Lambda-Calculus and Type Theory
Part III

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Section 1

Classical Logic and the $\lambda\mu$-Calculus

Propositional Classical Logic

- If we restrict our attention to the fragment with implication and $\bot$, we get the system $CPL_{\to,\bot}$, which is defined as follows:

$$\frac{}{\Gamma, \varphi \vdash \varphi}\ AX \qquad \frac{\Gamma, \varphi \to \bot \vdash \bot}{\Gamma \vdash \varphi}\ E\neg$$

$$\frac{\Gamma, \varphi \vdash \tau}{\Gamma \vdash \varphi \to \tau}\ I\to \qquad \frac{\Gamma \vdash \varphi \to \tau \quad \Gamma \vdash \varphi}{\Gamma \vdash \tau}\ E\to$$

- Rule $E\neg$ subsumes $E\bot$. Moreover, the following two rules are equivalent to it:

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} \qquad \frac{\Gamma, \varphi \to \tau \vdash \varphi}{\Gamma \vdash \varphi}$$

Theorem

Propositional classical logic is sound and complete for classical validity.

The $\lambda\mu$-Calculus

- Terms of the Church-style $\lambda\mu$-calculus are the following ones:

$$M ::= x \;\Big|\; (MM) \;\Big|\; (\lambda x : \varphi M) \;\Big|$$
$$([a]M) \;\Big|\; (\mu a : \neg\varphi M)$$

  where:
  - a ranges over a set Ξ of *addresses*;
  - φ ranges over $CPL_{\to,\bot}$ formulas.
- The notions of α-conversion and substitution can be generalized from the λ-calculus to the $\lambda\mu$-calculus, keeping in mind that:
  - $\mu a : \neg\varphi M$ acts as a binder for a in M.
  - Only variables can be substiuted for terms, while addresses are subsituted for more complicated objects, to be defined later.

The $\lambda\mu$-Calculus

- Environments now assigns types to variables and *negated* types to addresses.
- Typing rules are as follows:

$$\frac{}{\Gamma, x : \varphi \vdash x : \varphi}\ V$$

$$\frac{\Gamma, x : \varphi \vdash M : \tau}{\Gamma \vdash \lambda x M : \varphi \to \tau}\ \lambda \qquad \frac{\Gamma \vdash M : \varphi \to \tau \quad \Gamma \vdash N : \varphi}{\Gamma \vdash MN : \tau}\ @$$

$$\frac{\Gamma, a : \neg\varphi \vdash M : \bot}{\Gamma \vdash \mu a : \neg\varphi M : \varphi}\ \mu \qquad \frac{\Gamma, a : \neg\varphi \vdash M : \varphi}{\Gamma, a : \neg\varphi \vdash [a]M : \bot}\ [\cdot]$$

The $\lambda\mu$-Calculus: Reduction

- $\lambda\mu$-*contexts* are defined as follows:

$$C ::= [\cdot] \;\Big|\; CM \;\Big|\; [a]C$$

- Given a $\lambda\mu$-context C and a $\lambda\mu$-term M, the term $C[M]$ is defined as follows:

$$[\cdot][M] = M;$$
$$(CN)[M] = (C[M])N;$$
$$([a]C)[M] = [a](C[M]).$$

- The concept of free variable set is generalized to contexts as follows:

$$FV([\cdot]) = \emptyset;$$
$$FV(CM) = FV(C) \cup FV(M);$$
$$FV([a]C) = \{a\} \cup FV(C)$$

The $\lambda\mu$-Calculus: Reduction

- For a $\lambda\mu$-context C, an address a, and a $\lambda\mu$-term M, we define the susbstitution $M[a := C]$ as follows:

$$
\begin{aligned}
x[a := C] &= x \\
(\lambda y : \varphi M)[a := C] &= \lambda y : \varphi(M[a := C]) \\
(MN)[a := C] &= (M[a := C])(N[a := C]) \\
(\mu b : \neg\varphi M)[a := C] &= \mu b : \neg\varphi(M[a := C]) \\
([a]M)[a := C] &= C[M[a := C]] \\
([b]M)[a := C] &= [b](M[a := C])
\end{aligned}
$$

  where where $y, b \notin FV(C)$ and $a \neq b$.

