rangle = |\mathbf{0}\rangle \otimes |\mathbf{1}\rangle$.
  - The state

    $$\frac{1}{\sqrt{2}}|\mathbf{00}\rangle + \frac{1}{\sqrt{2}}|\mathbf{11}\rangle$$

    is entangled
- A unitary transform on $H_2 \otimes H_2$ which cannot be decomposed as two unitary transforms on H_2 is:

$$\mathbf{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

An Example

$$
\begin{aligned}
&(\mathbf{CNOT} \circ (\mathbf{H} \otimes \mathbf{I}))|\mathbf{00}\rangle \\
&= \mathbf{CNOT}((\mathbf{H} \otimes \mathbf{I})(|\mathbf{00}\rangle)) \\
&= \mathbf{CNOT}(\mathbf{H}(|\mathbf{0}\rangle) \otimes \mathbf{I}(|\mathbf{0}\rangle)) \\
&= \mathbf{CNOT}(\mathbf{H}(|\mathbf{0}\rangle) \otimes |\mathbf{0}\rangle) \\
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About Measurements, Again

- Suppose we want to work with the system $H_n \otimes H_m$. A state of the system can be expressed as:

$$\mathcal{Q} = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} |\mathbf{x}_i\rangle |\mathbf{x}_j\rangle.$$

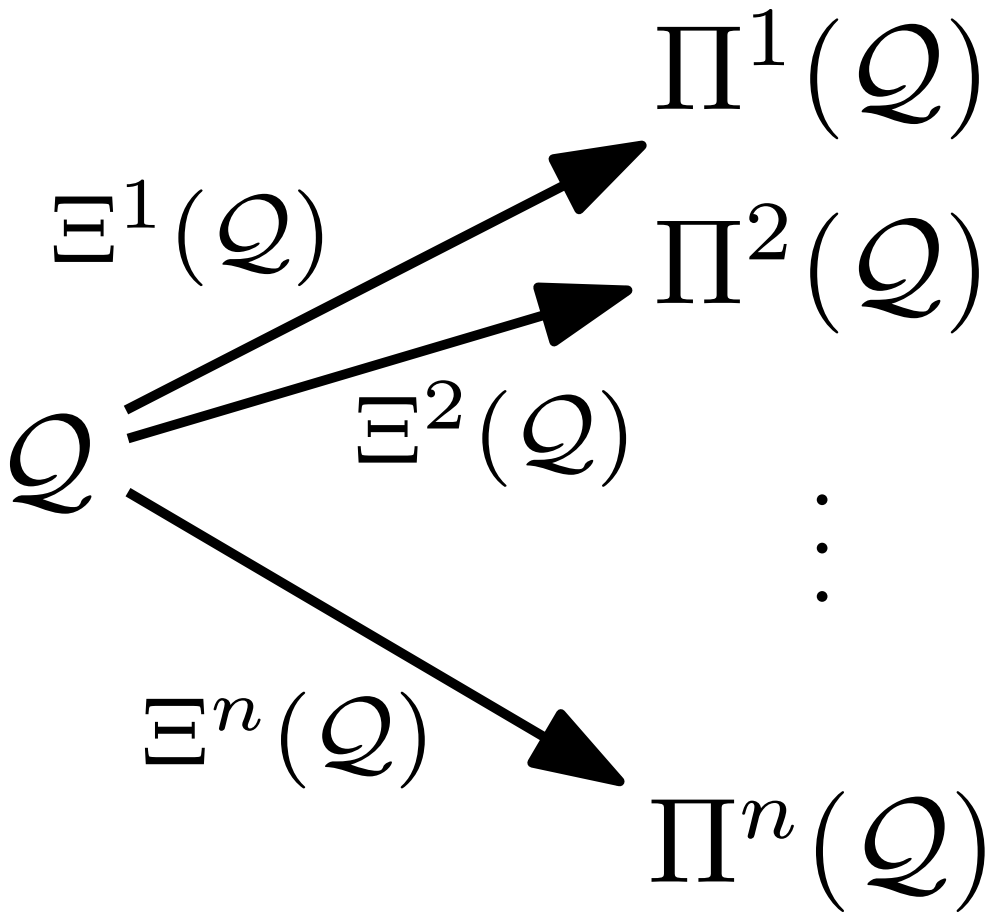
- If we observe the **first** component of the system (the "rightmost" component of the tensor product), we get that:
  - We will end up in a state $|\mathbf{x}_k\rangle \otimes |\mathbf{y}\rangle$ where $\mathbf{x}_k$ is an element of the computational base, while **y is not necessarily one such**.
  - The probability of reaching $|\mathbf{x}_k\rangle \otimes |\mathbf{y}\rangle$ is equal to

$$\Xi^k(\mathcal{Q}) = \sum_{j=1}^{m} ||c_{kj}||^2.$$

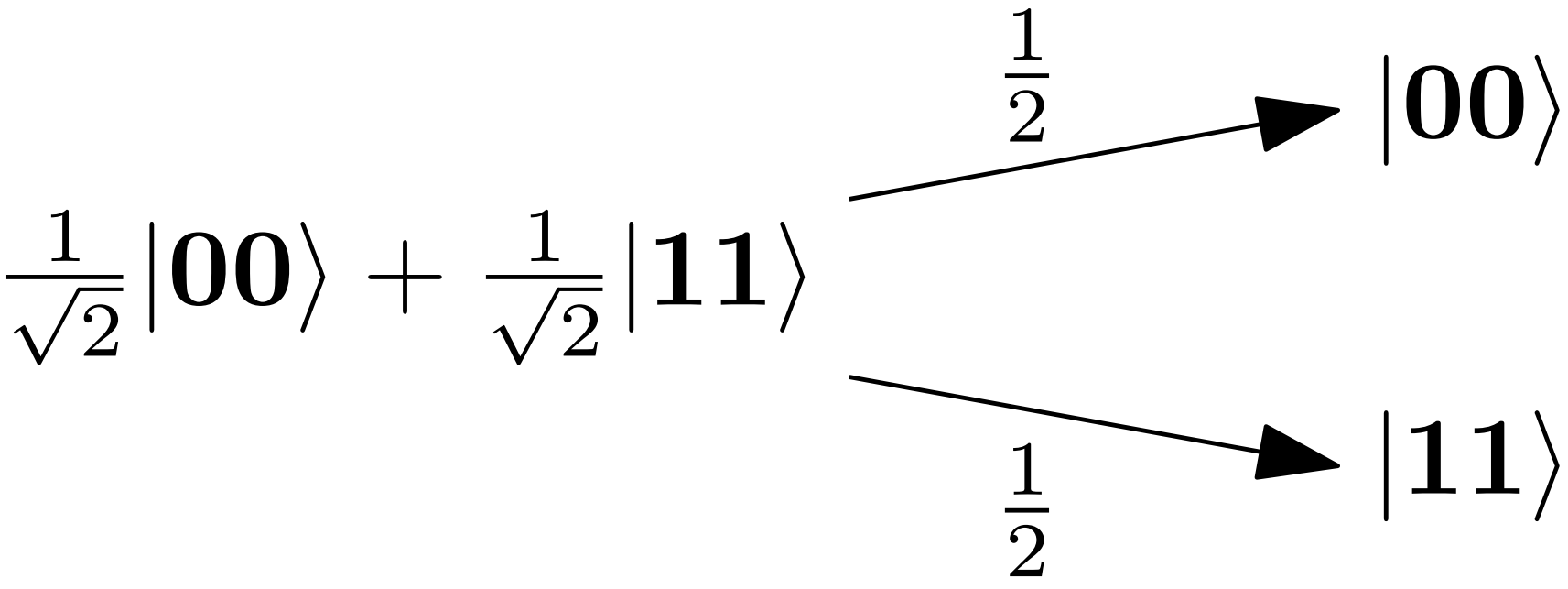
  - The state $|\mathbf{x}_k\rangle \otimes |\mathbf{y}\rangle$ can be expressed as follows:

$$\Pi^k(\mathcal{Q}) = \frac{1}{\sqrt{\mathbf{P}(k)}} \sum_{j=1}^{m} c_{kj} |\mathbf{x}_k\rangle |\mathbf{y}_j\rangle = \frac{|\mathbf{x}_k\rangle}{\sqrt{\mathbf{P}(k)}} \sum_{j=1}^{m} c_{kj} |\mathbf{y}_j\rangle$$

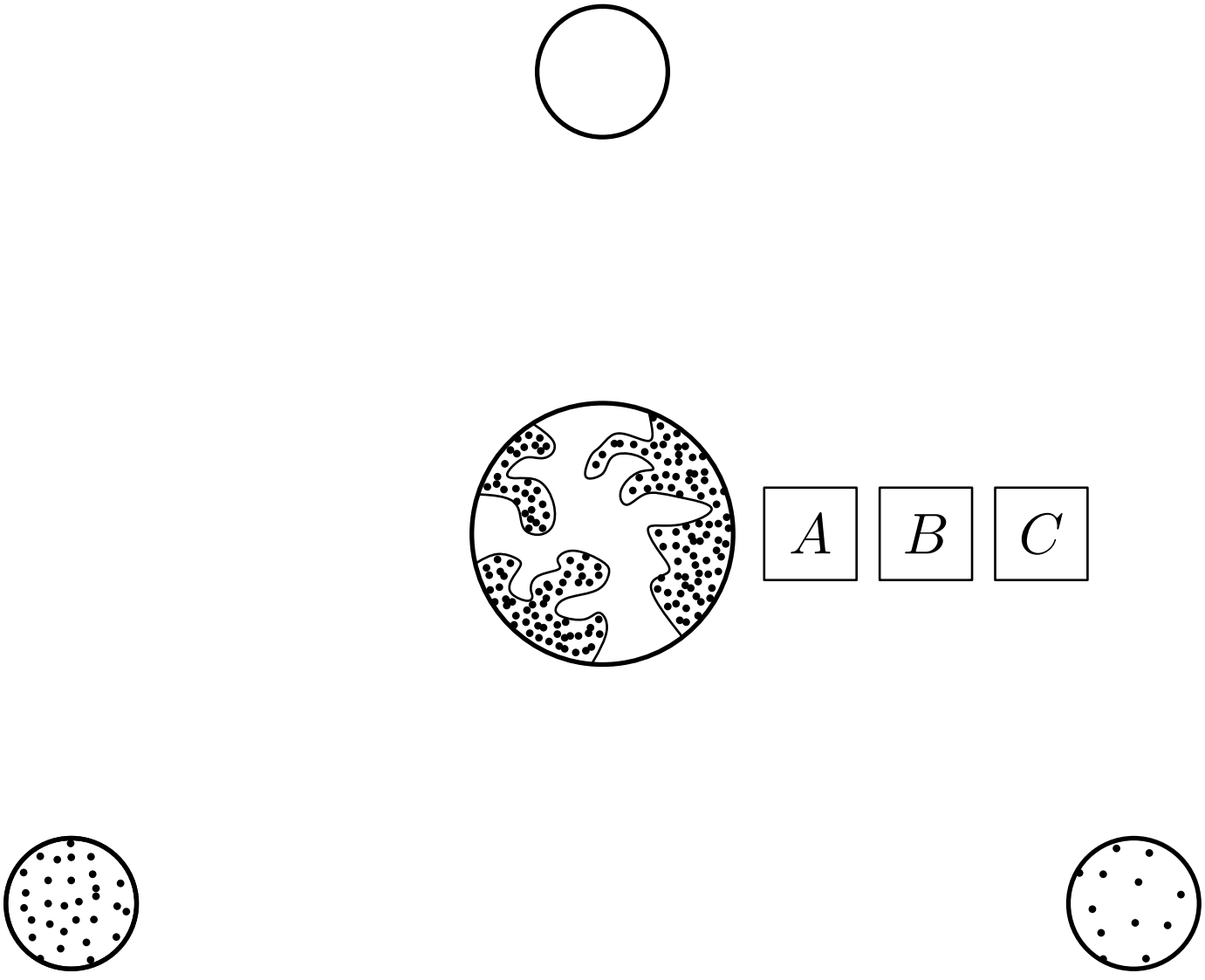
About Measurements, Again



About Measurements, Again



A Game



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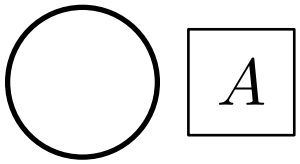

