Esempi di derivazioni

Novembre, 2004

LOGICHE ENUNCIATIVE

LOGICA POSITIVA $\mathbf{P}$

Schemi d'assiomi:

A1.1	$A \to (B \to A)$
A1.2	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
A2.1	$A \land B \to A$
A2.2	$A \land B \to B$
A2.3	$(A \to B) \to ((A \to C) \to (A \to B \land C))$
A3.1	$A \to A \lor B$
A3.2	$B \to A \lor B$
A3.3	$(A \to C) \to ((B \to C) \to (A \lor B \to C))$

Regola di inferenza:

$$\frac{A \qquad A \to B}{B} \; (\textit{Modus ponens})$$

LOGICA MINIMALE $\mathbf{M}$

$$\mathbf{P} + A5.1 \quad (A \to B) \to ((A \to \neg B)) \to \neg A)$$

LOGICA INTUIZIONISTA $\mathbf{I}$

$$\mathbf{P} + A5.2 \quad A \to (\neg A \to B)$$

LOGICA CLASSICA $\mathbf{C}$

$$\mathbf{I} + A5.3 \quad \neg\neg A \to A$$

Teorema 0.1 (di deduzione) *Sia* $L \supseteq \mathbf{P}$.

$$A_1, \ldots, A_n, A \vdash_L B \quad sse \quad A_1, \ldots, A_n \vdash_L A \to B$$

DIM. Vedi la dispensa sul teorema di deduzione.

Alcuni TEOREMI della logica POSITIVA

Usando il teorema di deduzione siamo in grado di ripresentare in modo più diretto le dimostrazioni di alcuni schemi che abbiamo già dimostrato nella dispensa su 'Logica del primo ordine' senza fare uso di tale teorema perchè tali schemi servivano proprio per la sua prova.

P1 $A \to A$

1	A	$\vdash$	A	Ass.
$1^\star$	$\emptyset$	$\vdash$	$A \to A$	Teor.deduz.

P2 $A \to ((A \to B) \to B)$

1	$A, A \to B$	$\vdash$	A	Ass.
2		$\vdash$	$A \to B$	Ass.
3		$\vdash$	B	MP:1,2
$1^\star$	A	$\vdash$	$(A \to B) \to B$	Teor.deduz.
$1^{\star^\star}$	$\emptyset$	$\vdash$	$A \to ((A \to B) \to B)$	Teor.deduz.

P3 $(A \to (A \to B)) \to (A \to B)$

1	$A, A \to (A \to B)$	$\vdash$	A	Ass.
2		$\vdash$	$A \to (A \to B)$	Ass.
3		$\vdash$	$A \to B$	MP:1,2
4		$\vdash$	B	MP:1,3
$1^\star$	$A \to (A \to B)$	$\vdash$	$A \to B$	Teor.deduz.
$1^{\star^\star}$	$\emptyset$	$\vdash$	$(A \to (A \to B)) \to (A \to B)$	Teor.deduz.

P4 $(A \to B) \to ((C \to A) \to (C \to B))$

1	$A \to B, C \to A, C$	$\vdash$	$C \to A$	Ass.
2		$\vdash$	C	Ass.
3		$\vdash$	A	MP:1,2
4		$\vdash$	$A \to B$	Ass.
3		$\vdash$	B	MP:3,4
$1^\star$	$A \to B, C \to A$	$\vdash$	$C \to B$	Teor.deduz.
$1^{\star^\star}$	$A \to B$	$\vdash$	$(C \to A) \to (C \to B)$	Teor.deduz.
$1^{\star^{\star^\star}}$	$\emptyset$	$\vdash$	$(A \to B) \to ((C \to A) \to (C \to B))$	Teor.deduz.

P5 $(A \to (B \to C)) \to (B \to (A \to C))$

1	$A \to (B \to C), B, A$	$\vdash$	$A \to (B \to C)$	Ass.
2		$\vdash$	B	Ass.
3		$\vdash$	A	Ass.
4		$\vdash$	$B \to C$	MP:1,3
3		$\vdash$	C	MP:2,4
$1^\star$	$A \to (B \to C), B$	$\vdash$	$A \to C$	Teor.deduz.
$1^{\star^\star}$	$A \to (B \to C)$	$\vdash$	$B \to (A \to C))$	Teor.deduz.
$1^{\star^{\star^\star}}$	$\emptyset$	$\vdash$	$(A \to (B \to C)) \to (B \to (A \to C))$	Teor.deduz.

P6 $(A \to B) \to ((B \to C) \to (A \to C))$

1	$A \to B, B \to C, A$	$\vdash$	$A \to B$	Ass.
2		$\vdash$	A	Ass.
3		$\vdash$	B	MP:1,2
4		$\vdash$	$B \to C$	Ass.
3		$\vdash$	C	MP:3,4
$1^\star$	$A \to B, B \to C$	$\vdash$	$A \to C$	Teor.deduz.
$1^{\star^\star}$	$A \to B$	$\vdash$	$(B \to C) \to (A \to C)$	Teor.deduz.
$1^{\star^{\star^\star}}$	$\emptyset$	$\vdash$	$(A \to B) \to ((B \to C) \to (A \to C))$	Teor.deduz.

