Lambda-Calculus and Type Theory
Part II

Ugo Dal Lago

Scuola Estiva AILA, Gargnano, August 2018

Propositional Intuitionistic Logic: Natural Deduction

- *Formulas* are derived by the grammar $\varphi ::= \bot \mid p \mid \varphi \to \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$, where p ranges over a set Θ of propositional variables.
- *Judgments* have the form $\Gamma \vdash \varphi$, where Γ is a finite set of formulas. Given two such sets Γ and Δ, their union is indicated as Γ, Δ.
- The *rules* of propositional intuitionistic logic are as follows:

$$\frac{}{\Gamma, \varphi \vdash \varphi}\ AX$$

$$\frac{\Gamma, \varphi \vdash \tau}{\Gamma \vdash \varphi \to \tau}\ I\to \qquad \frac{\Gamma \vdash \varphi \to \tau \quad \Gamma \vdash \varphi}{\Gamma \vdash \tau}\ E\to$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \tau}{\Gamma \vdash \varphi \wedge \tau}\ I\wedge \qquad \frac{\Gamma \vdash \varphi \wedge \tau}{\Gamma \vdash \varphi}\ E_L\wedge \qquad \frac{\Gamma \vdash \varphi \wedge \tau}{\Gamma \vdash \tau}\ E_R\wedge$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \tau}\ I_L\vee \qquad \frac{\Gamma \vdash \tau}{\Gamma \vdash \varphi \vee \tau}\ I_R\vee \qquad \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \tau \quad \Gamma, \varphi \vee \tau \vdash \rho}{\Gamma \vdash \rho}\ E\vee$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\ E\bot$$

Propositional Intuitionistic Logic: Semantics

- **Heyting Algebras**
  - Distributive lattices with top and bottom elements, in which relative pseudo-complement always exist.
  - Meet and joins interpret conjunctions and disjunctions, respectively. Implication is given semantics by way of pseudo-complements.
  - $\Gamma \models \varphi$ indicates that every Heyting Algebra validating Γ also validates φ.
- **Kripke Semantics**
  - Propositional variables are put in relation with the elements of a partial order or *possible worlds*.
  - Conjuction and disjuction are interpreted in Tarski-style, while implication is given semantics by way of the underlying partial order
  - $\Gamma \Vdash \varphi$ indicates that every Kripke model validating Γ also validates φ.

Theorem (Completeness)

$\Gamma \vdash \varphi$ *if and only if* $\Gamma \models \varphi$ *if and only if* $\Gamma \Vdash \varphi$.

Simply-Typed λ-Calculus *à la Curry*

- An implicational propositional formula is called a *simple type*. The set of all simple types is denoted by $\Phi_{\rightarrow}$.
- An *environment* is a finite set Γ of pairs of the form $\{x_1 : \varphi_1, \cdots, x_n : \varphi_n\}$, where the x_i are distinct variables and φ_i are simple types. In this case, $dom(\Gamma)$ is $\{x_1, \cdots, x_n\}$.
- A *typing judgement* is a triple $\Gamma \vdash M : \varphi$, consisting of an environment, a λ-term and a simple type.
- The rules are as follows:

$$\frac{}{\Gamma, x : \varphi \vdash x : \varphi}\ V$$

$$\frac{\Gamma, x : \varphi \vdash M : \tau}{\Gamma \vdash \lambda x M : \varphi \rightarrow \tau}\ \lambda \qquad \frac{\Gamma \vdash M : \varphi \rightarrow \tau \quad \Gamma \vdash N : \varphi}{\Gamma \vdash MN : \tau}\ @$$

- The obtained calculus is referred to as $ST_{\rightarrow}$.

Subject Reduction

Lemma (Generation Lemma)

Suppose that $\Gamma \vdash M : \varphi$. *Then:*

1. *If* M *is a variable* x, *then* $\Gamma(x) = \varphi$;
2. *If* M *is an application* NL, *then there is* τ *such that* $\Gamma \vdash N : \tau \to \varphi$ *and* $\Gamma \vdash L : \tau$;
3. *If* M *is an abstraction* $\lambda x N$ *and* $x \notin dom(\Gamma)$, *then* $\varphi = \tau \to \rho$, *where* $\Gamma, x : \tau \vdash N : \rho$.

Lemma (Substitution Lemma)

1. *If* $\Gamma \vdash M : \varphi$ *and* $\Gamma(x) = \Delta(x)$ *for every* $x \in FV(M)$, *then* $\Delta \vdash M : \varphi$
2. *If* $\Gamma, x : \varphi \vdash M : \tau$ *and* $\Gamma \vdash N : \varphi$, *then* $\Gamma \vdash M[x := N] : \tau$.