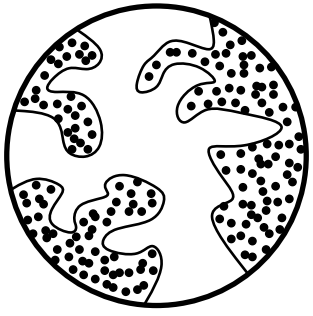

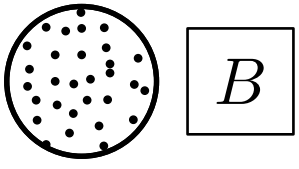

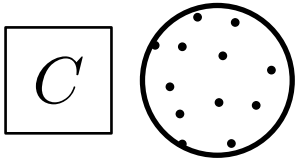

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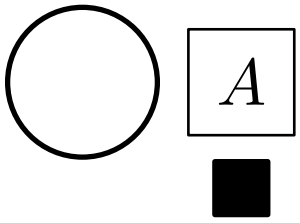

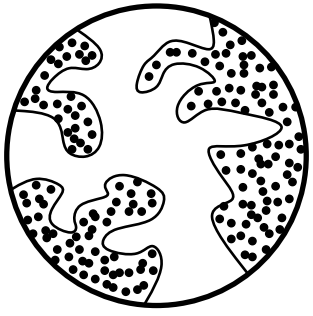

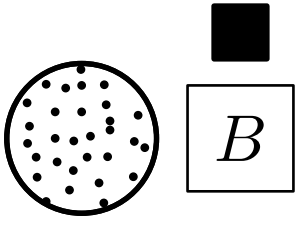

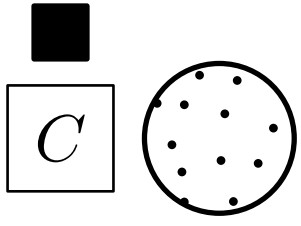

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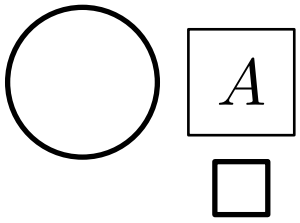

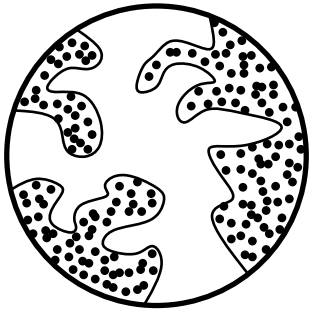

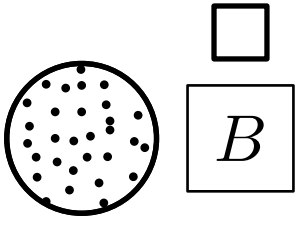

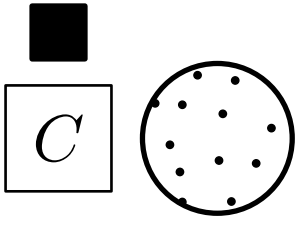

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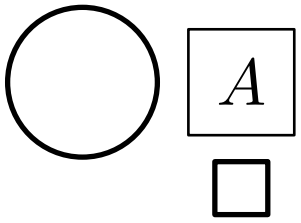

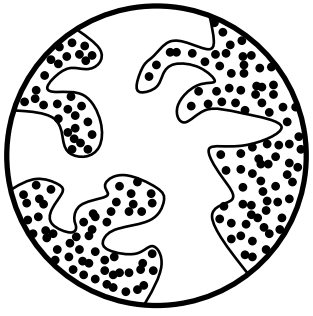

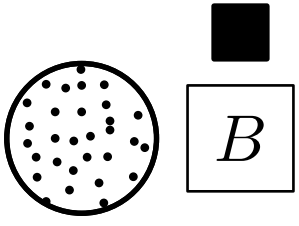

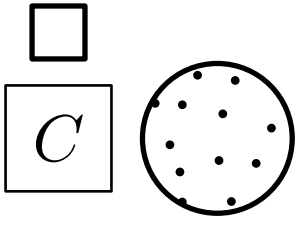

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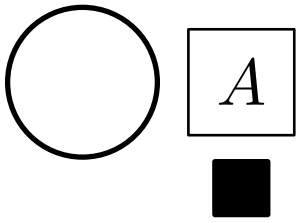

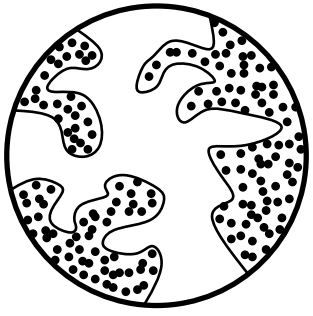

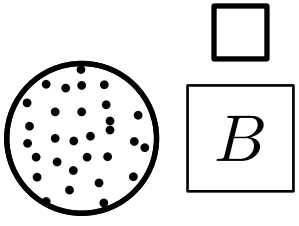

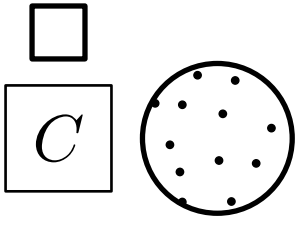

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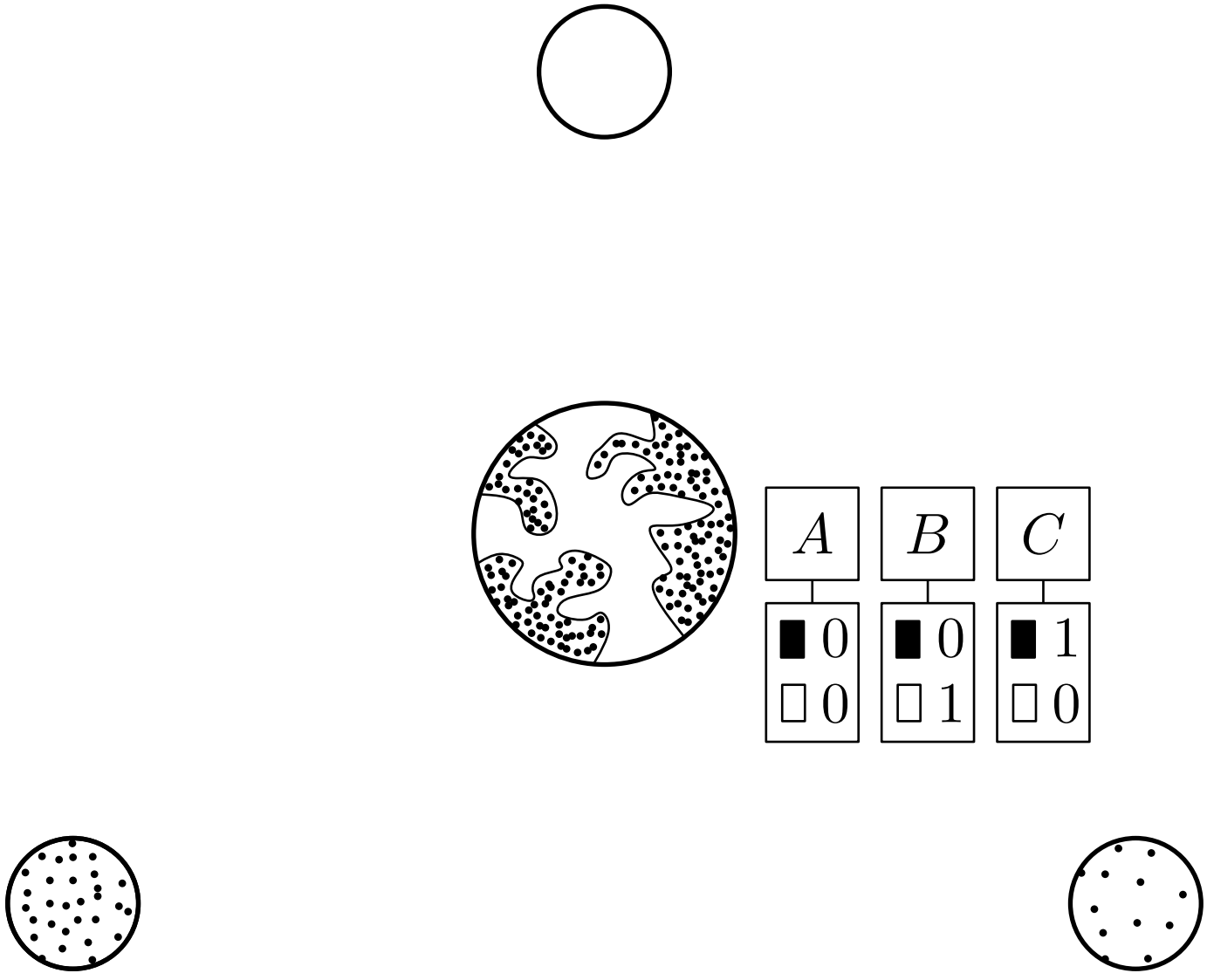

