Propositional Intuitionistic Logic: Natural Deduction

- **Formulas** are derived by the grammar
  \( \varphi ::= \bot \mid p \mid \varphi \rightarrow \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \), where \( p \) ranges over a set \( \Theta \) of propositional variables.

- **Judgments** have the form \( \Gamma \vdash \varphi \), where \( \Gamma \) is a finite set of formulas. Given two such sets \( \Gamma \) and \( \Delta \), their union is indicated as \( \Gamma, \Delta \).

- The **rules** of propositional intuitionistic logic are as follows:

\[
\begin{align*}
\frac{\text{AX}}{\Gamma, \varphi \vdash \varphi} \\
\frac{\Gamma, \varphi \vdash \tau}{\Gamma \vdash \varphi \rightarrow \tau} & \quad I \rightarrow \\
\frac{\Gamma \vdash \varphi \rightarrow \tau \quad \Gamma \vdash \varphi}{\Gamma \vdash \tau} & \quad E \rightarrow \\
\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \tau}{\Gamma \vdash \varphi \land \tau} & \quad I \land \\
\frac{\Gamma \vdash \varphi \land \tau}{\Gamma \vdash \varphi} & \quad E_{L \land} \\
\frac{\Gamma \vdash \varphi \land \tau}{\Gamma \vdash \tau} & \quad E_{R \land} \\
\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \lor \tau} & \quad I_{L \lor} \\
\frac{\Gamma \vdash \tau}{\Gamma \vdash \varphi \lor \tau} & \quad I_{R \lor} \\
\frac{\Gamma \vdash \varphi \lor \tau}{\Gamma \vdash \varphi} & \quad E_{\lor} \\
\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} & \quad E_{\bot}
\end{align*}
\]
Propositional Intuitionistic Logic: Semantics

**Heyting Algebras**
- Distributive lattices with top and bottom elements, in which relative pseudo-complement always exist.
- Meet and joins interpret conjunctions and disjunctions, respectively. Implication is given semantics by way of pseudo-complements.
- $\Gamma \models \varphi$ indicates that every Heyting Algebra validating $\Gamma$ also validates $\varphi$.

**Kripke Semantics**
- Propositional variables are put in relation with the elements of a partial order or *possible worlds*.
- Conjunction and disjunction are interpreted in Tarski-style, while implication is given semantics by way of the underlying partial order.
- $\Gamma \vdash \varphi$ indicates that every Kripke model validating $\Gamma$ also validates $\varphi$.

**Theorem (Completeness)**
$\Gamma \vdash \varphi$ if and only if $\Gamma \models \varphi$ if and only if $\Gamma \models \varphi$. 
Simply-Typed $\lambda$-Calculus à la Curry

- An implicational propositional formula is called a *simple type*. The set of all simple types is denoted by $\Phi \rightarrow$.

- An *environment* is a finite set $\Gamma$ of pairs of the form $\{x_1 : \varphi_1, \cdots, x_n : \varphi_n\}$, where the $x_i$ are distinct variables and $\varphi_i$ are simple types. In this case, $\text{dom}(\Gamma)$ is $\{x_1, \cdots, x_n\}$.

- A *typing judgement* is a triple $\Gamma \vdash M : \varphi$, consisting of an environment, a $\lambda$-term and a simple type.

- The rules are as follows:

  \[
  \begin{array}{c}
  \Gamma, x : \varphi \vdash x : \varphi \quad V \\
  \Gamma, x : \varphi \vdash M : \tau \quad \lambda \\
  \Gamma \vdash \lambda x M : \varphi \rightarrow \tau \\
  \Gamma \vdash M : \varphi \rightarrow \tau \quad \Gamma \vdash N : \varphi \\
  \Gamma \vdash MN : \tau \\
  \end{array}
  \]

- The obtained calculus is referred to as $ST \rightarrow$. 

\[\]

\[\]
Subject Reduction

Lemma (Generation Lemma)
Suppose that \( \Gamma \vdash M : \varphi \). Then:

1. If \( M \) is a variable \( x \), then \( \Gamma(x) = \varphi \);
2. If \( M \) is an application \( NL \), then there is \( \tau \) such that
   \( \Gamma \vdash N : \tau \rightarrow \varphi \) and \( \Gamma \vdash L : \tau \);
3. If \( M \) is an abstraction \( \lambda x N \) and \( x \notin \text{dom}(\Gamma) \), then
   \( \varphi = \tau \rightarrow \rho \), where \( \Gamma, x : \tau \vdash N : \rho \).

Lemma (Substitution Lemma)

1. If \( \Gamma \vdash M : \varphi \) and \( \Gamma(x) = \Delta(x) \) for every \( x \in \text{FV}(M) \), then
   \( \Delta \vdash M : \varphi \)
2. If \( \Gamma, x : \varphi \vdash M : \tau \) and \( \Gamma \vdash N : \varphi \), then \( \Gamma \vdash M[x := N] : \tau \).