Theorem (Subject Reduction Theorem)

If $\Gamma \vdash M : \varphi$ *and* $M \twoheadrightarrow_\beta N$, *then* $\Gamma \vdash N : \varphi$.

Simply-Typed λ-Calculus *à la Church*

- *Preterms* of the simply-typed λ-calculus *à la Church* are defined as follows:

$$M ::= x \;\Big|\; (MM) \;\Big|\; (\lambda x : \varphi M).$$

- The notions of substitution, α-conversion, term, and reduction can be generalised to terms *à la Church*.
- Typing rules are the obvious ones:

$$\frac{}{\Gamma, x : \varphi \vdash x : \varphi}\; V$$

$$\frac{\Gamma, x : \varphi \vdash M : \tau}{\Gamma \vdash \lambda x : \varphi M : \varphi \to \tau}\; \lambda \qquad \frac{\Gamma \vdash M : \varphi \to \tau \quad \Gamma \vdash N : \varphi}{\Gamma \vdash MN : \tau}\; @$$

- The Subject Reduction Theorem can be easily reproved.

Curry vs. Church

Proposition

In the Simply-Typed λ-calculus à la Church, if $\Gamma \vdash M : \varphi$ *and* $\Gamma \vdash M : \tau$*, then* $\varphi = \tau$*.*

- The erasing map $|\cdot|$ from terms *à la Church* to terms *à la Curry* is defined by induction on the structure of terms, as follows:

$$|x| := x \qquad |\lambda x : \varphi M| := \lambda x |M| \qquad |MN| := |M||N|$$

- Typability and reduction judgments in the two styles can be translated into each other relatively easily.

Weak Normalisation

Theorem

Every term typable in $ST_\rightarrow$ has a normal form.

- The proof is based on the following key **ideas**:
  - One can assign to each typable term M, a pair natural numbers $\mathbf{m}_M := (\delta_M, n_M)$ in such a way that if M is not a normal form, then there is N with $M \rightarrow_\beta N$ and $\mathbf{m}_M > \mathbf{m}_N$ in the lexicographic order.
  - Then, one proves the statement for every typable term M *by lexicographic induction* on $\mathbf{m}_M$.
- This is *not* a proof of strong normalisation.

Strong Normalisation

Theorem

Every term typable in $ST_{\rightarrow}$ is strongly normalising.

- **A Proof Based on Reducibility**.
  - For every type φ, a set of terms Red_{φ}, the *reducible* terms;
  - A proof that any term of type φ is in Red_{φ};
  - A proof that any term in Red_{φ} is strongly normalizing.
- **A Proof through λI**
  - η-reduction is the smaller compatible relation $\rightarrow_{\eta}$ including pairs of the form $\lambda x(Mx) \rightarrow_{\eta} M$ (where $x \notin FV(M)$. $\rightarrow_{\beta\eta}$ is the union of $\rightarrow_{\beta}$ and $\rightarrow_{\eta}$.
  - In the λI-calculus, one can form an abstraction λxM *only if* $x \in FV(M)$.
  - In the λI-calculus, $WN_{\beta} = SN_{\beta}$.

The Church-Rosser Property

- Let $\to$ be a binary relation on a set X.
  - $\to$ has the Church-Rosser property (CR) iff for all $a, b, c \in X$ such that $a \to^+ b$ and $a \to^+ c$ there is d such that $b \to^+ d$ and $c \to^+ d$.
  - $\to$ has the Weak Church-Rosser property (WCR) iff for all $a, b, c \in X$ such that $a \to b$ and $a \to c$ there is d such that $b \to^+ d$ and $c \to^+ d$.
  - $\to$ is strongly normalizing (SN) iff there is no infinite sequence $a_1 \to a_2 \to \cdots$.

Proposition (Newman's Lemma)

Let $\to$ be a binary relation satisfying SN. If $\to$ satisfies WCR, then $\to$ satisfies CR.

Theorem

Church-style $ST_\to$ is WCR, thus CR.

Expressivity

- The normal form of a term of length n can in the worst case have size

$$2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\Big\}\Theta(n) \text{ times}$$

which is higher (as a function on n) than any elementary function.

Theorem (Statman)

The problem of deciding whether any two given Church-style terms M and N of the same type are beta-equal is of nonelementary complexity.

Expressivity

- Let **int** $= (p \to p) \to (p \to p)$, where p is an arbitrary type variable. A function $f : \mathbb{N}^k \to \mathbb{N}$ is *$ST_{\to}$-definable* if there is a term M_f with $\emptyset \vdash M_f : \mathbf{int}^k \to \mathbf{int}$ such that

$$M_f \overline{n_1} \cdots \overline{n_k} \twoheadrightarrow_\beta \overline{f(n_1, \ldots, n_k)}$$

- The class of *extended polynomials* is the smallest class of functions over $\mathbb{N}$ which is closed under compositions and contains the constant functions, projections, addition, multiplication, and the conditional function

$$cond(n, m, p) = \begin{cases} m & \text{if } n = 0; \\ p & \text{otherwise.} \end{cases}$$

Theorem (Schwichtenberg)

The $ST_{\to}$-definable functions are exactly the extended polynomials.