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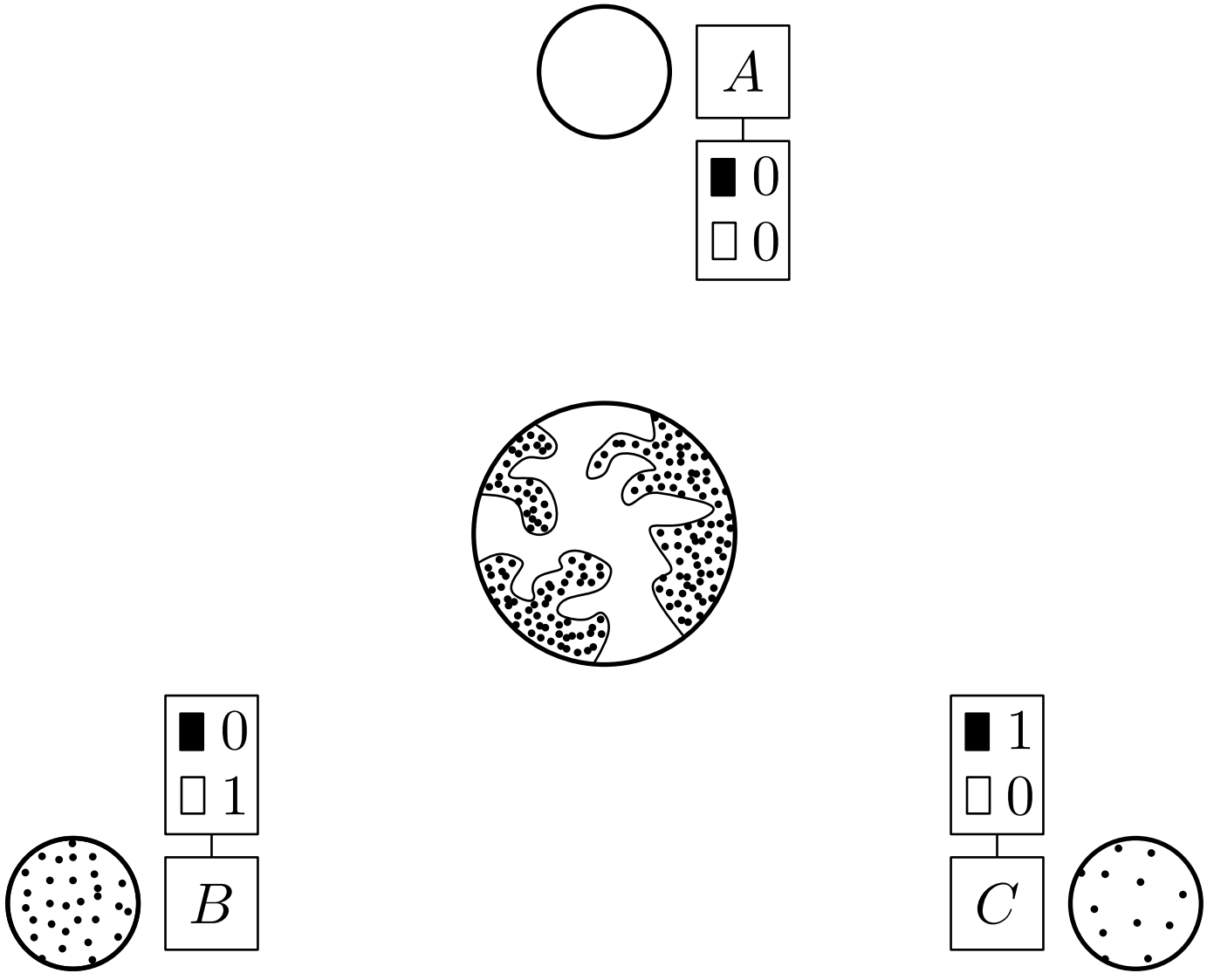

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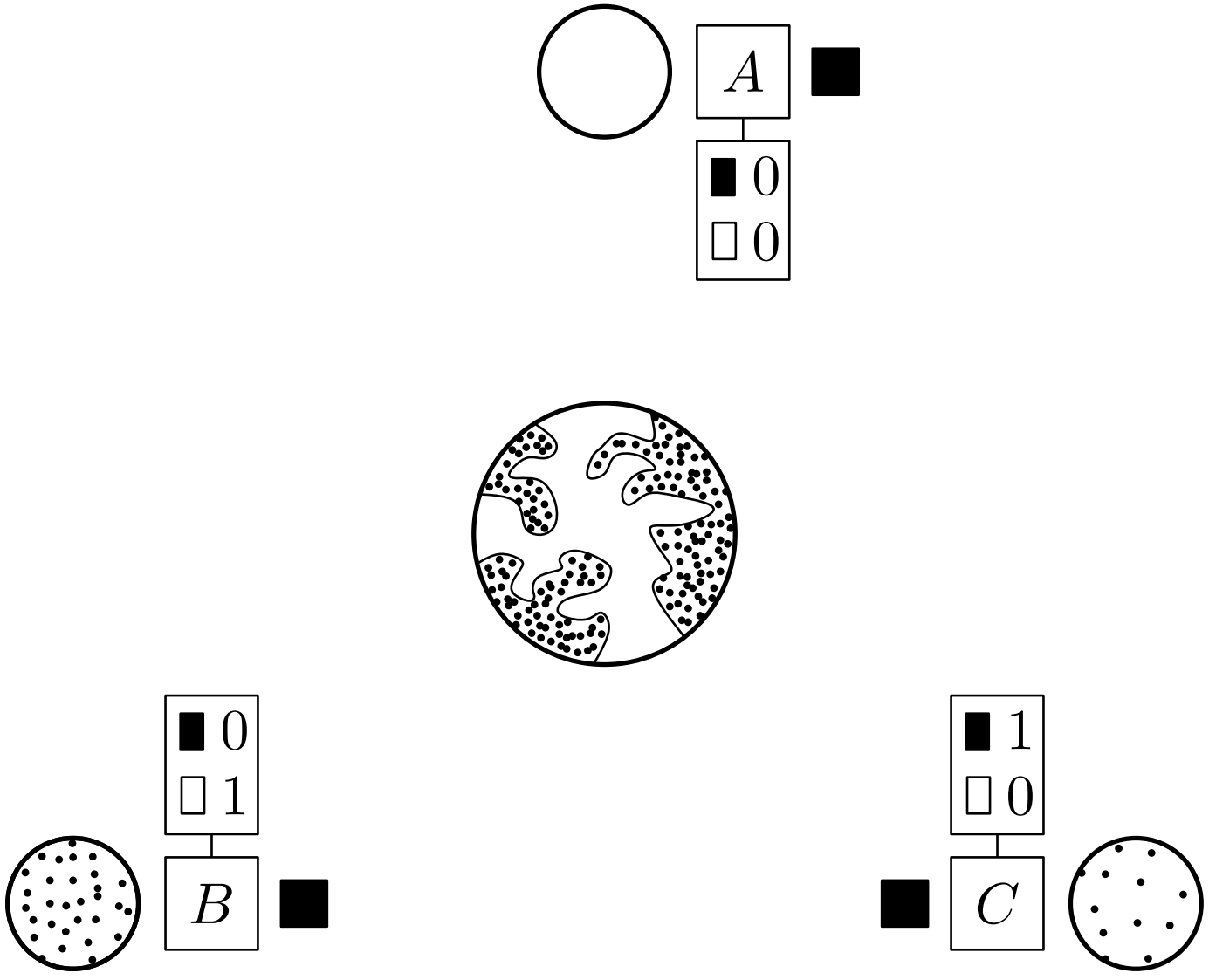

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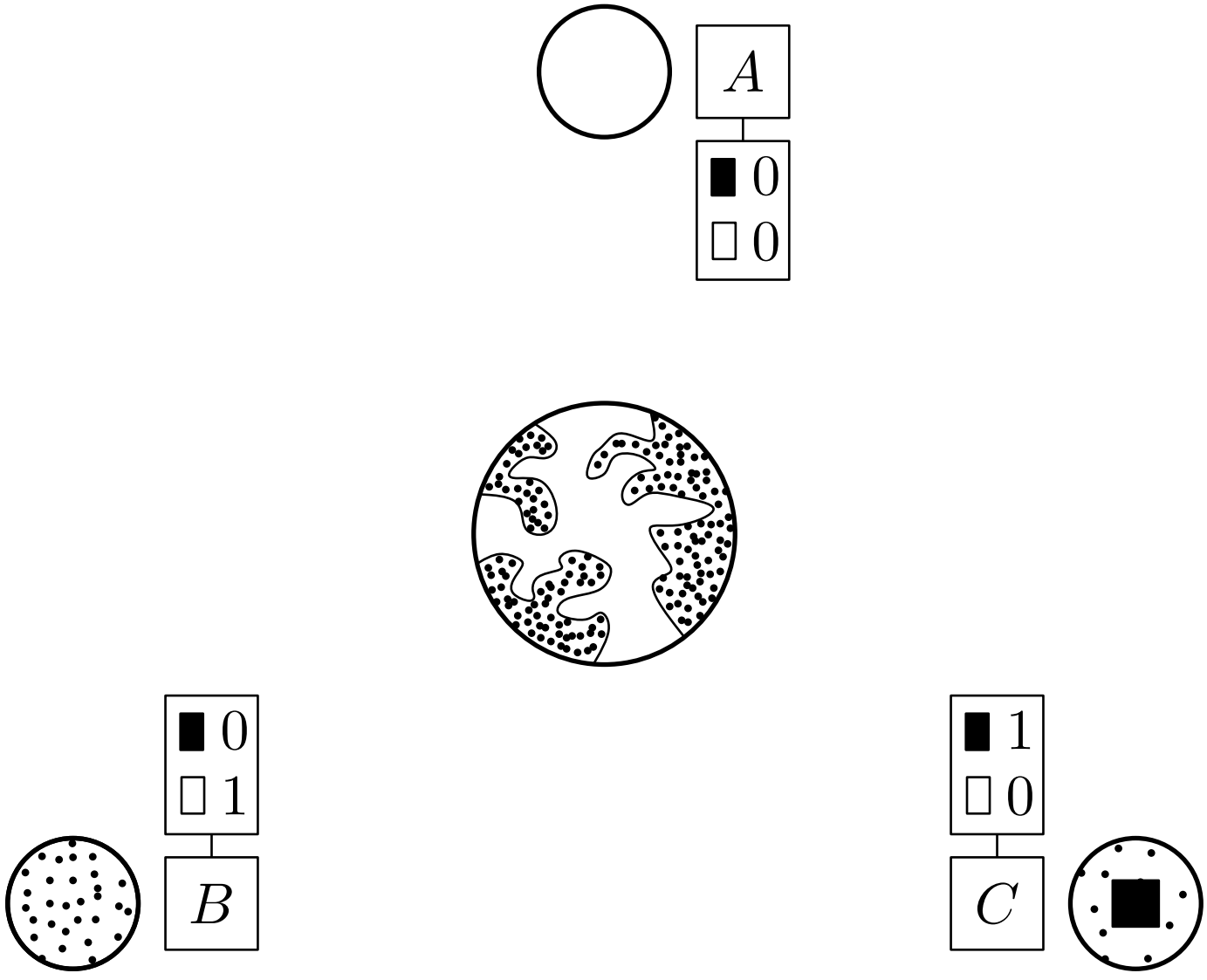

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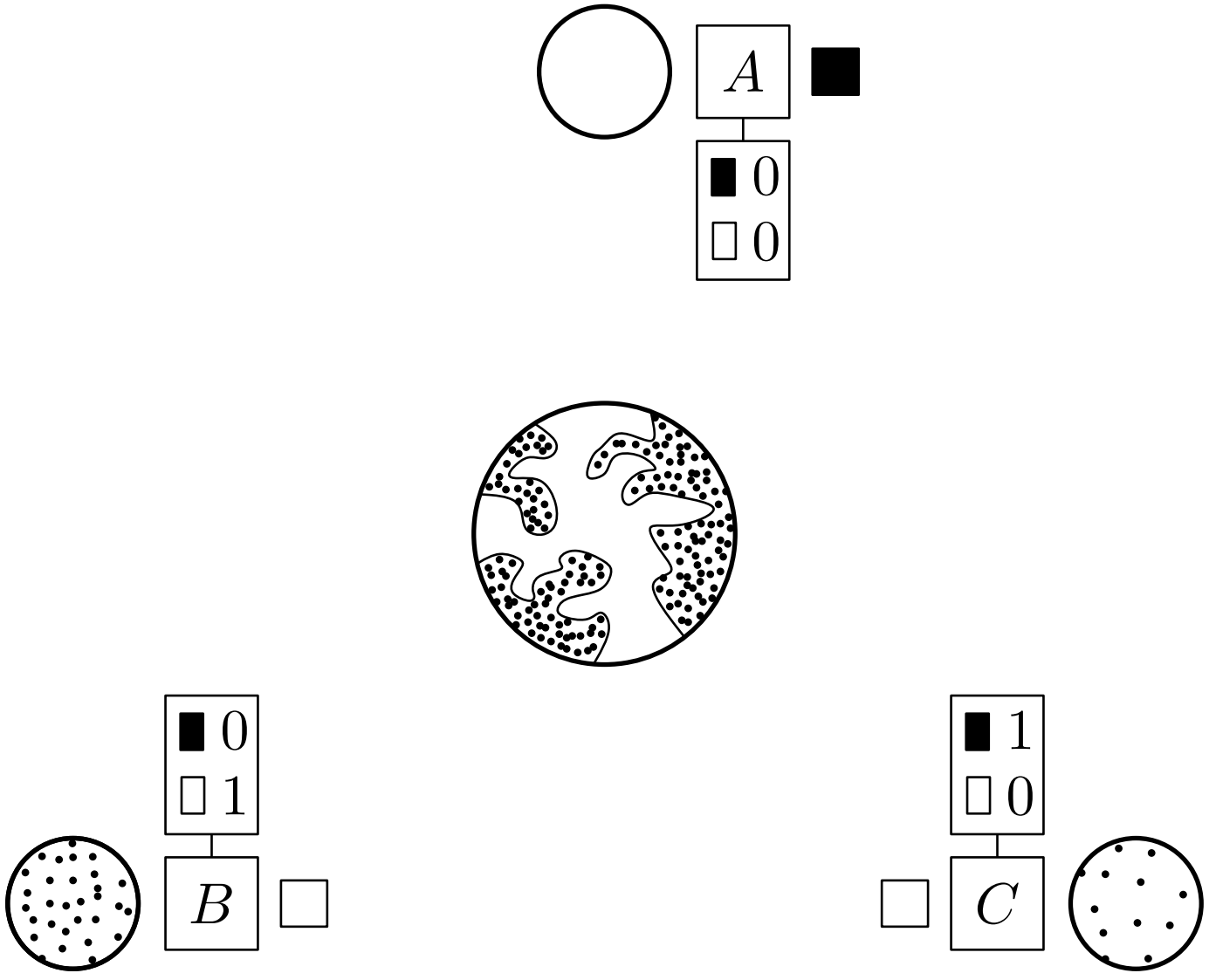

