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Revisiting Interactive Markov Chains

Mario Bravetti [1]

Dipartimento di Scienze dell'Informazione,
University of Bologna, Mura Anteo Zamboni 7, 40127 Bologna, Italy

Abstract

The usage of process algebras for the performance modeling and evaluation of concurrent systems turned out to be convenient due to their feature of compositionality. A particularly simple and elegant solution in this field is the calculus of Interactive Markov Chains (IMCs), where the behavior of processes is just represented by Continuous Time Markov Chains extended with action transitions representing process interaction. The main advantage of IMCs with respect to other existing approaches is that a notion of bisimulation which abstracts from τ transitions ("complete" interactions) can be defined which is a congruence. However in the original definition of the calculus of IMCs the high potentiality of compositionally minimizing the system state space given by the usage of a "weak" notion of equivalence and the elegance of the approach is somehow limited by the fact that the equivalence adopted over action transitions is a finer variant of Milner's observational congruence that distinguishes τ-divergent "Zeno" processes from non-divergent ones. In this paper we show that it is possible to reformulate the calculus of IMCs in such a way that we can just rely on simple standard observational congruence. Moreover we show that the new calculus is the first Markovian process algebra allowing for a new notion of Markovian bisimulation equivalence which is coarser than the standard one.

1 Introduction

The advantages of using process algebras for the performance modeling and evaluation of concurrent systems due to their feature of compositionality have been widely recognized (see [12,2,18,9,5,3] and the references therein). Particularly simple and successful has been the extension of standard process algebras with time delays whose duration follows an exponential probability distribution, called Markovian process algebras (see e.g. [12,2,18,9]). The "timed" behavior of systems specified with a Markovian process algebra can represented by a continuous time Markov chain (CTMC), i.e. a simple continuous time stochastic process where in each time point the future behavior of

[1] Email: bravetti@cs.unibo.it

the process is completely independent on its past behavior. Due to their simplicity CTMCs can be analyzed with standard mathematical techniques and software tools (see e.g. [19]) for deriving performance measures of systems.

1.1 Interactive Markov Chains

In [9] specifying concurrent systems as the parallel composition of interacting subsystems described by CTMCs is made possible simply by extending CTMCs with standard action transitions, thus giving rise to Interactive Markov Chains (IMCs). An IMC represents the behavior of a component by employing both *standard action transitions*, representing the interactive behavior of the component, and *exponentially timed transitions*, representing the timed probabilistic behavior of the component. Action transitions are just standard CCS/CSP [14,13] transitions labeled with an action "α", which can be either an internal τ action or a visible action "a". They are executed in zero time: when several action transitions are enabled in an IMC state, the choice among them is just performed non-deterministically and when IMCs are composed in parallel they synchronize following the CSP [13] approach, where the actions belonging to a given set S are required to synchronize. Exponentially timed transitions are, instead, labeled with a rate λ (the parameter of the exponential distribution) and represent timed choices performed according to a "race" between exponential delays. The interrelation between standard action transitions and exponentially timed transitions is governed by the so-called *maximal progress assumption* [17]: the possibility of executing τ transitions prevents the execution of exponentially timed transitions, thus expressing that the system cannot wait if it has something internal to do. Visible a transitions are, instead, interpreted as representing a "potential" interaction with the environment, hence their execution can be indefinitely delayed. Therefore in the IMC obtained from the specification of "complete concurrent system" no visible action transition occurs. In [9] a process algebra (called calculus of IMCs) is defined, which is just a simple extension of a standard process algebra (containing CCS [14] prefix "$\alpha.P$" and choice "$P + Q$" and CSP [13] parallel composition "$P \,\|_S\, Q$" and hiding "P/L") with a new form of prefix "$\lambda.P$" representing an exponential time delay. The semantics of the calculus of IMC derives IMCs from algebraic terms by using the standard CCS/CSP semantics for action transitions and by essentially using an interleaving semantics for "λ" delay prefixes (this is correct due to the memoryless property of exponential delays).

The notion of weak bisimulation for IMCs that is presented in [9] essentially matches exponentially timed transitions according to Markovian bisimulation [12] and abstracts from standard τ similarly to [14]. Since such an equivalence is shown to be a congruence for the calculus, it makes it possible to significantly and efficiently minimize the state-space of complete systems by abstracting from process interaction in a compositional way.

However the high potentiality and the elegance of the approach of [9] is somehow limited by the fact that the equivalence adopted over action transitions is a finer variant of Milner's observational congruence that distinguishes τ-divergent "Zeno" processes from non-divergent ones. In particular, similarly to [8], the additional requirement is introduced that two bisimilar terms must have the same opportunity to silently become stable terms, i.e. terms that cannot perform τ actions.