Theorem (Subject Reduction Theorem)

If \( \Gamma \vdash M : \varphi \) and \( M \rightarrow^\beta N \), then \( \Gamma \vdash N : \varphi \).
> *Preterms* of the simply-typed \( \lambda \)-calculus à la Church are defined as follows:

\[
M ::= x \mid (MM) \mid (\lambda x : \varphi M).
\]

> The notions of substitution, \( \alpha \)-conversion, term, and reduction can be generalised to terms à la Church.

> Typing rules are the obvious ones:

\[
\begin{align*}
\Gamma, x : \varphi & \vdash x : \varphi & V \\
\Gamma, x : \varphi & \vdash M : \tau & \lambda \\
\Gamma & \vdash \lambda x : \varphi M : \varphi \rightarrow \tau & \lambda \\
\Gamma & \vdash M : \varphi \rightarrow \tau & \Gamma, x : \varphi \vdash M N : \tau & @
\end{align*}
\]

> The Subject Reduction Theorem can be easily reproved.
Proposition

*In the Simply-Typed $\lambda$-calculus à la Church, if $\Gamma \vdash M : \varphi$ and $\Gamma \vdash M : \tau$, then $\varphi = \tau$.*

- The erasing map $|\cdot|$ from terms à la Church to terms à la Curry is defined by induction on the structure of terms, as follows:

  $|x| := x$ \quad $|\lambda x : \varphi M| := \lambda x|M|$ \quad $|MN| := |M||N|$

- Typability and reduction judgments in the two styles can be translated into each other relatively easily.
Weak Normalisation

Theorem

Every term typable in $ST \rightarrow$ has a normal form.

The proof is based on the following key ideas:

- One can assign to each typable term $M$, a pair natural numbers $m_M := (\delta_M, n_M)$ in such a way that if $M$ is not a normal form, then there is $N$ with $M \rightarrow^\beta N$ and $m_M > m_N$ in the lexicographic order.
- Then, one proves the statement for every typable term $M$ by lexicographic induction on $m_M$.
- This is not a proof of strong normalisation.
Strong Normalisation

Theorem

*Every term typable in $ST \rightarrow$ is strongly normalising.*

- **A Proof Based on Reducibility.**
  - For every type $\varphi$, a set of terms $Red_{\varphi}$, the reducible terms;
  - A proof that any term of type $\varphi$ is in $Red_{\varphi}$;
  - A proof that any term in $Red_{\varphi}$ is strongly normalizing.

- **A Proof through $\lambda I$**
  - $\eta$-reduction is the smaller compatible relation $\rightarrow_\eta$ including pairs of the form $\lambda x(Mx) \rightarrow_\eta M$ (where $x \notin FV(M)$. $\rightarrow_{\beta\eta}$ is the union of $\rightarrow_\beta$ and $\rightarrow_\eta$.
  - In the $\lambda I$-calculus, one can form an abstraction $\lambda xM$ only if $x \in FV(M)$.
  - In the $\lambda I$-calculus, $WN_\beta = SN_\beta$. 
The Church-Rosser Property

- Let $\rightarrow$ be a binary relation on a set $X$.
  - $\rightarrow$ has the Church-Rosser property (CR) iff for all $a, b, c \in X$ such that $a \rightarrow^+ b$ and $a \rightarrow^+ c$ there is $d$ such that $b \rightarrow^+ d$ and $c \rightarrow^+ d$.
  - $\rightarrow$ has the Weak Church-Rosser property (WCR) iff for all $a, b, c \in X$ such that $a \rightarrow b$ and $a \rightarrow c$ there is $d$ such that $b \rightarrow^+ d$ and $c \rightarrow^+ d$.
  - $\rightarrow$ is strongly normalizing (SN) iff there is no infinite sequence $a_1 \rightarrow a_2 \rightarrow \cdots$.

Proposition (Newman’s Lemma)

*Let $\rightarrow$ be a binary relation satisfying SN. If $\rightarrow$ satisfies WCR, then $\rightarrow$ satisfies CR.*

Theorem

*Church-style $ST \rightarrow$ is WCR, thus CR.*
Expressivity

- The normal form of a term of length $n$ can in the worst case have size

$$2 \cdot 2 \cdot 2 \ldots \Theta(n) \text{ times}$$

which is higher (as a function on $n$) than any elementary function.