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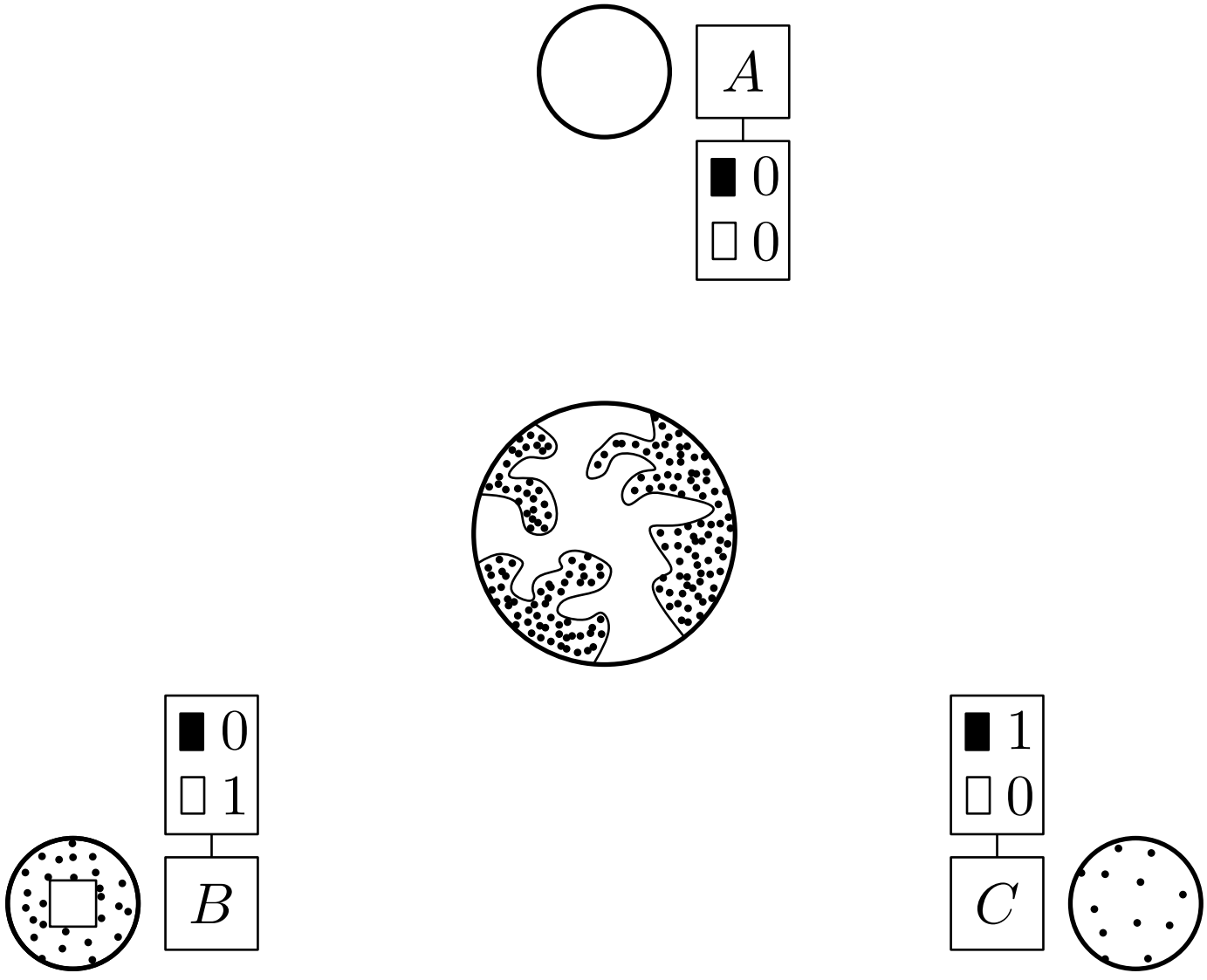

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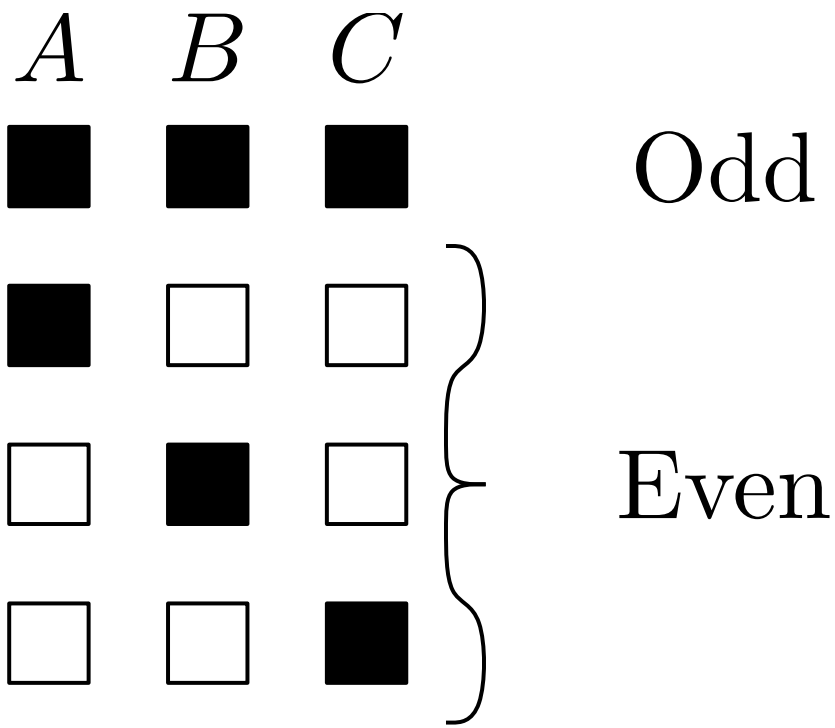

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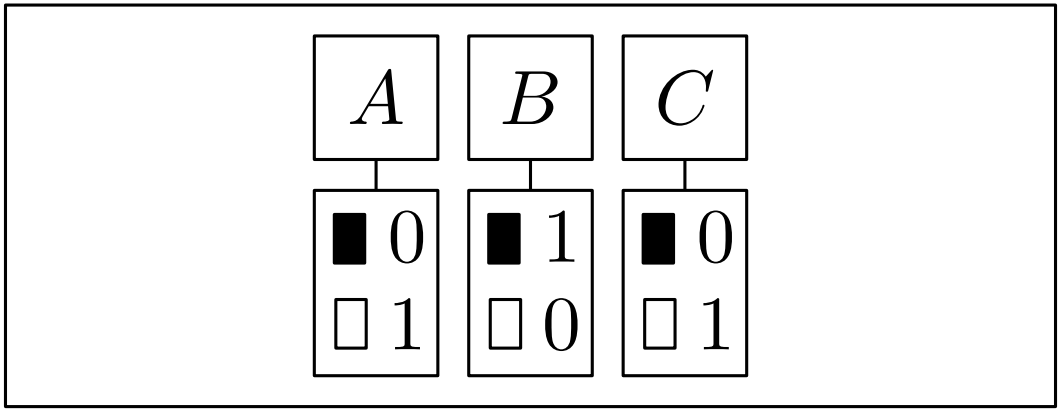

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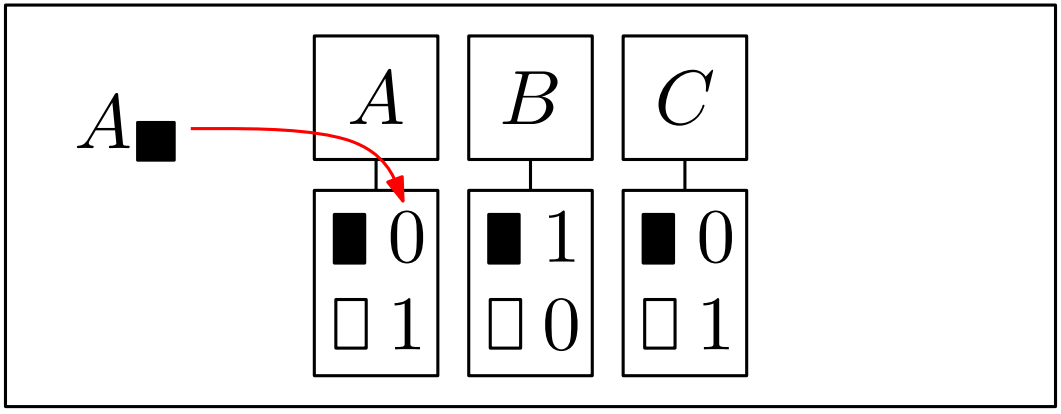