The Curry-Howard Correspondence

- If $\Gamma = \{x_1 : \varphi_1, \cdots, x_n : \varphi_n\}$, then $rg(\Gamma)$ is the set of implicational propositional formulas $\{\varphi_1, \ldots, \varphi_n\}$.

Proposition (Curry-Howard Isomorphism)

1. *If* $\Gamma \vdash M : \varphi$ *in* $ST_{\rightarrow}$, *then* $rg(\Gamma) \vdash \varphi$ *in* $IPL_{\rightarrow}$.
2. *If* $\Gamma \vdash \varphi$ *in* $IPL_{\rightarrow}$, *then there are* Δ, M *with* $rg(\Delta) = \Gamma$ *and* $\Delta \vdash M : \varphi$.

Corollary

$IPL_{\rightarrow}$ *is consistent.*

The Curry-Howard Correspondence

- The one we presented is not an isomorphism between proofs of $IPL_{\rightarrow}$ and terms of $ST_{\rightarrow}$.
- Getting an exact isomorphism requires altering the way we presented natural deduction:

$$\dfrac{\begin{array}{c}[\varphi]^i \\ \vdots \\ \tau\end{array}}{\varphi \rightarrow \tau}\ (i) \qquad\qquad \dfrac{\varphi \rightarrow \tau \quad \varphi}{\tau}$$

- What corresponds to β-reduction is the following rule:

$$\dfrac{\begin{array}{c}\vdots \\ \varphi\end{array} \quad \dfrac{\begin{array}{c}[\varphi]^i \\ \vdots \\ \tau\end{array}}{\varphi \rightarrow \tau}\ (i)}{\tau} \qquad \Rightarrow \qquad \begin{array}{c}\vdots \\ \varphi \\ \vdots \\ \tau\end{array}$$

Hilbert-Style Proofs

- *Logical axioms* are defined as all those instances of the following t two schemes:

$$\varphi \to \tau \to \varphi; \quad (A1)$$
$$(\varphi \to \tau \to \rho) \to (\varphi \to \tau) \to \varphi \to \rho; \quad (A2)$$

- The Hilbert-Style rules for propositional intuitionistic logic are as follows:

$$\frac{}{\Gamma, \varphi \vdash_H \varphi} \qquad \frac{\varphi \text{ is a logical axiom}}{\Gamma \vdash_H \varphi} \qquad \frac{\Gamma \vdash_H \varphi \to \tau \quad \Gamma \vdash_H \varphi}{\Gamma \vdash_H \tau}$$

Theorem (Deduction Theorem)

If $\Gamma, \varphi \vdash_H \tau$, *then* $\Gamma \vdash_H \varphi \to \tau$.

Theorem

$\Gamma \vdash_H \varphi$ *iff* $\Gamma \vdash \varphi$

Combinatory Logic

- *Terms* of combinatory logic are defined as follows:

$$M ::= x \mid (MM) \mid \mathsf{K} \mid \mathsf{S}.$$

- The relation $\to_w$ is the least compatible relation on combinatory logic terms such that

$$\mathsf{K}MN \to_w M;$$
$$\mathsf{S}MNL \to_w ML(NL).$$

  As usual, $\twoheadrightarrow_w$ is the reflexive and transitive closure of $\to_w$.
- The notions of normal forms, weak normalization and strong normalization are defined as usual.

Typed Combinatory Logic

- Typing Rules for Combinatory Logic Terms are defined as follows:

$$\frac{}{\Gamma, x : \varphi \vdash x : \varphi}\ V \qquad \frac{}{\Gamma \vdash \mathsf{S} : (\varphi \to \tau \to \rho) \to (\varphi \to \tau) \to \varphi \to \rho}\ \mathtt{S}$$

$$\frac{}{\Gamma \vdash \mathsf{K} : \varphi \to \tau \to \varphi}\ \mathtt{K} \qquad \frac{\Gamma \vdash M : \varphi \to \tau \quad \Gamma \vdash N : \varphi}{\Gamma \vdash MN : \tau}\ @$$

Theorem (Subject Reduction)

If $\Gamma \vdash M : \varphi$ *and* $M \twoheadrightarrow_w N$, *then* $\Gamma \vdash N : \varphi$.

Theorem (Strong Normalization)

$\Gamma \vdash M : \varphi$, *then* M *is strongly normalizing.*