P7 $(A \to B) \to ((A \to (B \to C) \to (A \to C))$

1	$A \to B, A \to (B \to C), A$	$\vdash$	$A \to B$	Ass.
2		$\vdash$	A	Ass.
3		$\vdash$	B	MP:1,2
4		$\vdash$	$A \to (B \to C)$	Ass.
5		$\vdash$	$B \to C$	MP:2,4
6		$\vdash$	C	MP:3,4
$1^\star$	$A \to B, A \to (B \to C)$	$\vdash$	$A \to C$	Teor.deduz.
$1^{\star^\star}$	$A \to B$	$\vdash$	$(A \to (B \to C) \to (A \to C)$	Teor.deduz.
$1^{\star^{\star^\star}}$	$\emptyset$	$\vdash$	$(A \to B) \to ((A \to (B \to C) \to (A \to C))$	Teor.deduz.

P8 $(A \to (B \to C)) \to (A \wedge B \to C)$

1	$A \to (B \to C), A \wedge B$	$\vdash$	$A \wedge B$	Ass.
2		$\vdash$	A	A2.1:1
3		$\vdash$	B	A2.2:2
4		$\vdash$	$A \to (B \to C)$	Ass.
5		$\vdash$	$B \to C$	MP:2,4
6		$\vdash$	C	MP:3,5
$1^\star$	$A \to (B \to C)$	$\vdash$	$A \wedge B \to C$	Teor.deduz.
$1^{\star^\star}$	$\emptyset$	$\vdash$	$A \to (B \to C) \to (A \wedge B \to C)$	Teor.deduz.

P9 $A \to (B \to A \wedge B)$

1	A, B	$\vdash$	A	Ass.
2		$\vdash$	B	Ass.
3		$\vdash$	$A \wedge B$	A2.3:1,2
$1^{\star}$	A	$\vdash$	$B \to (A \wedge B)$	Teor.deduz.
$1^{\star^{\star}}$	$\emptyset$	$\vdash$	$A \to (B \to A \wedge B)$	Teor.deduz.

P10 $(A \to (B \to C)) \to ((C \to D) \to (A \to (B \to D)))$

1	$A \to (B \to C), A, B, C \to D$	$\vdash$	$A \to (B \to C)$	Ass.
2		$\vdash$	A	Ass.
3		$\vdash$	$B \to C$	MP:1,2
4		$\vdash$	B	Ass.
5		$\vdash$	C	MP:3,4
6		$\vdash$	$C \to D$	Ass.
7		$\vdash$	D	MP:5,6
$1^{\star}$	$A \to (B \to C), A, C \to D$	$\vdash$	$B \to D$	Teor.deduz.
$1^{\star^{\star}}$	$A \to (B \to C), C \to D$	$\vdash$	$A \to (B \to D)$	Teor.deduz.
$1^{\star^{\star}}$	$A \to (B \to C)$	$\vdash$	$(C \to D) \to (A \to (B \to D))$	Teor.deduz.
$1^{\star^{\star^{\star}}}$	$\emptyset$	$\vdash$	$(A \to (B \to C)) \to ((C \to D) \to (A \to (B \to D)))$	Teor.deduz.

P11 $(A \wedge B \to C) \to (A \to (B \to C))$

1	$A \wedge B \to C, A, B$	$\vdash$	$A \wedge B \to C$	Ass.
2		$\vdash$	A	Ass.
3		$\vdash$	B	Ass.
4		$\vdash$	$A \wedge B$	A2.3:2,3
5		$\vdash$	C	MP:1,4
6	$A \wedge B \to C, A$	$\vdash$	$B \to C$	Teor.deduz.
$1^{\star}$	$A \wedge B \to C$	$\vdash$	$A \to (B \to C)$	Teor.deduz.
$1^{\star^{\star}}$	$\emptyset$	$\vdash$	$(A \wedge B \to C) \to (A \to (B \to C))$	Teor.deduz.

P12 $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

1		$\vdash$	$A \to (B \to A \wedge B)$	P9
2		$\vdash$	$A \to (C \to A \wedge C)$	P9
3		$\vdash$	$A \to [(B \vee C) \to (A \wedge B) \vee (A \wedge C)]$	A3.3:1,2
4		$\vdash$	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$	IMP:3

P13 $(A \wedge B) \vee (A \wedge C) \to A \wedge (B \vee C)$

1	$\vdash$	$A \wedge B \to A$	A2.1
2	$\vdash$	$A \wedge B \to B$	A2.2
3	$\vdash$	$A \wedge B \to B \vee C$	A3.1:2
4	$\vdash$	$A \wedge B \to A \wedge (B \vee C)$	A2.3:1,3
5	$\vdash$	$A \wedge C \to A$	A2.1
6	$\vdash$	$A \wedge C \to C$	A2.2
7	$\vdash$	$A \wedge C \to B \vee C$	AA3.2:6
8	$\vdash$	$A \wedge C \to A \wedge (B \vee C)$	A2.3:5,7
9	$\vdash$	$(A \wedge B) \vee (A \wedge C) \to A \wedge (B \vee C)$	A3.3:4,8