- The relation $\to_\mu$ denotes the least compatible relation containing all the following pairs:

$$
\begin{aligned}
(\lambda x : \varphi M)N &\to_\mu M[x := N]; \\
\mu a : \neg\varphi[a]M &\to_\mu M \text{ if } a \notin FV(M); \\
[b](\mu a : \neg\varphi M) &\to_\mu M[a := ([b][\cdot])]; \\
(\mu a : \neg(\varphi \to \tau)M)N &\to_\mu \mu b : \neg\tau(M[a := ([b]([\cdot]N))]) \text{ if } a \neq b \notin FV(MN)
\end{aligned}
$$

The $\lambda\mu$-Calculus: Confluence and Normalisation

Theorem (Subject Reduction Theorem)

If $\Gamma \vdash M : \varphi$ and $M \to_\mu^* N$, then $\Gamma \vdash N : \varphi$.

Theorem (Strong Normalization)

The relation $\to_\mu$ on $\lambda\mu$-terms is strongly normalizing.

Theorem (Confluence)

The relation $\to_\mu$ on $\lambda\mu$-terms is weakly Church-Rosser, thus Church-Rosser.

Section 2

Gödel's $\mathbb{T}$ and Arithmetic

Gödel's $\mathbb{T}$: Statics

- It can be seen as an extension of the simply-typed λ-calculus natural numbers and a form of primitive recursion are directly available, rather than encoded.
- Types are extended as follows:

$$\varphi ::= \cdots \;\Big|\; \mathbf{int} \;\Big|\; \mathbf{unit}$$

- Terms are extended as follows:

$$M :: \cdots \;\Big|\; \mathsf{R} \;\Big|\; \mathsf{X} \;\Big|\; \mathsf{Z} \;\Big|\; *$$

where the new constants can be typed by the following rules:

$$\frac{}{\Gamma \vdash \mathsf{R} : \mathbf{int} \to \varphi \to (\mathbf{int} \to \varphi \to \varphi) \to \varphi}\ \mathtt{R} \qquad \frac{}{\Gamma \vdash \mathsf{X} : \mathbf{int} \to \mathbf{int}}\ \mathtt{X}$$

$$\frac{}{\Gamma \vdash \mathsf{Z} : \mathbf{int}}\ \mathtt{Z} \qquad \frac{}{\Gamma \vdash * : \mathbf{unit}}\ *$$

Gödel's $\mathbb{T}$: Dynamics

- Two new reduction rules are necessary:

$$\mathsf{R}\,\mathsf{Z}\,M\,N \to_\beta M;$$
$$\mathsf{R}\,(\mathsf{X}L)\,M\,N \to_\beta N\,L(\mathsf{R}\,L\,M\,N)$$

Theorem (Strong Normalization)

The relation $\twoheadrightarrow_\beta$ on terms of Gödel's $\mathbf{T}$ is strongly normalizing.

- A function $f : \mathbb{N}^k \to \mathbb{N}$ is *definable in* $\mathbb{T}$ if there is a term M_f with $\emptyset \vdash M_f : \mathbf{int}^k \to \mathbf{int}$ such that

$$M_f\overline{n_1}\cdots\overline{n_k} \twoheadrightarrow_\beta \overline{f(n_1,\ldots,n_k)}$$

where $\overline{n}$ is the encoding of n via Z and X.

Theorem

The class of total functions on $\mathbb{N}$ which are λ-definable in $\mathbb{T}$ coincides with those which are provably total in Peano's arithmetic.

