Lambda-Calculus and Type Theory
Part I

Ugo Dal Lago

Scuola Estiva AILA, Gargnano, August 2018

Preterms

- Let Υ denote a countably infinite set of symbols, called *variables*.
- A *preterm* is an expression derived by way of the following grammar:

$$M ::= x \;\Big|\; (MM) \;\Big|\; (\lambda x M).$$

  where $x \in \Upsilon$.
- We use the following notational conventions:
  1. The outermost parentheses in a preterm are omitted.
  2. Application associates to the left; $((MN)L)$ is abbreviated (MNL).
  3. Abstraction associates to the right: $(\lambda x(\lambda y M))$ is abbreviated $(\lambda x \lambda y M)$.
  4. A sequence of abstractions $(\lambda x_1(\lambda x_2 \ldots (\lambda x_n M) \ldots))$ can be written as $(\lambda x_1 \ldots x_n . M)$.

Substitution on Preterms

- Define the set $FV(M)$ of *free variables* of M as follows:

$$\begin{aligned} FV(x) &= \{x\}; \\ FV(\lambda x M) &= FV(M) - \{x\}; \\ FV(MN) &= FV(M) \cup FV(N). \end{aligned}$$

- An *occurrence* of a term M in another term N is a subexpression of N which is syntactically equal to M.
- The *substitution* of N for x in M, written $M[x := N]$, is defined iff no free occurrence of x in M is in an occurrence of a term $\lambda y L$ in M, where $y \in FV(N)$:

$$\begin{aligned} x[x := N] &= N; \\ y[x := N] &= y; \\ (PQ)[x := N] &= P[x := N]Q[x := N]; \\ (\lambda x P)[x := N] &= \lambda x P; \\ (\lambda y P)[x := N] &= \lambda y P[x := N]. \end{aligned}$$

α-Equivalence

- The relation $=_\alpha$ (aka α-conversion) is the smallest transitive and reflexive relation on preterms satisfying the following:
  - If $y \notin FV(M)$ and $M[x := y]$ is defined, then $\lambda x M =_\alpha \lambda y M[x := y]$;
  - If $M =_\alpha N$, then $\lambda x M =_\alpha \lambda x N$ for all variable x;
  - If $M =_\alpha N$, then $ML =_\alpha NL$ and $LM =_\alpha LN$.

Lemma

The relation $=_\alpha$ is an equivalence relation.

Lemma

If $M =_\alpha N$ and $L =_\alpha P$, then $M[x := L] =_\alpha N[x := P]$, provided both sides are defined.

Lemma

For all M, N and for every variable x, there exists L such that $M =_\alpha L$ and the substitution $L[x := N]$ is defined.

Terms

- Define the set Λ of λ-terms as the quotient set of $=_\alpha$ over preterms:
$$\Lambda = \{[M]_\alpha \mid M \text{ is a preterm}\},$$
where $[M]_\alpha = \{N \mid M =_\alpha N\}$.
- For λ-terms M and N, we define the *substitution* $M[x := N]$ as follows: let $M = [L]_\alpha$ and $N = [P]_\alpha$ where $Q = L[x := P]$ is defined, then $M[x := N]$ is $[Q]_\alpha$.
- Substitution is well-defined but also a *total* operation.
- In the following:
  - We will work with **terms**.
  - But we will give definitions and proofs following the structure of the underlying **preterms**, being sure that what we do "makes sense". We will thus use the following notation for λ-terms:

$$\begin{aligned} LP &= [MN]_\alpha \text{ where } L = [M]_\alpha \text{ and } P = [N]_\alpha \\ \lambda x L &= [\lambda x P]_\alpha \text{ where } L = [P]_\alpha \\ x &= [x]_\alpha \end{aligned}$$

β-Reduction

- A relation $\succ$ on Λ is *compatible* iff it satisfies the following two conditions for all $M, N, L \in \Lambda$:
  1. If $M \succ N$ then $\lambda x M \succ \lambda x N$ for every x.
  2. If $M \succ N$ then $ML \succ NL$ and $LM \succ LN$.
- The least compatible relation $\to_\beta$ on Λ such that

$$(\lambda x M)N \to_\beta M[x := N]$$

  is called β-reduction.
- Any term of the form $(\lambda x M)N$ is called a *β-redex*, and $M[x := N]$ is the result of *contracting* the redex.
- A β-normal form is a term $M \in \Lambda$ such that there is no N with $M \to_\beta N$. The set of normal forms is NF_β.
- A *β-reduction sequence* is any finite or infinite sequence of terms $M_0, M_1, \ldots$ such that

$$M_0 \to_\beta M_1 \to_\beta M_2 \to_\beta \cdots$$

- The transitive and reflexive closure of $\to_\beta$ is indicated with $\twoheadrightarrow_\beta$.