P14 $\quad A \vee (B \wedge C) \to (A \vee B) \wedge (A \vee C)$

1	$\vdash$	$A \to A \vee B$	A3.1
2	$\vdash$	$A \to A \vee C$	A3.1
3	$\vdash$	$A \to (A \vee B) \wedge (A \vee C)$	A2.3:1,2
4	$\vdash$	$B \wedge C \to C$	A2.2
5	$\vdash$	$B \wedge C \to A \vee C$	A3.2
6	$\vdash$	$B \wedge C \to B$	A2.1
7	$\vdash$	$B \wedge C \to A \vee B$	A3.2
8	$\vdash$	$B \wedge C \to (A \vee B) \wedge (A \vee C)$	A2.3
9	$\vdash$	$A \vee (B \wedge C) \to (A \vee B) \wedge (A \vee C)$	A3.3:3,8

P15 $\quad (A \vee B) \wedge (A \vee C) \to A \vee (B \wedge C)$

1	$\vdash$	$C \to (B \to C \wedge B))$	P9
2	$\vdash$	$C \to (B \to (A \vee (C \wedge B)))$	A3.2:1
3	$\vdash$	$A \to (A \vee (C \wedge B))$	A3.1
4	$\vdash$	$C \to (A \to (A \vee (C \wedge B)))$	A1.1:3
5	$\vdash$	$C \to (A \vee B \to (A \vee (C \wedge B)))$	A3.3:4,2
6	$\vdash$	$A \to (A \vee B \to (A \vee (C \wedge B)))$	A3.1
7	$\vdash$	$(A \vee C) \to (A \vee B \to (A \vee (C \wedge B)))$	A3.3:5,6
8	$\vdash$	$(A \vee C) \wedge (A \vee B) \to (A \vee (C \wedge B))$	SP:7

Alcuni TEOREMI della logica MINIMALE

Le seguenti regole sono *direttamente ammissibili* in M:

$$\frac{A \to B \qquad A \to \neg B}{\neg A} \quad \text{(Regola associata all'assioma A5.1)}$$

M2 $$\frac{C \to (A \to B) \qquad D \to (A \to \neg B)}{C \wedge D \to \neg A}$$

M3 $\quad (A \to (B \wedge \neg B)) \to \neg A$

1	$A \to (B \wedge \neg B)$	$\vdash$	$A \to (B \wedge \neg B)$	Ass.
2		$\vdash$	$A \to B$	A2.1 + T
3		$\vdash$	$A \to \neg B$	A2.2 + T
4		$\vdash$	$\neg A$	A5.1:2,3
$1^\star$	$\emptyset$	$\vdash$	$(A \to (B \wedge \neg B)) \to \neg A$	Teor.deduz.

M4 $(A \to \neg A) \to \neg A$

1	$A \to \neg A$	$\vdash$	$A \to \neg A$	Ass.
2		$\vdash$	$A \to A$	identità
3		$\vdash$	$\neg A$	A5.1:1,2
$1^\star$	$\emptyset$	$\vdash$	$(A \to \neg A) \to \neg A$	Teor.deduz.

M5 $(\neg A \to A) \to \neg\neg A$

1	$\neg A \to A$	$\vdash$	$\neg A \to A$	Ass.
2		$\vdash$	$\neg A \to \neg A$	identità
3		$\vdash$	$\neg\neg A$	A5.1:1,2
$1^\star$	$\emptyset$	$\vdash$	$(\neg A \to A) \to \neg\neg A$	Teor.deduz.

M6 $(A \to \neg\neg A)$

1		$\vdash$	$A \to (\neg A \to A)$	A1.1
2		$\vdash$	$(\neg A \to A) \to \neg\neg A$	M5
3		$\vdash$	$A \to \neg\neg A$	T:1,2

M7 $(A \to B) \to (\neg B \to \neg A)$

1	$A \to B, \neg B$	$\vdash$	$\neg B$	Ass.
2		$\vdash$	$A \to \neg B$	A1.1:1
3		$\vdash$	$A \to B$	Ass.
4		$\vdash$	$\neg A$	A5.1:2,3
$1^\star$	$A \to B$	$\vdash$	$\neg B \to \neg A$	Teor.deduz.
$1^{\star^\star}$	$\emptyset$	$\vdash$	$(A \to B) \to (\neg B \to \neg A)$	Teor.deduz.

M8 $(A \to \neg B) \to (B \to \neg A)$

$(\neg A \to B) \to (B \to \neg\neg A)$

$(\neg A \to \neg B) \to (B \to \neg\neg A)$

Da M7 (per rimpiazzamento) e M6. Esercizio.

