Bisimulations of open games

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Abstract

We define a notion of bisimulation between open games of the same type, and study the category whose morphisms are bisimulation classes of open games. This has much richer structure than the previously-considered category whose morphisms are isomorphism classes of open games: It is compact closed, self-dual and is moreover a hypergraph category (all objects have a special commutative Frobenius algebra structure in a compatible way). The crucial ingredient of these new structures are 'virtual agents' whose goal is to predict a future value, which appeared in earlier work on selection functions in game theory.

For sets X, S, Y, R, a *bimorphic lens* $\lambda : (X, S) \to (Y, R)$ consists of functions $v_\lambda : X \to Y$ and $u_\lambda : X \times R \to S$ [5]. These form the morphisms of a symmetric monoidal category **BLens** [3]. An *open game* $\mathcal{G} : (X, S) \to (Y, R)$ consists of a set $\Sigma_\mathcal{G}$ of *strategy profiles*, an indexed family of bimorphic lenses $\mathcal{G}(-) : \Sigma_\mathcal{G} \to \hom_{\mathbf{BLens}}((X, S), (Y, R))$ and a *best response relation* $\mathbf{B}_\mathcal{G}(h, k) \subseteq \Sigma_\mathcal{G}^2$ for each *context* $(h, k) : X \times (Y \to R)$. Open games modulo isomorphisms of $\Sigma_\mathcal{G}$ form the morphisms of another symmetric monoidal category, which provides a strongly compositional foundation to game theory [2] (*not* game semantics). This category has a curious partially-defined duality, with 'counits' $\varepsilon : (X, X) \to I$ but no corresponding 'units' [3].

Given open games $\mathcal{G}, \mathcal{H} : (X, S) \to (Y, R)$, strategy profiles $\sigma : \Sigma_\mathcal{G}$ and $\tau : \Sigma_\mathcal{H}$ and a context $(h, k) : X \times (Y \to R)$, we say that σ, τ are *observationally equivalent in context* (h, k), and write $\sigma \sim_{(h,k)} \tau$, if $v_{\mathcal{G}(\sigma)}(h) = v_{\mathcal{H}(\tau)}(h) =: y$ and $u_{\mathcal{G}(\sigma)}(h, k(y)) = u_{\mathcal{H}(\tau)}(h, k(y))$. A *bisimulation* between $\mathcal{G}$ and $\mathcal{H}$ is a relation $R \subseteq (X \times (Y \to R)) \times \Sigma_\mathcal{G} \times \Sigma_\mathcal{H}$ such that if $(\sigma, \sigma) \in \mathbf{B}_\mathcal{G}(c)$ then there exists $(c, \sigma, \tau) \in R$ with $\sigma \sim_c \tau$ and $(\tau, \tau) \in \mathbf{B}_\mathcal{H}(c)$, and conversely if $(\tau, \tau) \in \mathbf{B}_\mathcal{H}(c)$ then there exists $(c, \sigma, \tau) \in R$ with $\sigma \sim_c \tau$ and $(\sigma, \sigma) \in \mathbf{B}_\mathcal{G}(c)$. Informally, bisimilar open games have observationally equivalent Nash equilibria in every context.

These is a symmetric monoidal category whose morphisms are bisimulation classes of open games. In this category the previous partial duality can be completed, with a dual to the counit ε_X being provided by lifting the fixpoint agents from [4]. By similar methods the category can be shown to be self-dual, and open game comonoids on $(X, 1)$ and monoids on $(1, X)$ can be extended to a special commutative Frobenius algebra on every object [1]. This is similar to categories such as **Rel** and **FdHilb** from categorical quantum mechanics, and provides a much better-behaved foundation on which to build game theory.

References

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