Section 3

Second-Order Logic and Polymorphism

Second-Order Propositional Logic

- *Formulas* are the expressions derived from the following grammar:

$$\varphi ::= \bot \;\Big|\; p \;\Big|\; \varphi \to \varphi \;\Big|\; \varphi \wedge \varphi \;\Big|\; \varphi \vee \varphi$$
$$\Big|\; \exists p \varphi \;\Big|\; \forall p \varphi$$

- Usual concepts, like those of α-conversion, free variables, and the like are generalized easily to this new setting.
- Classically, the following two logical equivalences hold:

$$\exists p\varphi \iff \varphi[p := \neg\bot] \vee \varphi[p := \bot]$$
$$\forall p\varphi \iff \varphi[p := \neg\bot] \wedge \varphi[p := \bot]$$

  where $\neg\varphi := \varphi \to \bot$.
- Intuitionistically, the same does *not hold*.

Second-Order Propositional Logic

- Getting a natural deduction system for second-order propositional logic boils down to extending natural deduction for propositional logic with the following four rules:

$$\frac{\Gamma \vdash \varphi[p := \tau]}{\Gamma \vdash \exists p \varphi}\ I\exists \qquad \frac{\Gamma \vdash \exists p \varphi \quad \Gamma, \varphi \vdash \tau \quad p \notin FV(\Gamma, \tau)}{\Gamma \vdash \tau}\ E\exists$$

$$\frac{\Gamma \vdash \varphi \quad p \notin FV(\Gamma)}{\Gamma \vdash \forall p \varphi}\ I\forall \qquad \frac{\Gamma \vdash \forall p \varphi}{\Gamma \vdash \varphi[p := \tau]}\ E\forall$$

- Both Heyting and Kripke Semantics can be generalized to second-order quantifiers.

Theorem (Completeness)

$\Gamma \vdash \varphi$ *if and only if* $\Gamma \models \varphi$ *if and only if* $\Gamma \Vdash \varphi$.

The Polymorphic λ-Calculus

- Types are taken as second-order propositional formulas including only the operators $\rightarrow$ and $\forall$.
- Terms are generated by the following grammar:

$$M ::= x \;\Big|\; \lambda x : \varphi M \;\Big|\; (MM) \;\Big|\; \Lambda p M \;\Big|\; M\varphi$$

  $\Lambda p M$ binds p in M.
- Typing rules are the following ones:

$$\frac{}{\Gamma, x : \varphi \vdash x : \varphi}\; V$$

$$\frac{\Gamma, x : \varphi \vdash M : \tau}{\Gamma \vdash \lambda x M : \varphi \rightarrow \tau}\; \lambda \qquad \frac{\Gamma \vdash M : \varphi \rightarrow \tau \quad \Gamma \vdash N : \varphi}{\Gamma \vdash MN : \tau}\; @$$

$$\frac{\Gamma \vdash M : \varphi \quad p \notin FV(\Gamma)}{\Gamma \vdash \Lambda p M : \forall p \varphi}\; \Lambda \qquad \frac{\Gamma \vdash M : \forall p \varphi}{\Gamma \vdash (M\tau) : \varphi[p := \tau]}\; @_{\Lambda}$$

The Polymorphic λ-Calculus: Reduction

- The relation $\to_\Lambda$ denotes the least compatible relation such that

$$(\lambda x : \varphi M)N \to_\Lambda M[x := N];$$
$$(\Lambda p M)\varphi \to_\Lambda M[p := \varphi].$$

Theorem (Subject Reduction Theorem)

If $\Gamma \vdash M : \varphi$ and $M \to_\Lambda^* N$, then $\Gamma \vdash N : \varphi$.

Theorem (Strong Normalization)

The relation $\to_\Lambda$ on $\lambda\mu$-terms is strongly normalizing.

Theorem (Confluence)

The relation $\to_\Lambda$ on $\lambda\mu$-terms is weakly Church-Rosser, thus Church Rosser.