M9 $A \to (\neg A \to \neg B)$

1		$\vdash$	$A \to (B \to A)$	A1.1
2		$\vdash$	$(B \to A) \to (\neg A \to \neg B)$	M7
3		$\vdash$	$A \to (\neg A \to \neg B)$	T:1,2

M10 $\neg(A \wedge \neg A)$

1	$\vdash$	$A \wedge \neg A \to A$	A2.1
2	$\vdash$	$A \wedge \neg A \to \neg A$	A2.2
3	$\vdash$	$\neg(A \wedge \neg A)$	A5.1:1,2

M11 $\neg\neg(A \vee \neg A)$

1	$\vdash$	$A \to A \vee \neg A$	A3.1
2	$\vdash$	$\neg A \to A \vee \neg A$	A3.2
3	$\vdash$	$\neg(A \vee \neg A) \to \neg A$	M7:1
4	$\vdash$	$\neg(A \vee \neg A) \to \neg\neg A$	M7:2
5	$\vdash$	$\neg\neg(A \vee \neg A)$	A5.1:3,4

M12 $(\neg A \vee \neg B) \to \neg(A \wedge B)$

1	$\vdash$	$A \wedge B \to A$	A2.1
2	$\vdash$	$\neg A \to \neg(A \wedge B)$	M7:1
3	$\vdash$	$A \wedge B \to B$	A2.2
4	$\vdash$	$\neg B \to \neg(A \wedge B)$	M7:3
5	$\vdash$	$(\neg A \vee \neg B) \to \neg(A \wedge B)$	A3.3:2,4

M13 $(\neg A \wedge \neg B) \leftrightarrow \neg(A \vee B)$

1	$\vdash$	$\neg A \wedge \neg B \to \neg A$	A2.1
2	$\vdash$	$A \to \neg(\neg A \wedge \neg B)$	M8:1
3	$\vdash$	$\neg A \wedge \neg B \to \neg B$	A2.2
4	$\vdash$	$B \to \neg(\neg A \wedge \neg B)$	M8:3
5	$\vdash$	$(A \vee B) \to \neg(\neg A \wedge \neg B)$	A3.3:2,4
6	$\vdash$	$(\neg A \wedge \neg B) \to \neg(A \vee B))$	M8:5

1	$\vdash$	$A \to A \vee B$	A3.1
2	$\vdash$	$\neg(A \vee B) \to \neg B$	M7:1
3	$\vdash$	$B \to A \vee B$	A3.2
4	$\vdash$	$\neg(A \vee B) \to \neg B$	M7:3
5	$\vdash$	$\neg(A \vee B) \to \neg A \neg B$	A3.3:2,4

M14 $A \wedge B \to \neg(\neg A \vee \neg B)$

1	$\vdash$	$(\neg A \vee \neg B) \to \neg(A \wedge B)$	M12
2	$\vdash$	$A \wedge B \to \neg(\neg A \vee \neg B)$	M8:1

M15 $A \vee B \to \neg(\neg A \wedge \neg B)$

1	$\vdash$	$\neg A \wedge \neg B \to \neg(A \vee B)$	M.13
2	$\vdash$	$A \vee B \to \neg(\neg A \wedge \neg B)$	M7 + M6

M16 $(A \to B) \to \neg(A \wedge \neg B)$

1	$A \wedge \neg B$	$\vdash$	$A \wedge \neg B$	Ass.
2		$\vdash$	A	A2.1:1
3		$\vdash$	$\neg B$	A2.2:2
4		$\vdash$	$A \to ((A \to B) \to B)$	*Modus Ponens*X
5		$\vdash$	$(A \to B) \to B$	MP:2,4
6		$\vdash$	$\neg B \to \neg(A \to B)$	M7:5
7		$\vdash$	$\neg(A \to B)$	MP:3,6
1*	$\emptyset$	$\vdash$	$A \wedge \neg B \to \neg(A \to B)$	Teor.deduz.
2*		$\vdash$	$(A \to B) \to \neg(A \wedge \neg B)$	M8:1*

M17 $\vdash_M A \wedge \neg B \to \neg(A \to B)$ Esercizio.

M18 $\vdash_M A \vee B \to (\neg A \to \neg\neg B)$ Esercizio.

M19 $\vdash_M (A \to B) \to \neg\neg(\neg A \vee B)$ Esercizio.

M20 $\neg\neg\neg A \to \neg A$

1	$\neg\neg\neg A$	$\vdash$	$A \to \neg\neg A$	M6
2		$\vdash$	$\neg\neg\neg A$	Ass.
3		$\vdash$	$A \to \neg\neg\neg A$	A1.1:2
4		$\vdash$	$\neg A$	A5.1:1,3
1*	$\emptyset$	$\vdash$	$\neg\neg\neg A \to \neg A$	Teor.deduz.

M21 $(\neg\neg(A \to B) \wedge \neg\neg A) \to \neg\neg B$

1		$\vdash$	$(A \to B) \to (\neg B \to \neg A)$	M7
2		$\vdash$	$\neg B \to [(A \to B) \to \neg A]$	SP:1
3		$\vdash$	$\neg\neg A \to ((A \to B) \to \neg\neg A)$	A1.1
4		$\vdash$	$\neg\neg A \wedge \neg B \to \neg(A \to B)$	M2:2,3
5		$\vdash$	$\neg\neg A \to (\neg B \to \neg(A \to B))$	EXP:4
6		$\vdash$	$\neg\neg(A \to B) \to (\neg B \to \neg\neg(A \to B))$	A1.1
7		$\vdash$	$\neg\neg A \wedge \neg\neg(A \to B) \to \neg\neg B$	M2:5,6

