Bisimulations of open games

Josef Bolt Jules Hedges

Abstract

We define a notion of bisimulation between open games of the same type, and study the category whose morphisms are bisimulation classes of open games. This has much richer structure than the previously-considered category whose morphisms are isomorphism classes of open games: It is compact closed, self-dual and is moreover a hypergraph category (all objects have a special commutative Frobenius algebra structure in a compatible way). The crucial ingredient of these new structures are ‘virtual agents’ whose goal is to predict a future value, which appeared in earlier work on selection functions in game theory.

For sets $X, S, Y, R$, a bimorphic lens $\lambda : (X, S) \to (Y, R)$ consists of functions $v_\lambda : X \to Y$ and $u_\lambda : X \times R \to S$ \footnote{These form the morphisms of a symmetric monoidal category $\mathbf{BLens}$ \footnote{An open game $G : (X, S) \to (Y, R)$ consists of a set $\Sigma_G$ of strategy profiles, an indexed family of bimorphic lenses $G(-) : \Sigma_G \to \text{Hom}_{\mathbf{BLens}}((X, S), (Y, R))$ and a best response relation $B_G(h, k) \subseteq \Sigma_G^2$ for each context $(h, k) : X \times (Y \to R)$.}$. These form the morphisms of a symmetric monoidal category $\mathbf{BLens}$ \footnote{An open game $G : (X, S) \to (Y, R)$ consists of a set $\Sigma_G$ of strategy profiles, an indexed family of bimorphic lenses $G(-) : \Sigma_G \to \text{Hom}_{\mathbf{BLens}}((X, S), (Y, R))$ and a best response relation $B_G(h, k) \subseteq \Sigma_G^2$ for each context $(h, k) : X \times (Y \to R)$.}$. An open game $G : (X, S) \to (Y, R)$ consists of a set $\Sigma_G$ of strategy profiles, an indexed family of bimorphic lenses $G(-) : \Sigma_G \to \text{Hom}_{\mathbf{BLens}}((X, S), (Y, R))$ and a best response relation $B_G(h, k) \subseteq \Sigma_G^2$ for each context $(h, k) : X \times (Y \to R)$. Open games modulo isomorphisms of $\Sigma_G$ form the morphisms of another symmetric monoidal category, which provides a strongly compositional foundation to game theory \footnote{This category has a curious partially-defined duality, with ‘counits’ $\varepsilon : (X, X) \to I$ but no corresponding ‘units’}.\footnote{Given open games $G, H : (X, S) \to (Y, R)$, strategy profiles $\sigma : \Sigma_G$ and $\tau : \Sigma_H$ and a context $(h, k) : X \times (Y \to R)$, we say that $\sigma, \tau$ are observationally equivalent in context $(h, k)$, and write $\sigma \sim_{(h, k)} \tau$, if $v_{G(\sigma)}(h) = v_{H(\tau)}(h) =: y$ and $u_{G(\sigma)}(h, k(y)) = u_{H(\tau)}(h, k(y))$. A bisimulation between $G$ and $H$ is a relation $R \subseteq (X \times (Y \to R)) \times \Sigma_G \times \Sigma_H$ such that if $(\sigma, \sigma) \in B_G(c)$ then there exists $(c, \sigma, \tau) \in R$ with $\sigma \sim_c \tau$ and $(\tau, \tau) \in B_H(c)$, and conversely if $(\tau, \tau) \in B_H(c)$ then there exists $(c, \sigma, \tau) \in R$ with $\sigma \sim_c \tau$ and $(\sigma, \sigma) \in B_G(c)$. Informally, bisimilar open games have observationally equivalent Nash equilibria in every context.}. This category has a curious partially-defined duality, with ‘counits’ $\varepsilon : (X, X) \to I$ but no corresponding ‘units’ \footnote{These is a symmetric monoidal category whose morphisms are bisimulation classes of open games. In this category the previous partial duality can be completed, with a dual to the counit $\varepsilon_X$ being provided by lifting the fixpoint agents from \footnote{By similar methods the category can be shown to be self-dual, and open game comonoids on $(X, 1)$ and monoids on $(1, X)$ can be extended to a special commutative Frobenius algebra on every object \footnote{This is similar to categories such as $\mathbf{Rel}$ and $\mathbf{FdHilb}$ from categorical quantum mechanics, and provides a much better-behaved foundation on which to build game theory.}}. By similar methods the category can be shown to be self-dual, and open game comonoids on $(X, 1)$ and monoids on $(1, X)$ can be extended to a special commutative Frobenius algebra on every object \footnote{This is similar to categories such as $\mathbf{Rel}$ and $\mathbf{FdHilb}$ from categorical quantum mechanics, and provides a much better-behaved foundation on which to build game theory.}. This is similar to categories such as $\mathbf{Rel}$ and $\mathbf{FdHilb}$ from categorical quantum mechanics, and provides a much better-behaved foundation on which to build game theory.}. This is similar to categories such as $\mathbf{Rel}$ and $\mathbf{FdHilb}$ from categorical quantum mechanics, and provides a much better-behaved foundation on which to build game theory.}.
References