Confluence

- Given any relation $\succ$, its transitive closure is indicated as $\succ^+$, while its reflexive and transitive closure is indicated as $\succ^*$.
- A relation $\succ$ is said to be *confluent* iff for every M, N, L, the following holds

$$M \succ^* N \land M \succ^* L \quad \text{implies} \quad \exists P.N \succ^* P \land L \succ^* P$$

Theorem

The relation $\rightarrow_\beta$ on Λ is confluent.

Corollary

If $M \twoheadrightarrow_\beta N$ and $M \twoheadrightarrow_\beta L$ where $N, L \in NF_\beta$, then $N = L$.

Proof of Confluence: The Main Ingredients

- The relation $\Rightarrow_\beta$ is the least relation on Λ such that:
  1. $x \Rightarrow_\beta x$ for every variable x;
  2. If $M \Rightarrow_\beta N$, then $\lambda x M \Rightarrow_\beta \lambda x N$;
  3. If $M \Rightarrow_\beta N$ and $L \Rightarrow_\beta P$, then $ML \Rightarrow_\beta NP$;
  4. If $M \Rightarrow_\beta N$ and $L \Rightarrow_\beta P$, then $(\lambda x M)L \Rightarrow_\beta N[x := P]$.
- The *complete development* of a term M is defined as follows:

$$
\begin{aligned}
x^\dagger &= x \\
(\lambda x M)^\dagger &= \lambda x M^\dagger \\
(MN)^\dagger &= M^\dagger N^\dagger \text{ if } M \text{ is not an abstraction;} \\
((\lambda x M)N)^\dagger &= M^\dagger[x := N^\dagger]
\end{aligned}
$$

Proposition

If $M \Rightarrow_\beta N$, then $N \Rightarrow_\beta M^\dagger$.

Normalization

- A term M is *normalizing* iff $M \twoheadrightarrow_\beta N$, where $N \in NF_\beta$. In that case N is said to be *the normal form* of M. The set of normalizing terms is WN_β.
- A term M is said to be *strongly normalizing* iff all reduction sequences starting at M are finite. The set of strongly normalizing terms is SN_β.
- A *reduction strategy* F is a map from Λ to Λ such that $F(M) = M$ whenever $M \in NF_\beta$, and $M \to_\beta F(M)$ otherwise.
- A reduction strategy F is *normalizing* iff for every normalizing M there is a natural number $n \in \mathbb{N}$ such that $F^n(M) \in NF_\beta$.
- In the *leftmost reduction strategy* $L : \Lambda \to \Lambda$, $L(M)$ is obtained by contracting the leftmost redex in M, if any.

Theorem

L is a normalizing reduction strategy.

Expressiveness

- The boolean values T and F can be encoded as λ-terms as follows:
$$\overline{T} := \lambda xy.x \qquad \overline{F} := \lambda xy.y$$
- If $M, N \in \Lambda$, then $\langle M, N \rangle$ stands for the term $\lambda x(xMN)$, where $x \notin FV(M) \cup FV(N)$.
- Any natural number $n \in \mathbb{N}$ can be encoded as follows:
$$\overline{n} := \lambda xy.x^n y$$
where $x^n(y)$ stands for $x(x(\cdots(y)\cdots))$, with n occurrences of x.
- A partial function $f : \mathbb{N}^k \to \mathbb{N}$ is λ-*definable* iff there is $M_f \in \Lambda$ such that:
  1. If $f(n_1, \ldots, n_k) = m$, then $M_f \overline{n_1} \cdots \overline{n_k} \twoheadrightarrow_\beta \overline{m}$.
  2. If $f(n_1, \ldots, n_k)$ is undefined, then $M_f \overline{n_1} \cdots \overline{n_k}$ has no normal form.

Universality

Theorem

The λ-definable functions are precisely the partial recursive ones.

- Some functions and operators which are useful are the following ones:

$$\begin{aligned} \textbf{if } M \textbf{ then } N \textbf{ else } L &:= MNL; \\ \pi_1 &:= \lambda x(x\overline{T}); \\ \pi_2 &:= \lambda x(x\overline{F}); \\ \textbf{zero}? &:= \lambda x.x(\lambda y\overline{F})\overline{T}. \end{aligned}$$

Corollary

The problem of checking whether M has a normal form is undecidable.