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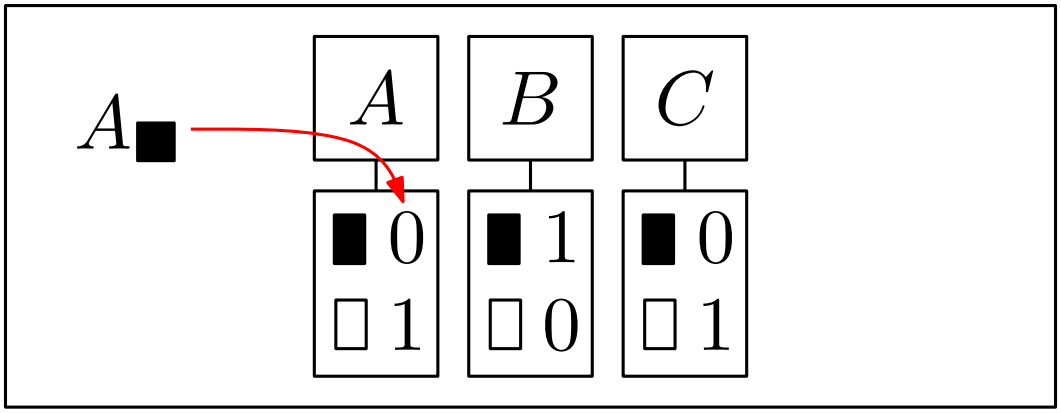

$$A_{\blacksquare} + B_{\blacksquare} + C_{\blacksquare} = \text{Odd}$$

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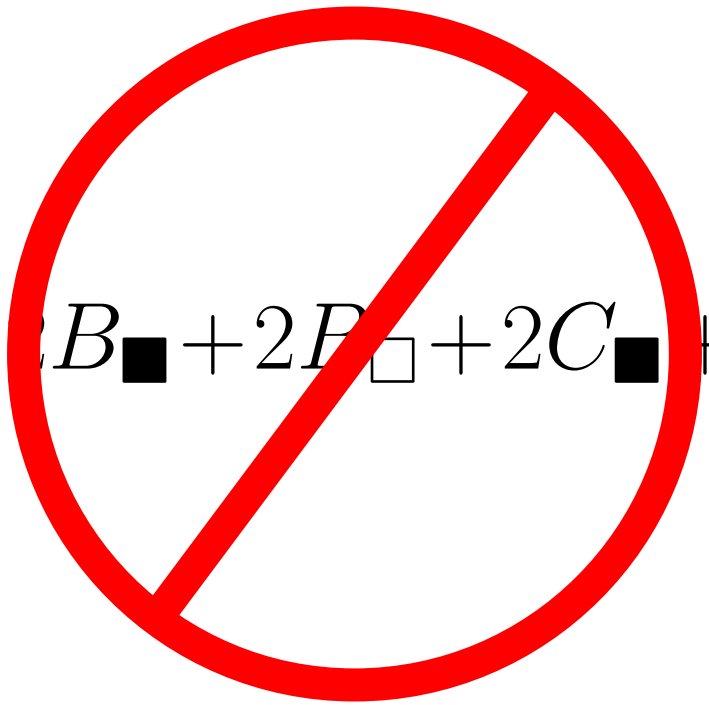
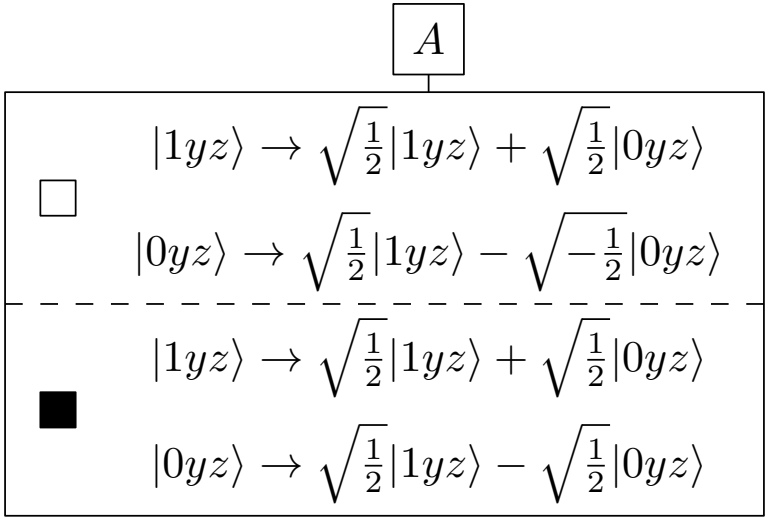


Quantum Strategies



If ■ arrives. . .

$$\sqrt{\frac{1}{2}}|111\rangle + \sqrt{\frac{1}{2}}|000\rangle \to \frac{1}{2}|111\rangle + \frac{1}{2}|011\rangle + \frac{1}{2}|100\rangle - \frac{1}{2}|000\rangle$$

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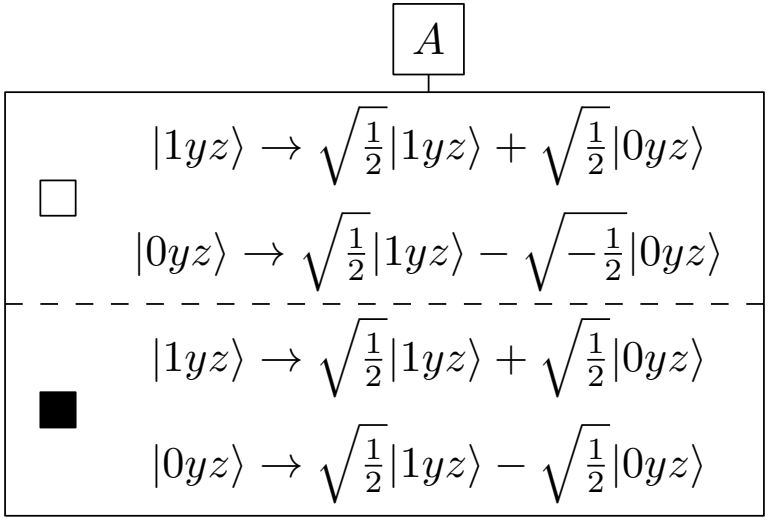

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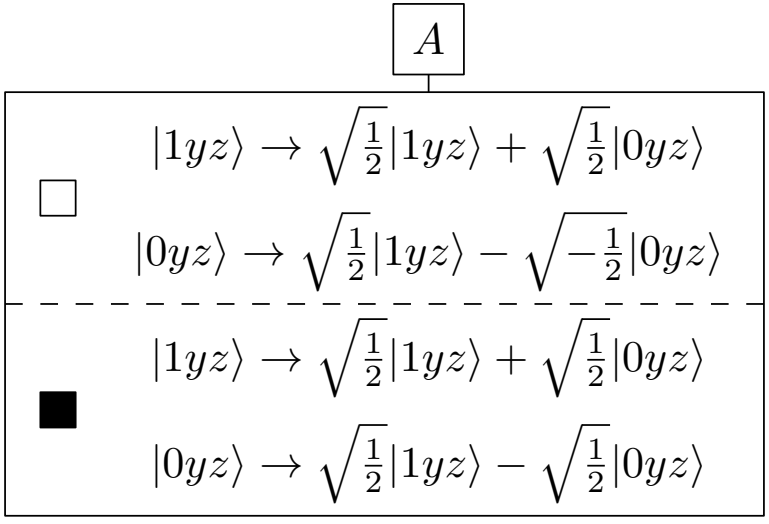

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Section 4

Probabilistic Computational Models

Probabilistic Turing Machines

- Generalizing ordinary Turing Machines to the probabilistic setting is a relatively easy task.
- A **Turing Machine** is a quadruple (Q, Σ, δ, F).
  - In presence of **determinism**,

$$\delta : Q \times \Sigma \to Q \times \Sigma \times \{\leftarrow, \downarrow, \rightarrow\}.$$

  - When evolution is **nondeterministic**,

$$\delta : Q \times \Sigma \to \mathcal{P}(Q \times \Sigma \times \{\leftarrow, \downarrow, \rightarrow\}).$$

  - When evolution is **probabilistic**,

$$\delta : Q \times \Sigma \to \mathcal{D}(Q \times \Sigma \times \{\leftarrow, \downarrow, \rightarrow\}).$$

    where $\mathcal{D}(A)$ is the set of all functions f from A to $\mathbb{R}$ such that $\sum_{a \in A} f(a) = 1$.
- In PTMs, any computation $\rho : C_1 \to C_2 \to \ldots \to C_n$ has a probability.

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PTMs, Languages and Functions

- In PTM, **termination** and **acceptance of a string** become probabilistic events.
- A PTM can terminate with probability 1 even if there is the *possibility* of nontermination.
- A given string $s \in \Sigma^*$ is accepted *with a probability* between 0 and 1, included.
- How can we define language recognition?
  - One can consider different, alternative, constraints on the **error** probability:
    - It must be 0.
    - It must be smaller than $\frac{1}{2}$;
    - It must be smaller than a given fixed constant ε, itself smaller than $\frac{1}{2}$.
  - One can stipulate that the given result is the one with **highest probability**.

Section 5

Quantum Computational Models

No-Cloning Theorem

- Let us consider a quantum system H_n with computational base $\mathbf{x}_1, \ldots, \mathbf{x}_n$.
- A unitary transformation $\mathbf{Q}$ on $H_n \otimes H_n$ is a **cloning map** if $\mathbf{Q}|\mathbf{x}\rangle|\mathbf{x}_1\rangle = |\mathbf{x}\rangle|\mathbf{x}\rangle$.

Theorem (No-Cloning)

For $n \geq 2$, there is no cloning map.

- This does *not* mean that states **of the computational base** cannot be cloned.
- There is an analogous theorem about **erasing maps**.

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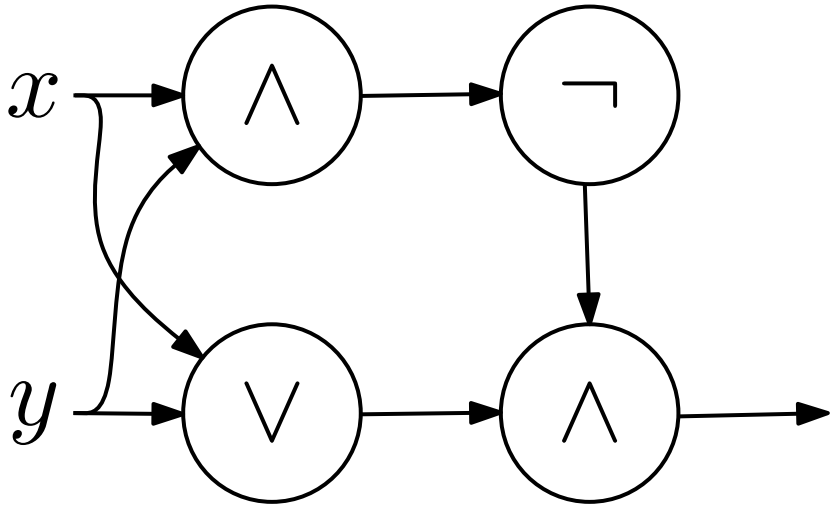
Boolean Circuits

- As we all know, they are **networks** of gates which compute very simple functions on $\mathbb{F} = \{0, 1\}$.
- Example:



- *Every* function $f : \mathbb{F}^n \to \mathbb{F}^m$ is computable by a boolean circuit C_f.
- Could we have finite, **universal**, set of gates?
  - Yes, e.g., $\{\wedge, \vee, \neg\}$.

But... How About Turing Machines?

- Boolean circuits can be seen as a **minimal** computational model, whose expressive power turns out to be **very poor**.
- There are fundamental differences with Turing Machines:
  - In boolean circuits, everything is (even potentially) finite.
  - Functions computed by circuits and by TM are functions from different domains: $\mathbb{F}^n \to \mathbb{F}^m$ vs. $\mathbb{F}^* \to \mathbb{F}^*$.
  - Boolean circuits are somehow **complete**, while TM are not, as we all know.
- Can we somehow reconcile the two worlds?
  - A **circuit family** is a collection $\{C_n\}_{n \in \mathbb{N}}$.
  - A circuit family **computes** a function from $\mathbb{F}^*$ to $\mathbb{F}^*$.
  - A circuit family is said to be **uniform** if there is a TM which, on input n, outputs (an enconding) of C_n.

Theorem

The class of functions computed by TMs equals the class of functions computed by uniform circuit families.