M22 $(\neg\neg A \to \neg A) \to \neg A$

1		$\vdash$	$(\neg\neg A \to \neg A) \to \neg\neg\neg A$	M5
2		$\vdash$	$\neg\neg\neg A \to \neg A$	M20
3		$\vdash$	$(\neg\neg A \to \neg A) \to \neg A$	T:1,2

Sia $\top =_{df} p \to p$ per qualche prefissata costante enunciativa p e $\bot =_{df} \neg\top$

M23 $(A \to \bot) \to \neg A$

1		$\vdash \top$	def. di $\top$ e identità
2		$\vdash A \to \top$	def. di $\top$ e identità
3		$\vdash \neg \bot$	def di $\bot$ e M6:1
4		$\vdash A \to \neg \bot$	A1.1:2
5		$\vdash (A \to \bot) \to \neg A$	A5.1:4

Dai lemmi M4 e M22 si ottiene che se è teorema della logica $A \to B \vee B \to A$ e dunque anche $(\neg A \to \neg\neg A) \vee (\neg\neg A \to \neg A)$ allora è teorema $\neg A \vee \neg\neg A$.

Alcuni TEOREMI della logica INTUIZIONISTA

I1 $\quad \neg A \to (A \to B)$ Esercizio.

I2 $\quad (A \wedge \neg A) \to B$ Esercizio.

I3 $\quad (A \vee B) \to (\neg A \to B)$

1		$\vdash A \to (\neg A \to B)$	A5.2
2		$\vdash B \to (\neg A \to B)$	A1.1
3		$\vdash (A \vee B) \to (\neg A \to B)$	A3.3

I4 $\quad (\neg A \vee B) \to (A \to B)$

1		$\vdash \neg A \to (A \to B)$	I1
2		$\vdash B \to (A \to B)$	A1.1
3		$\vdash (\neg A \vee B) \to (A \to B)$	A3.3

I5 $\quad A \vee (B \wedge \neg B) \to A$

1		$\vdash A \to A$	identità
2		$\vdash (B \wedge \neg B) \to A$	I2
3		$\vdash A \vee (B \wedge \neg B) \to A$	A3.3:1,2

I6 $\quad (A \to \bot) \leftrightarrow \neg A$

1		$\vdash (A \to \bot) \to \neg A$	M21
2		$\vdash \neg A \to (A \to \bot)$	I1

I8 $\quad \neg\neg(\neg\neg A \to A)$

1	$\neg(\neg\neg A \to A)$	$\vdash \neg(\neg\neg A \to A)$	Ass.
2		$\vdash A \to \neg(\neg\neg A \to A)$	A1.1:1
3		$\vdash A \to (\neg\neg A \to A)$	A1.1
4		$\vdash \neg A$	A5.1:2,3
5		$\vdash \neg A \to (\neg\neg A \to A)$	A5.2
6		$\vdash \neg\neg A \to A$	MP:4,5
1*	$\emptyset$	$\vdash \neg(\neg\neg A \to A) \to (\neg\neg A \to A)$	Teor.deduz.
2*		$\vdash \neg\neg(\neg\neg A \to A)$	

Alcuni TEOREMI della logica CLASSICA

noindent C1 $A \vee \neg A$ Terzo escluso

1	$\neg(A \vee \neg A)$	$\vdash A \to A \vee \neg A$	A3.1
2		$\vdash \neg(A \vee \neg A)$	Ass.
3		$\vdash A \to \neg(A \vee \neg A)$	A1.1:2
4		$\vdash \neg A$	A5.1:1,3
5		$\vdash A \vee \neg A$	A3.2:4
$1^\star$		$\vdash \neg(A \vee \neg A) \to (A \vee \neg A)$	Teor.deduz.
$2^\star$		$\vdash \neg(A \vee \neg A) \to \neg(A \vee \neg A)$	identità
$3^\star$		$\vdash \neg\neg(A \vee \neg A)$	A5.1:$1^\star$,$2^\star$
$4^\star$		$\vdash A \vee \neg A$	A5.3:$3^\star$

C2 $A \vee (A \to B)$ Legge di Tarski

1	$\vdash A \vee \neg A$	C1
2	$\vdash \neg A \to (A \to B)$	I1
3	$\vdash A \vee (A \to B)$	T:1,2

C3 $((A \to B) \to A) \to A$ Legge di Peirce

1	$\vdash A \to [((A \to B) \to A) \to A]$	A1.1
2	$\vdash ((A \to B) \to A) \wedge (A \to B)) \to A$	*Modus Ponens*
3	$\vdash (A \to B) \to [((A \to B) \to A) \to A]$	EXP + SP
4	$\vdash [(A \vee (A \to B)] \to [((A \to B) \to A) \to A]$	A3.3:1,3
5	$\vdash A \vee (A \to B)$	Legge di Tarski
6	$\vdash ((A \to B) \to A) \to A$	MP: 4,5

C4 $(A \to B) \leftrightarrow (\neg A \vee B)$ Legge di Filone

1		$\vdash (\neg A \vee B) \to (A \to B)$	I4
1	$A \to B, A$	$\vdash A \to B$	Ass.
2		$\vdash A$	Ass.
3		$\vdash B$	MP:1,2
4		$\vdash \neg A \vee B$	A3.2:3
$1^\star$	$A \to B$	$\vdash A \to \neg A \vee B$	Teor.deduz.
$2^\star$		$\vdash \neg A \to \neg A \vee B$	A3.1
$3^\star$		$\vdash (A \vee \neg A) \to (\neg A \vee B)$	A3.3:5,6
$4^\star$		$\vdash (A \vee \neg A)$	Terzo escluso
$5^\star$		$\vdash \neg A \vee B$	MP:$3^\star$, $4^\star$
$1^{\star^\star}$	$\emptyset$	$\vdash (A \to B) \to (\neg A \vee B)$	Teor.deduz.