**Theorem (Statman)**

The problem of deciding whether any two given Church-style terms $M$ and $N$ of the same type are beta-equal is of nonelementary complexity.
Expressivity

- Let ∫ = (p → p) → (p → p), where p is an arbitrary type variable. A function f : \( \mathbb{N}^k \rightarrow \mathbb{N} \) is \( ST \rightarrow \)-definable if there is a term \( M_f \) with \( \emptyset \vdash M_f : \int^k \rightarrow \int \) such that

\[
M_f \overline{n_1} \cdots \overline{n_k} \rightarrow^\beta f(n_1, \ldots, n_k)
\]

- The class of extended polynomials is the smallest class of functions over \( \mathbb{N} \) which is closed under compositions and contains the constant functions, projections, addition, multiplication, and the conditional function

\[
\text{cond}(n, m, p) = \begin{cases} 
  m & \text{if } n = 0; \\
  p & \text{otherwise.}
\end{cases}
\]

Theorem (Schwichtenberg)

The \( ST \rightarrow \)-definable functions are exactly the extended polynomials.
The Curry-Howard Correspondence

If $\Gamma = \{x_1 : \varphi_1, \ldots, x_n : \varphi_n\}$, then $rg(\Gamma)$ is the set of implicational propositional formulas $\{\varphi_1, \ldots, \varphi_n\}$.

Proposition (Curry-Howard Isomorphism)

1. If $\Gamma \vdash M : \varphi$ in $ST\rightarrow$, then $rg(\Gamma) \vdash \varphi$ in $IPL\rightarrow$.
2. If $\Gamma \vdash \varphi$ in $IPL\rightarrow$, then there are $\Delta, M$ with $rg(\Delta) = \Gamma$ and $\Delta \vdash M : \varphi$.

Corollary

$IPL\rightarrow$ is consistent.
The Curry-Howard Correspondence

- The one we presented is not an isomorphism between proofs of $IPL \rightarrow$ and terms of $ST \rightarrow$.
- Getting an exact isomorphism requires altering the way we presented natural deduction:

$$
\frac{\phi^i \ldots \tau}{\phi \rightarrow \tau} (i) \quad \frac{\phi \rightarrow \tau \ \phi}{\tau}$$

- What corresponds to $\beta$-reduction is the following rule:

$$
\frac{\phi^i \ldots \tau}{\phi \rightarrow \tau} (i) \quad \Rightarrow \quad \phi \quad \tau
$$
Hilbert-Style Proofs

- **Logical axioms** are defined as all those instances of the following two schemes:

  \[ \varphi \rightarrow \tau \rightarrow \varphi; \quad (A1) \]
  \[ (\varphi \rightarrow \tau \rightarrow \rho) \rightarrow (\varphi \rightarrow \tau) \rightarrow \varphi \rightarrow \rho; \quad (A2) \]

- The Hilbert-Style rules for propositional intuitionistic logic are as follows:

  \[
  \begin{array}{c}
  \hline
  \Gamma, \varphi \vdash H \varphi \\
  \hline
  \Gamma \vdash H \varphi \\
  \hline
  \Gamma \vdash H \varphi \rightarrow \tau \\
  \hline
  \Gamma, \varphi \vdash H \varphi \\
  \hline
  \Gamma \vdash H \varphi \\
  \hline
  \Gamma \vdash H \tau \\
  \end{array}
  \]

**Theorem (Deduction Theorem)**

If \( \Gamma, \varphi \vdash H \tau \), then \( \Gamma \vdash H \varphi \rightarrow \tau \).

**Theorem**

\( \Gamma \vdash H \varphi \) iff \( \Gamma \vdash \varphi \)
Combinatory Logic

▸ Terms of combinatory logic are defined as follows:

\[ M ::= x \mid (MM) \mid K \mid S. \]

▸ The relation \( \rightarrow_w \) is the least compatible relation on combinatory logic terms such that

\[
\begin{align*}
KMN & \rightarrow_w M; \\
SMLN & \rightarrow_w ML(NL).
\end{align*}
\]

As usual, \( \rightarrowto_w \) is the reflexive and transitive closure of \( \rightarrow_w \).

▸ The notions of normal forms, weak normalization and strong normalization are defined as usual.
Typed Combinatory Logic

- Typing Rules for Combinatory Logic Terms are defined as follows:

\[
\begin{align*}
\Gamma, x : \varphi \vdash x : \varphi & \quad V \\
\Gamma \vdash S : (\varphi \to \tau \to \rho) \to (\varphi \to \tau) \to \varphi \to \rho & \quad S \\
\Gamma \vdash K : \varphi \to \tau \to \varphi & \quad K \\
\Gamma \vdash M : \varphi \to \tau \quad \Gamma \vdash N : \varphi & \quad @ \\
\Gamma \vdash MN : \tau & \quad @
\end{align*}
\]

Theorem (Subject Reduction)
If \(\Gamma \vdash M : \varphi\) and \(M \to^w N\), then \(\Gamma \vdash N : \varphi\).

Theorem (Strong Normalization)
\(\Gamma \vdash M : \varphi\), then \(M\) is strongly normalizing.