On the Expressivity of the Propositional Fragment of Independence Friendly Logic

Ondrej Majer and Chris Fermüller)

Independence friendly (IF) logic, introduced by Hintikka and Sandu [2], extends classical first order logic by considering imperfect information in otherwise classical evalution games. The semantics of IF logics is thus based on a game theoretical interpretation of logical connectives, where the truth of a formula is defined as existence of a winning strategy for the initial player (Proponent) and its falsity by existence of a winning strategy of the second player (Opponent). Imperfect information entails a loss of determinacy; i.e., none of the player may have a winning strategy. Equilibrium semantics (see [4]) interprets IF logic as a many-valued logic, where the truth values are identified with euqilibrium pay off values that arise if the players are allow to play mixed strategies.

Imperfect information was originally introduced for the game theoretical interpretation of quantifiers: e.g., an existential quantifier occurrence corresponds to a move, where Proponent chooses a witnessing element from the domain of the model. But it can be applied already on the propositional level. Here, we confine ourselves to the minimal version of propositional IF logic introduced, e.g., in [5], which introduces formulas expressing independence of disjunctions from immediately preceding conjunctions and, likewise, independence of conjunctions from immediately preceding disjunctions. This is accomplished by extending the standard propositional syntax by slashed formula. E.g., $(\varphi_1 \vee / \wedge \psi_1) \wedge (\varphi_2 \vee / \wedge \psi_2)$ expresses that Proponent has to choose either the right or the left disjcunct without knowing to which of the conjuncts was previously choosen by the Opponent. The semantic game for this formula is the simplest (non-trivial) case of incomplete information as we can see from its strategic form (Opponent chooses a row, Proponent independently chooses a column and the payoff is given by the value of the correspondning formula):

$$\begin{array}{lcc} \mathbf{O}\backslash\mathbf{P} & L & R \\ L & \left(\begin{array}{c} ||\varphi_1|| \\ ||\varphi_2|| \end{array}\right. & \left. \begin{array}{c} ||\psi_1|| \\ ||\psi_2|| \end{array} \right) \\ R & & \end{array}$$

In [1] we combined such games with Giles's game for Łukasiewicz logic in order to obtain meet a standard objection regarding intermediary 'degrees of truth', namely that they are intrinsically arbitrary. By identifying atomic formluas of Łukasiewicz logic with complex IF formulas, we obtain an interpretation of intermediary truth values as equilibrium values arising in impefect information games involving only classical atomic formulas.

An important question in this context is, whether IF logic is rich enough to cover a sufficient range of truth values. Equilibrium semantics for IF logic refers to constant-sum, two-player games with 0 and 1 as the only possible payoff values. It is a well known game-theoretic fact that the value of every such game is rational, hence the values of IF formulas under equilibrium semantics must be rationals from the interval $[0, 1]$. We therefore have to ask, whether there exists for any $q \in [0, 1] \cap \mathbb{Q}$ an IF formula φ such that the value of φ is q. Mann, Sandu and Sevenster in [3] deal with this question within the framework of first order IF logic.

Our first result is a solution of the above problem within the framework of minimal propositional IF logic. We prove that for any $q \in [0, 1] \cap \mathbb{Q}$ there is a propositional IF formula φ such that the value of φ is q. In fact we need just a fragment of propositional IF $\{W, \neg, \top, \bot\}$, where W is the formula corresponding to the slashed disjunction mentioned above, $W(\varphi_1, \psi_1, \varphi_2.\psi_2) = (\varphi_1 \vee / \wedge \psi_1) \wedge (\varphi_2 \vee / \wedge \psi_2)$. Our second result shows that this set of connectives of propositional IF logic is functionally complete with respect to all functions from $\{0, 1\}^n$ to $[0, 1] \cap \mathbb{Q}$, i.e. for any such function $f(x_1, ..., x_n)$ there is a formula φ containing just $W, \neg, \top, \bot$ such that for any classical valuation v it holds that $f(v(x_1), ..., v(x_n)) = ||\varphi||_v$. We think that the corresponding proofs are of wider interest and will discuss various prospects for extending our findings.

References

[1] C. Fermueller, O. Majer Equilibrium Semantics for IF Logic and Many-Valued Connectives. In: Hansen H., Murray S., Sadrzadeh M., Zeevat H. (eds) *Logic, Language, and Computation. TbiLLC 2015*. Lecture Notes in Computer Science, vol 10148: 290–312 Springer, 2017 .

[2] J. Hintikka, G. Sandu. Game-theoretical semantics. In *Handbook of Logic and Language*. Elsevier, 2010.

[3] A. L. Mann, G. Sandu, M. Sevenster. *Independence-friendly logic: A game-theoretic approach*. Cambridge University Press, 2011.

[4] M. Sevenster, G. Sandu. Equilibrium semantics of languages of imperfect information. Annals of Pure and Applied Logic 161(5):618–631, 2010.

[5] G. Sandu, A. Pietarinen. Partiality and games: propositional logic. Logic Journal of IGPL 9(1): 101–121, 2001.