C5 $(\neg A \to A) \to A$ *Consequentia mirabilis*

1	$\neg A \to A$	$\vdash \neg A \to A$	Ass.
2		$\vdash \neg A \to \neg A$	Identitaà
3		$\vdash \neg\neg A$	A5.1:1,2
4		$\vdash A$	A5.3:3
1*	$\emptyset$	$\vdash (\neg A \to A) \to A$	Teor.deduz.

C6 $A \to B \vee B \to A$ Legge di confrontabilità

1		$\vdash A \to (B \to A)$	A1.1
2		$\vdash \neg A \to (A \to B)$	I1
3		$\vdash (A \vee \neg A) \to (A \to B) \vee (B \to A)$	A3.3:1,2
4		$\vdash A \vee \neg A$	Terzo escluso
5		$\vdash A \to B \vee B \to A$	MP:3,4

C7 $(A \to B) \leftrightarrow (\neg B \to \neg A)$ Contrapposizione

1		$\vdash (A \to B) \to (\neg B \to \neg A)$	M7

1		$\vdash (\neg B \to \neg A) \to (\neg\neg A \to \neg\neg B)$	M7
2		$\vdash (\neg B \to \neg A) \to (A \to \neg\neg B)$	M6:1
3		$\vdash (\neg B \to \neg A) \to (A \to B)$	A5.3:2

C8 $\neg(A \wedge B) \to (\neg A \vee \neg B)$

1	$\neg(A \wedge B)$	$\vdash \neg(A \wedge B)$	Ass.
2		$\vdash (A \wedge B) \to \perp$	I1:1
3		$\vdash A \to (B \to \perp)$	EXP:2
4		$\vdash A \to \neg B$	M21 + T:3
5		$\vdash A \to (\neg A \vee \neg B)$	A3.2 + T:4
6		$\vdash \neg A \to (\neg A \vee \neg B)$	A3.1
7		$\vdash (A \vee \neg A) \to (\neg A \vee \neg B)$	A3.3:5,6
8		$\vdash A \vee \neg A$	Terzo escluso
9		$\vdash \neg A \vee \neg B$	MP:7,8
1*	$\emptyset$	$\vdash \neg(A \wedge B) \to (\neg A \vee \neg B)$	Teor.deduz.

C9 $(\neg A \to \perp) \to A$

1		$\vdash (\neg A \to \perp) \to \neg\neg A$	M23
2		$\vdash \neg\neg A \to A$	A5.3
3		$\vdash (\neg A \to \perp) \to A$	T:1,2

noindent
I seguenti schemi NON sono teoremi della logica intuizionista:
$(\neg B \to \neg A) \to (A \to B)$
$(\neg A \to B) \to (\neg B \to A)$
$\neg(\neg A \vee \neg B) \to (A \wedge B)$
$\neg(\neg A \wedge \neg B) \to (A \vee B)$
$\neg(A \wedge \neg B) \to (A \to B)$

$(A \to B) \to (\neg A \vee B)$
$A \to B \vee B \to A$
$A \vee \neg A$
$A \vee (A \to B)$
$(\neg A \to A) \to A$
$((A \to B) \to A) \to A$
$\neg\neg A \to A$
$(\neg A \to \perp) \to A$
$((B \wedge \neg A) \to \perp) \to (B \to A$

LOGICHE PREDICATIVE

Con PQ, MQ, IQ e K denotiamo il calcolo predicativo ottenuto aggiungendo agli assiomi enunciativi di, rispettivamente P, M, I e C gli assiomi e le regole per i quantificatori qui riportate.

Assiomi

A6	$\forall xA \to A$
A7	$A \to \exists xA$

$\dfrac{A \to B}{A \to \forall xB}$	purché x non sia libera in A	Generalizzazione posteriore (GP)
$\dfrac{A \to B}{\exists xA \to B}$	purché x non sia libera in B	Particolarizzazione anteriore (PA)
$\dfrac{A}{A[t/x]}$	purché t sia libero per x in A	Sostituzione di variabili libere (SV)

noindent PQ $\quad \forall x\forall yA \to \forall y\forall xA$

1	$\vdash\ \forall yA \to A$	A6
2	$\vdash\ \forall x\forall yA \to A$	GA:1
3	$\vdash\ \forall x\forall yA \to \forall xA$	GP:2
4	$\vdash\ \forall x\forall yA \to \forall y\forall xA$	GP:3

MQ $\quad \exists x\neg A \to \neg\forall xA$

1	$\vdash\ \forall xA \to A$	A6
2	$\vdash\ \neg A \to \neg\forall xA$	M7:1
3	$\vdash\ \exists x\neg A \to \neg\forall xA$	PA:2