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- They themselves consist in **gate networks**, which can be of three different kinds:
  - **Quantum gates**, which compute unitary transforms on $H_2 \otimes \ldots \otimes H_2$.
  - **Measurement gates**, which observe the value of one qubit.
- The wires connecting gates are themselves of two types:
  - **Quanum channels**, which connect quantum gates to other quantum gates or to measurement gates.
  - **Classic channels**, which are the output wires of measurement gates and which control the application of quantum gates.
- Qubits cannot be duplicated.
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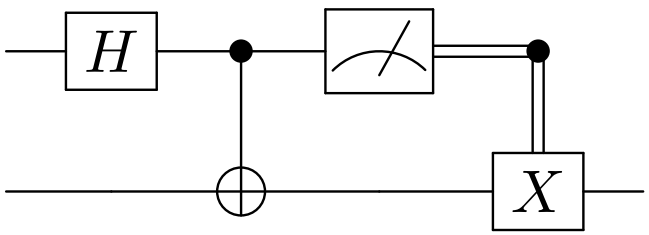
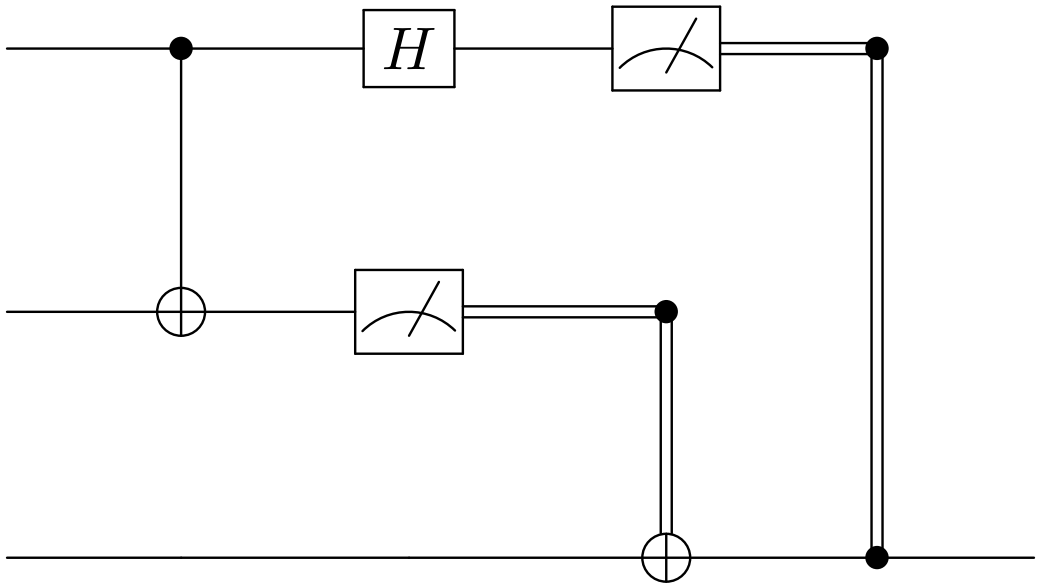


Quantum Teleportation



Boolean Circuits as Quantum Circuits

- When can a boolean gate be considered a quantum gate?
- How to simulate copying?
- First of all, we need to allow boolean gates to have **more than one** outputs.
- Restrict to **invertible** gates.
  - Indeed, every unitary transform $\mathbf{Q}$ has an inverse!
- Every boolean circuit can then be seen as a quantum circuit.
- Are we loosing anything?

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Every boolean function can be computed by a quantum circuit.

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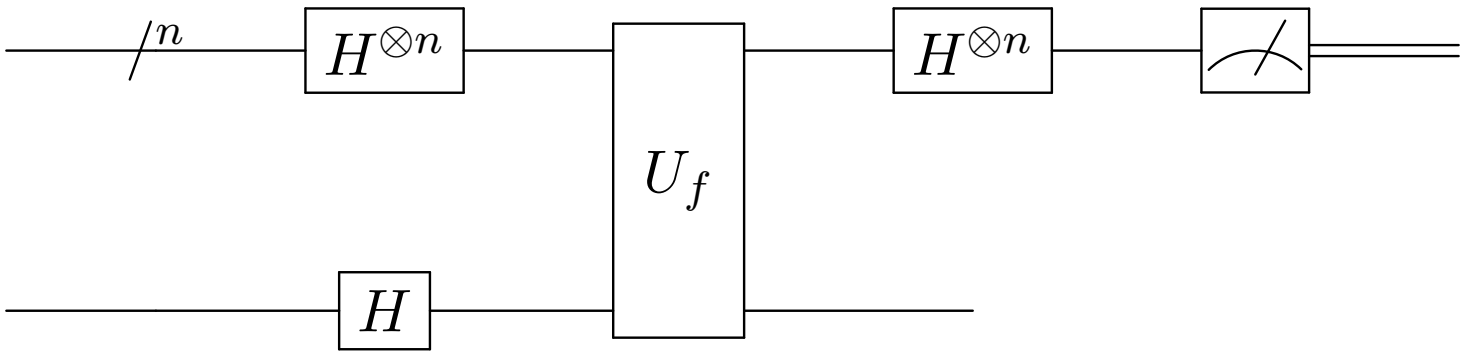

Universal Collections of Gates

- Is it possible to single out a **finite** set of quantum gates such that **all** quantum circuits can be expressed by way of these ports?

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Every quantum circuit is equivalent to a circuit built from ***unary*** *quantum gates and the* **CNOT** *port.*

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Section 6

(Probabilistic and Quantum) Computational Complexity

Computing Functions and Solve Problems

- How could we formalize that a function $f : \mathbb{F}^* \to \mathbb{F}^*$ can be computed **mechanically**?
  - The answer is simple: is there a TM computing f?
  - A TM **computes** f iff whenever $s \in \mathbb{F}^*$ the machine always halts producing $f(s) \in \mathbb{F}^*$ in output.
  - $\mathcal{FR}$ is the corresponding class of functions.
- What could we say about **computational problems**, i.e., about any $L \subseteq \mathbb{F}^*$?
  - L is said to be **recursive** iff the function f_L defined as follows is computable:

$$f_L(x) = \begin{cases} 0 & \text{if } x \in L \\ 1 & \text{otherwise} \end{cases}$$

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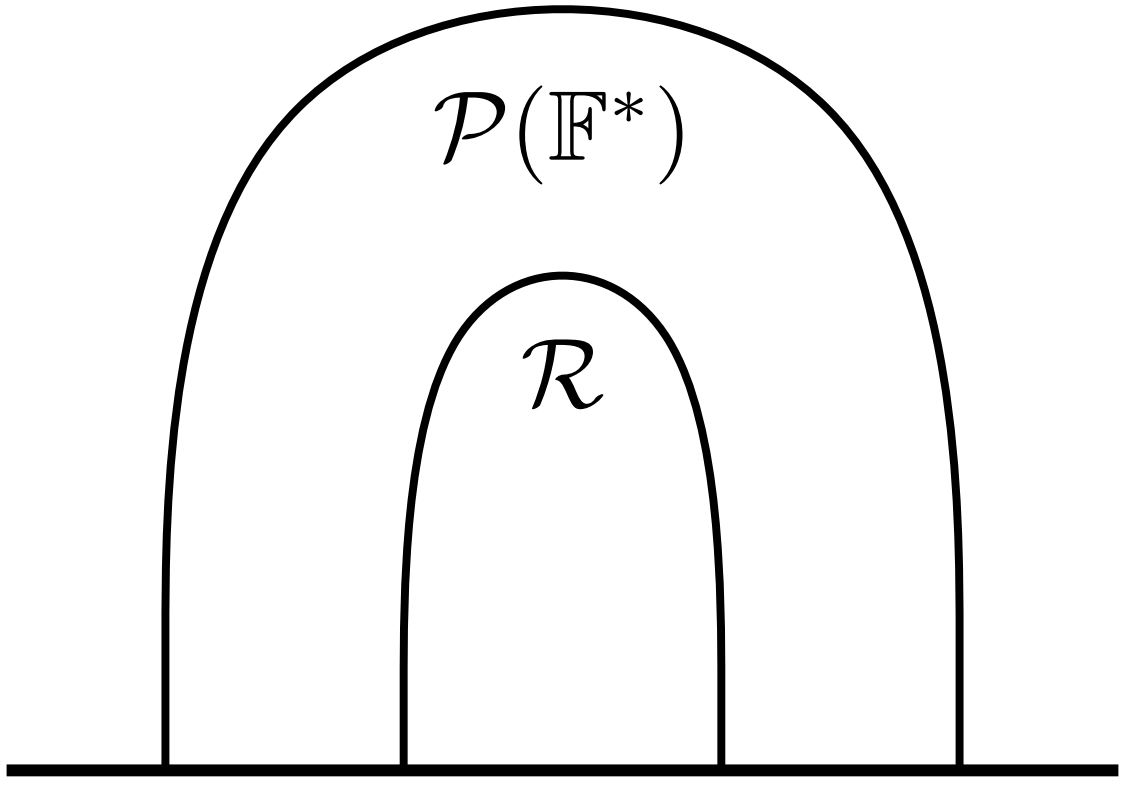
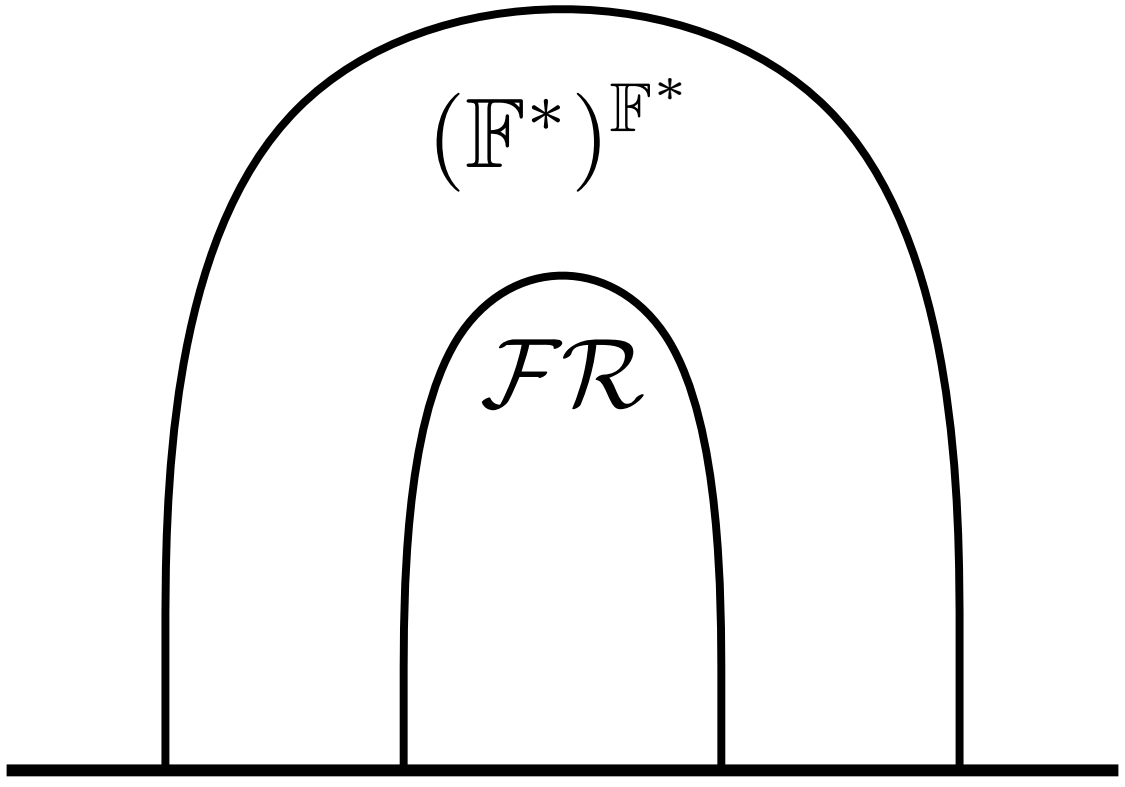
  - $\mathcal{R}$ is the corresponding class of problems.