K $\quad \neg\forall xA \to \exists x\neg A$

1	$\vdash\ \neg A \to \exists x\neg A$	A7
2	$\vdash\ \neg\exists x\neg A \to \neg\neg A$	M7:1
3	$\vdash\ \neg\exists x\neg A \to A$	A5.3
4	$\vdash\ \neg\exists x\neg A \to \forall xA$	PA:3
5	$\vdash\ \neg\forall xA \to \neg\neg\exists x\neg A$	M7:4
6	$\vdash\ \neg\forall xA \to \exists x\neg A$	A5.3

MQ $\quad \forall x\neg A \to \neg\exists xA$

1	$\vdash$	$A \to \exists x A$	A7
2	$\vdash$	$\neg \exists x A \to \neg A$	M7:1
3	$\vdash$	$\neg \exists x A \to \forall x \neg A$	GP

MQ $\quad \neg \exists x A \to \forall x \neg A$

1	$\vdash$	$\forall x \neg A \to \neg A$	A6
2	$\vdash$	$\neg\neg A \to \neg \forall x \neg A$	M7:1
3	$\vdash$	$A \to \neg \forall x \neg A$	M6:2
4	$\vdash$	$\exists x A \to \neg \forall x \neg A$	PA:3
5	$\vdash$	$\neg\neg \forall x \neg A \to \neg \exists x A$	M7:4
6	$\vdash$	$\forall x \neg A \to \neg \exists x A$	M6:5

MQ $\quad \forall x A \to \neg \exists x \neg A$

1	$\vdash$	$\exists x \neg A \to \neg \forall x A$	MQ
2	$\vdash$	$\neg\neg \forall x A \to \neg \exists x \neg A$	M7:1
3	$\vdash$	$\forall x A \to \neg \exists x \neg A$	M6:2

K $\quad \neg \exists x \neg A \to \forall x A$

1	$\vdash$	$\neg \exists x \neg A \to \forall x \neg\neg A$	MQ
2	$\vdash$	$\neg \exists x \neg A \to \forall x A$	A5.3

MQ $\quad \exists x A \to \neg \forall x \neg A$

1	$\vdash$	$\forall x \neg A \to \neg \exists x A$	MQ
2	$\vdash$	$\neg\neg \exists x A \to \neg \forall x \neg A$	M7:1
3	$\vdash$	$\exists x A \to \neg \forall x \neg A$	M6:2

K $\quad \neg \forall x \neg A \to \exists x A$

1	$\vdash$	$\neg \forall x \neg A \to \exists x \neg\neg A$	MQ
2	$\vdash$	$\neg \forall x \neg A \to \exists x A$	A5.3

PQ $\quad \forall x(A \land B) \leftrightarrow \forall x A \land \forall x B$ Esercizio.

PQ $\quad \exists x(A \land B) \to \exists x A \land \exists x B$ Esercizio.

PQ $\quad \exists x(A \land B) \to \exists x A \land B$ purché x non libera in B. Esercizio.

PQ $\quad \exists x(A \lor B) \leftrightarrow \exists x A \lor \exists x B$ Esercizio.

PQ $\quad \forall x A \lor \forall x B \to \forall x(A \lor B)$ Esercizio.

PQ $\quad \forall x(A \lor B) \to \forall x A \lor B$ purché x non libera in B.

1	$\neg B, \forall x(A \vee B) \vdash \neg B$	Ass.
2	$\neg B, \forall x(A \vee B) \vdash \forall x(A \vee B)$	Ass.
3	$\neg B, \forall x(A \vee B) \vdash A \vee B$	A6.2
4	$\neg B, \forall x(A \vee B) \vdash (A \vee B) \wedge \neg B$	da 1 e 3
5	$\neg B, \forall x(A \vee B) \vdash (A \vee B) \wedge \neg B \to A$	tautologia
6	$\neg B, \forall x(A \vee B) \vdash A$	MP: 4,5
7	$\neg B, \forall x(A \vee B) \vdash \forall x A$	G: 6
1*	$\forall x(A \vee B) \vdash \neg B \to \forall x A$	Teor.deduz.
2*	$\forall x(A \vee B) \vdash \neg\neg B \vee \forall x A$	da 1*
3*	$\forall x(A \vee B) \vdash B \vee \forall x A$	da 2*
4*	$\forall x(A \vee B) \vdash \forall x A \vee B$	da 3*
1**	$\vdash \forall x(A \vee B) \to \forall x A \vee B$	Teor.deduz.

$\vdash (A \to \forall x B) \longleftrightarrow \forall x(A \to B)$, x non libera in A

1	$A \to \forall x B(x) \vdash A \to \forall x B(x)$	Ass.
2	$A \to \forall x B(x) \vdash \forall x B(x) \to B(x)$	A6
3	$A \to \forall x B(x) \vdash A \to B(x)$	T:1,2
4	$A \to \forall x B(x) \vdash \forall x(A \to B(x))$	G: 3
1*	$\vdash A \to \forall x B(x) \to \forall x(A \to B(x))$	Teor.deduz.