$\mathcal{P}(\mathbb{F}^*)$
$\mathcal{R}$


$(\mathbb{F}^*)^{\mathbb{F}^*}$
$\mathcal{FR}$

Computing with Limited Resources

- Some computable functions seem to require too much **time** (and/or **space**) to be computed.
- Let us consider the function $A : \mathbb{N}^2 \to \mathbb{N}$ defined as follows:

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0; \\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0; \\ A(m-1,A(m,n-1)) & \text{if } m > 0 \text{ and } n > 0. \end{cases}$$

- The value $A(m,n)$ increases **too fast** when m and n grow, e.g.,

$$A(4,3) \approx 10^{6031 \cdot 10^{19727}}.$$

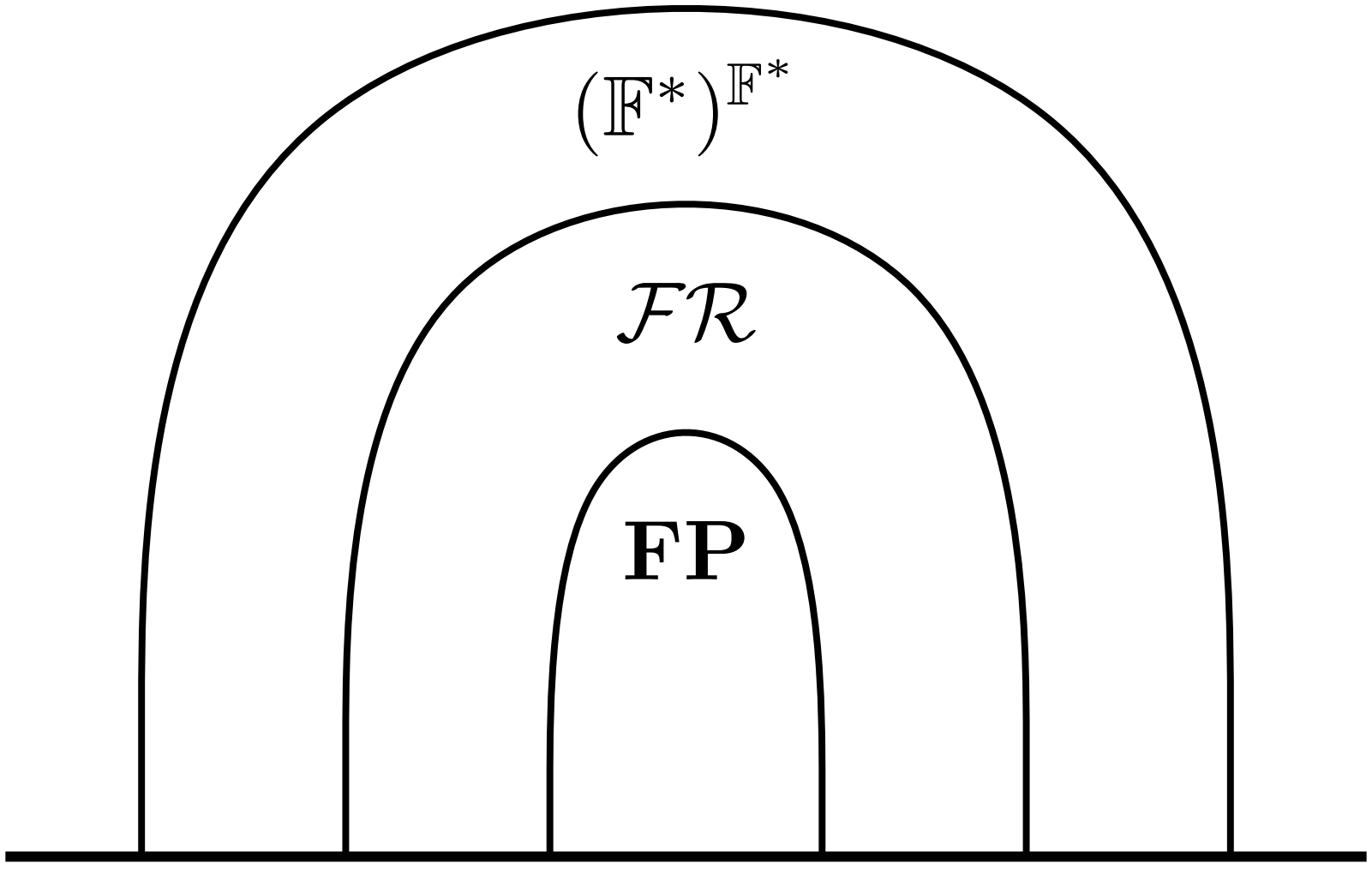
- Ideally, we would like the time necessary to compute a function on $s \in \mathbb{F}^*$ to be bounded by a **fixed** polynomial on $|s|$.

Polynomial Time

- A TM M **works** in time $g : \mathbb{N} \to \mathbb{N}$ iff for every $s \in \mathbb{F}^*$, the computation of M on input s takes at most $g(s)$ steps. In this case, we will write that $M \in Time(g)$.
- The complexity class **FP** is the collection of all functions from $\mathbb{F}^*$ in $\mathbb{F}^*$ which are computable by a TM M in

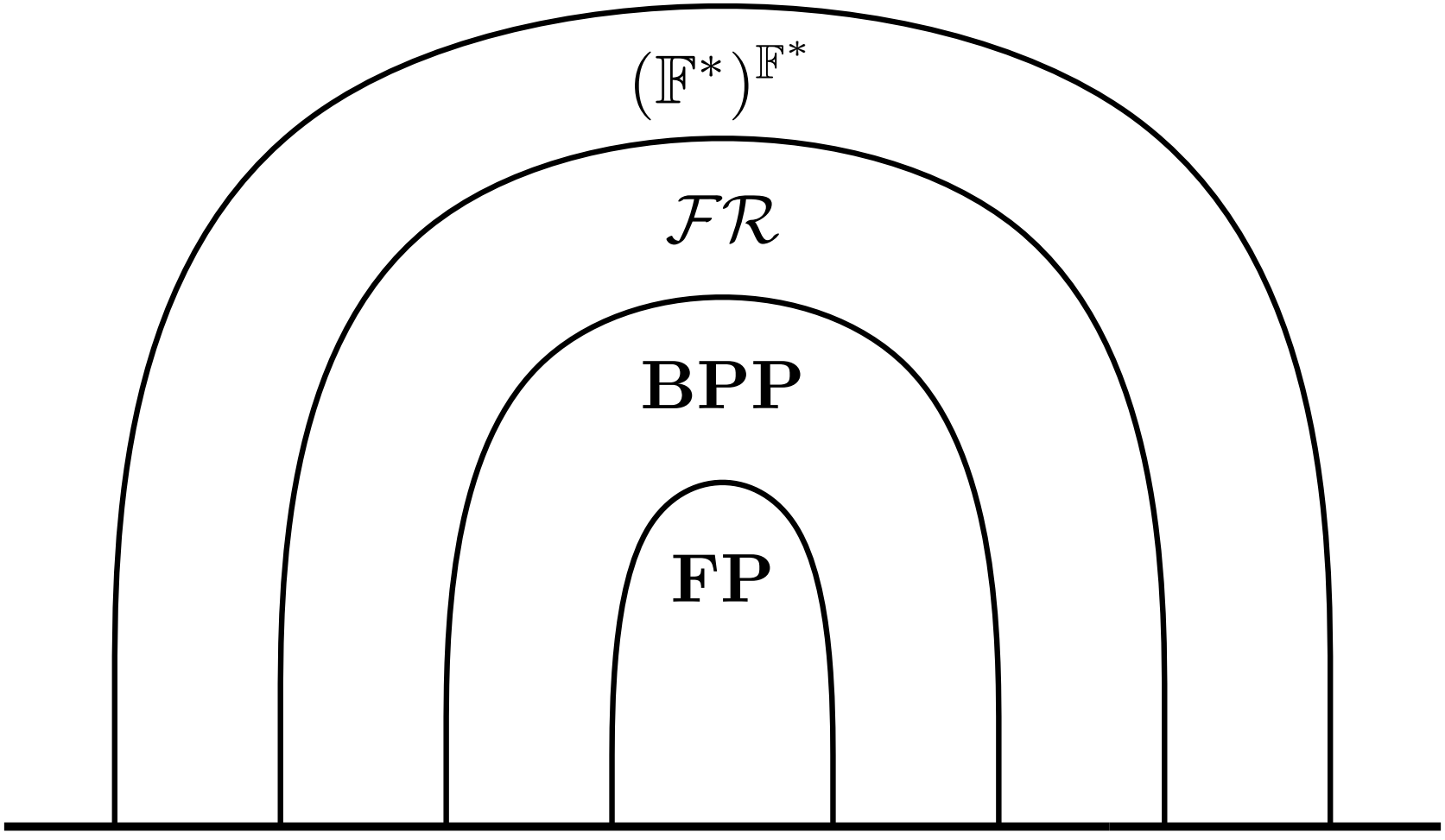
$$\bigcup_{g \in \mathsf{POLY}} Time(g).$$

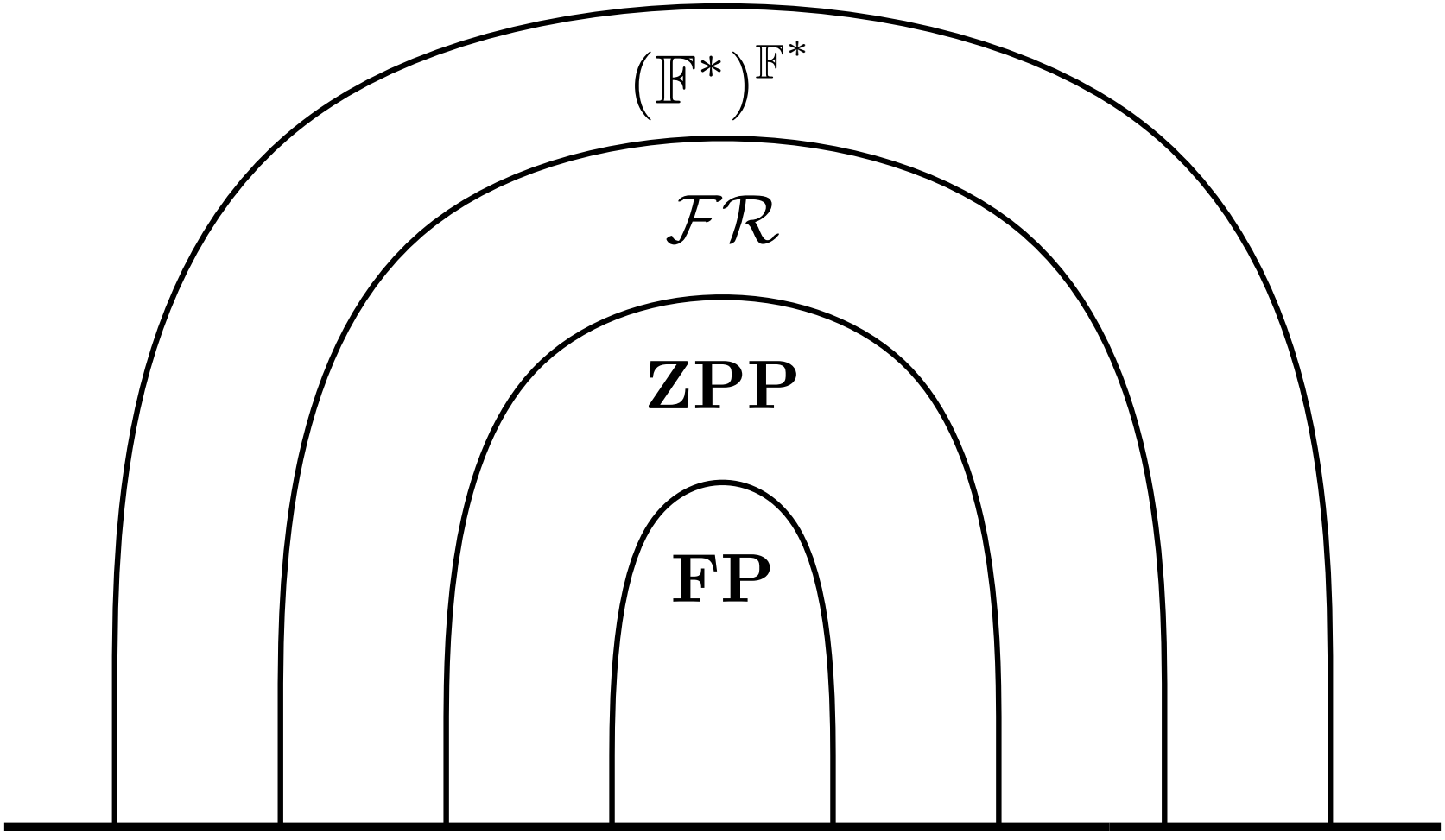
  where POLY is the space of all polynomials from $\mathbb{N}$ to $\mathbb{N}$.
- Knowing that a function is **not** in **FP** can sometime help a lot.....


$(\mathbb{F}^*)^{\mathbb{F}^*}$
$\mathcal{FR}$
FP

Probabilistic Polynomial Time

- How should we capture the concept of feasibility in a probabilistic setting?
- There is a tradeoff between bounding running times and bounding the error...
  - We can insist on having **no** error, but allow computation to take polynomial time only on the average, obtaining the class **ZPP**.
  - We can allow the error to be smaller than a constant $\varepsilon < \frac{1}{2}$, but insist on polynomial time bounds to hold independently on the random choices, getting the class **BPP**.
- There are many other possibilities, but these are somehow less interesting, giving rise to classes which are too large.


$(\mathbb{F}^*)^{\mathbb{F}^*}$
$\mathcal{FR}$
BPP
FP


$(\mathbb{F}^*)^{\mathbb{F}^*}$
$\mathcal{FR}$
ZPP
FP

Quantum Polynomial Time

- If, when defining **FP**, we replace the TM M with a **quantum** TM N, then the underlying complexity class is, again different at least in principle.
- It is very convenient, however, to consider **quantum circuit families** which are generated by deterministic Turing Machines.
  - Timing constraints are captured by imposing that the generating TM works in polynomial time.
- The result of a quantum computation, however, is not **a** string in $\mathbb{F}^*$ but, instead, a **probability distribution** on $\mathbb{F}^*$.
- But of course, we should also take the **error** into account
  - It suffices to impose that the error is strictly smaller than a constant $\varepsilon < \frac{1}{2}$, independently on the size of the input.
  - This way we obtain the complexity class **BQP**.

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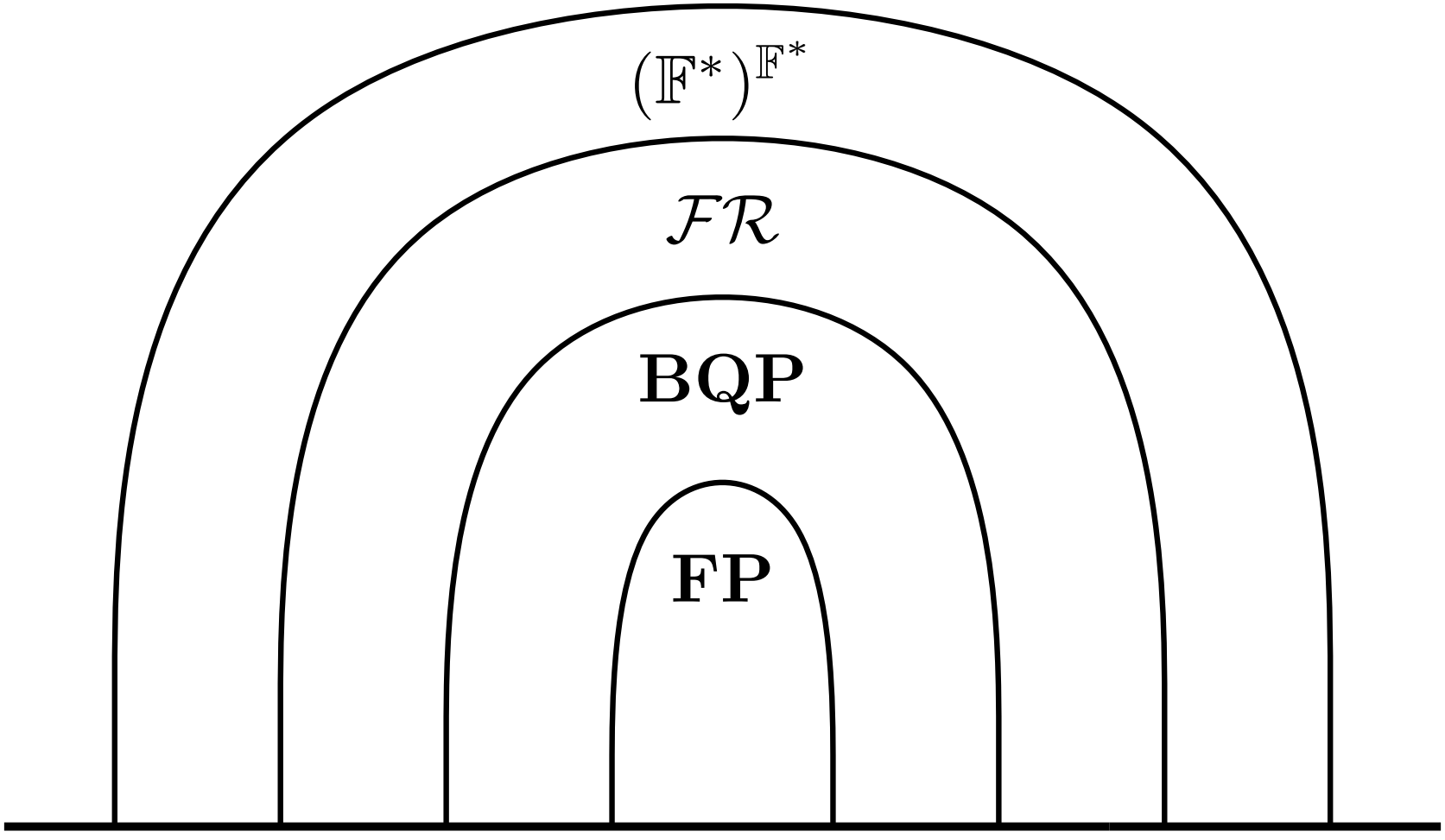
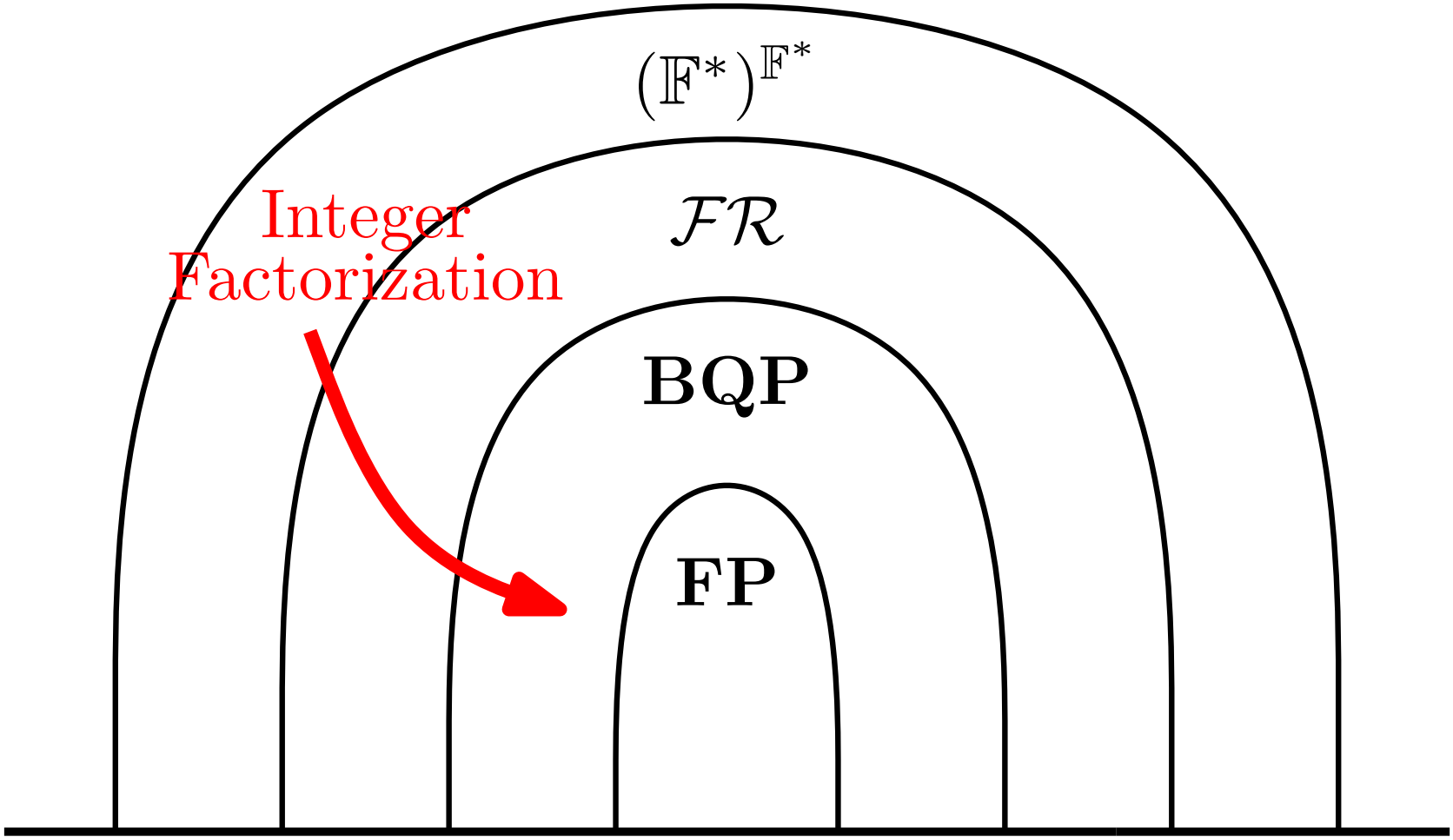

$(\mathbb{F}^*)^{\mathbb{F}^*}$
$\mathcal{FR}$
BQP
FP


$(\mathbb{F}^*)^{\mathbb{F}^*}$
$\mathcal{FR}$
Integer
Factorization
BQP
FP

Thank You!

Questions?