1	$\forall x(A \to B(x)) \vdash \forall x(A \to B(x))$	Ass.
2	$\forall x(A \to B(x)) \vdash A \to B(x)$	A6:1
3	$\forall x(A \to B(x)) \vdash A \to \forall x B(x)$	GP: 2
1*	$\vdash \forall x(A \to B(x)) \to (A \to \forall x B(x))$	Teor.deduz.

$\vdash (A \to \exists x B) \longleftrightarrow \exists x(A \to B)$, x non libera in A

1	$A \to B(x) \vdash A \to B(x)$	Ass.
2	$A \to B(x) \vdash A \to \exists x B(x)$	PP: 1
1*	$\vdash (A \to B(x)) \to (A \to \exists x B(x))$	Teor.deduz.
2*	$\vdash \exists x(A \to B(x)) \to (A \to \exists x B(x))$	PA: 1

1	$A \to \exists x B(x) \vdash A \vee (A \to B(x))$	Tautologia
2	$A \to \exists x B(x) \vdash \exists x(A \vee (A \to B(x)))$	A7: 1
3	$A \to \exists x B(x) \vdash A \vee \exists x(A \to B(x))$	Distr. $\exists$ su $\vee$: 2
4	$A \to \exists x B(x) \vdash A \to \exists x B(x)$	Ass.
5	$A \to \exists x B(x) \vdash \exists x B(x) \vee \exists x(A \to B(x))$	da 4 e 3
6	$A \to \exists x B(x) \vdash B(x) \to (A \to B(x))$	a fortiori
7	$A \to \exists x B(x) \vdash B(x) \to \exists x(A \to B(x))$	PP: 6
8	$A \to \exists x B(x) \vdash \exists x B(x) \to \exists x(A \to B(x))$	PP: 6
9	$A \to \exists x B(x) \vdash \exists x(A \to B(x))$	da 5 e 8
1*	$\vdash (A \to \exists x B(x)) \to \exists x(A \to B(x))$	Teor.deduz.

$\vdash \forall x(A(x) \to B) \longleftrightarrow (\exists x A(x) \to B)$, x non libera in B

1	$\forall x(A(x) \to B) \vdash \forall x(A(x) \to B)$	Ass.
2	$\forall x(A(x) \to B) \vdash A(x) \to B$	A6: 1
3	$\forall x(A(x) \to B) \vdash \exists x A(x) \to B$	PA: 2
1*	$\vdash \forall x(A(x) \to B) \to (\exists x A(x) \to B)$	Teor.deduz.

1	$\exists x(A(x) \to B) \vdash \exists x(A(x) \to B)$	Ass.
2	$\exists x(A(x) \to B) \vdash A(x) \to \exists x A(x)$	A.7
3	$\exists x(A(x) \to B) \vdash A(x) \to B$	T:1,2
4	$\exists x(A(x) \to B) \vdash \forall x(A(x) \to B)$	G: 3
1*	$\vdash \exists x(A(x) \to B) \to \forall x(A(x) \to B)$	Teor.deduz.

$\vdash \exists x(A(x) \to B) \longleftrightarrow (\forall x A(x) \to B)$, x non libera in B

1	$A(x) \to B \vdash A(x) \to B$	Ass.
2	$A(x) \to B \vdash \forall x A(x) \to A$	A.6
3	$A(x) \to B \vdash \forall x A(x) \to B$	T:1,2
1*	$\vdash (A(x) \to B) \to (\forall x A(x) \to B)$	Teor.deduz.
2*	$\vdash \exists x(A(x) \to B) \to (\forall x A(x) \to B)$	PA: 1

1	$\forall x A(x) \to B \vdash \forall x A(x) \to B$	Ass.
2	$\forall x A(x) \to B \vdash \neg B \to \neg \forall x A(x)$	M7: 1
3	$\forall x A(x) \to B \vdash \neg \forall x A(x) \to \exists x \neg A(x)$	
4	$\forall x A(x) \to B \vdash \neg B \to \exists x \neg A(x)$	T:2,3
5	$\forall x A(x) \to B \vdash (\neg B \to \exists x \neg A(x)) \to \exists x(\neg B \to \neg A(x)$	già dimostrato
6	$\forall x A(x) \to B \vdash \exists x(\neg B \to \neg A(x))$	T:4,5
7	$\forall x A(x) \to B \vdash \exists x(A(x) \to B)$	C7: 6
1*	$\vdash (\forall x A(x) \to B) \to \exists x(A(x) \to B)$	Teor.deduz.

$\vdash \exists x(\exists x B(x) \to B(x))$

1	$\vdash (\exists x B(x) \to \exists x B(x)) \to \exists x(\exists x B(x) \to B(x))$	già dimostrato
2	$\vdash \exists x B(x) \to \exists x B(x)$	Tautologia
3	$\vdash \exists x(\exists x B(x) \to B(x))$	MP: 1,2

$\vdash \exists x(B(x) \to \forall x B(x))$

1	$\vdash (\forall x B(x) \to \forall x B(x)) \to \exists x(B(x) \to \forall x B(x))$	già dimostrato
2	$\vdash \forall x B(x) \to \forall x B(x)$	Tautologia
3	$\vdash \exists x(B(x) \to \forall x B(x))$	MP: 1,2