In [11] it is claimed that, since the maximal progress assumption generates a priority mechanism, it is somehow necessary to have such a τ-divergence sensitive equivalence. In particular [11] shows how to adapt the standard Milner's sound and complete axiomatization of observational congruence for a basic algebra with prefix, choice and recursion, when exponential delay prefixes are introduced (in such a way that the corresponding operators of the calculus of IMCs are obtained) and the τ-divergence sensitive equivalence of [9] is considered.

1.2 Simplifying the Notion of Weak Equivalence

In [6] we made a first step in the direction of eliminating the condition about stability from the equivalences of [8,9] in the context of interactive timed processes. We showed that maximal progress and Milner's standard notion of observational congruence are indeed compatible: it is possible to obtain a complete axiomatization for the basic interactive timed algebra of [11] even if the equivalence considered is not sensible to τ divergence.

Moreover, it is worth noting that in [6] we express priority arising from maximal progress by cutting transitions which cannot be performed directly in the operational semantics, instead of capturing such priority in the definition of equivalence as done in [9] (a solution also hinted in [10]). This technique allows us to obtain smaller system models and to further simplify the notion of equivalence considered in [9].

Unfortunately the results obtained in [6] for the basic interactive timed algebra do not scale to the full calculus of IMC [9]. This because the equivalence, being it not sensible to τ divergence, would not be a congruence for the parallel operator. The problem with congruence is that, e.g., while $\tau.\underline{0} \simeq \mathit{rec}X.\tau.X$, since the parallel operator behaves in such a way that the *presence* of a τ action within the actions immediately executable by a process pre-empts the other process from executing a timed action λ (global pre-emption [7]) we have that $\tau.\underline{0} \,\|_\emptyset\, \lambda.\underline{0} \not\simeq \mathit{rec}X.\tau.X \,\|_\emptyset\, \lambda.\underline{0}$. [2] This because the semantics of $\tau.\underline{0} \,\|_\emptyset\, \lambda.\underline{0}$ is that of $\tau.\lambda.\underline{0}$, while the semantics of $\mathit{rec}X.\tau.X \,\|_\emptyset\, \lambda.\underline{0}$ is that of $\mathit{rec}X.\tau.X$ (where no λ action can be executed). Note that this problem arises both in the case we capture priority in the notion of equivalence as done in [9] and in the case we enforce it in the definition of the operational semantics of the

[2] Here and in the rest of the paper we assume the following operator binding precedence: prefix > recursion > parallel composition > choice.

parallel operator with the technique of [6]. In the following we will suppose priority to be captured in the semantics of operators and equivalence to be "neutral" with respect to priority.

Conceptually, the problem above derives from the fact that the parallel operator deals with the terminated process $\underline{0}$ (and in general with processes which cannot execute neither τ actions nor λ actions) as if it let time pass. For example $\underline{0} \,\|_\emptyset\, \lambda$ may execute λ and become $\underline{0} \,\|_\emptyset\, \underline{0}$. This is obviously in contrast with the fact that $\underline{0}$ is weakly bisimilar to $recX.\tau.X$, which is clearly a process that does not let time pass (it represents a so-called time deadlock).

1.3 A New Markovian Calculus

As a consequence of the previous discussion, a very clean solution is to consider as processes which can let time pass only processes which can actually execute λ actions. In this way $\underline{0}$ is interpreted not as a terminated process which may let time pass, but as a time deadlock. As a consequence the definition of the parallel operator changes. In particular the parallel operator must be defined, similarly as in [8], in such a way that the *absence* of λ actions within the actions executable by a process (which means that the process cannot let time pass) pre-empts the other process from executing a timed action λ. Pre-emption caused by the *absence* of λ actions differs from pre-emption caused by the *presence* of τ actions exactly for the class of processes that were misinterpreted, i.e. processes which cannot execute neither τ actions nor λ actions. The new interpretation of such processes (as in [8]) is that, consistently with weak bisimilarity, either they immediately execute a visible action or they cause a time deadlock.

Based on this idea, in this paper we will define the new calculus of "Revisited" IMCs (RIMCs). In particular, the difference between IMCs and RIMCs at the transition system level is just in the meaning of states which cannot execute neither τ actions nor λ actions: RIMCs do not allow time to elapse in such states as, instead, IMCs do. Note that, as for IMCs, we can derive a CTMC from a complete system specification only if the derived RIMC cannot incur time deadlocks, i.e. states executing infinite sequences of τ transitions (as for IMCs) or equivalently states with no outgoing transitions (for RIMCs only).

As already explained, in the calculus we will define the rules for the parallel operator in such a way that, when we derive an exponentially timed move of $P \,\|_S\, Q$ from a corresponding move of P we require that also Q may perform an exponentially timed move, instead of requiring that Q must not perform a τ move. Note that, differently from [8], even if we require that Q may perform an exponentially timed move, we do not actually perform it because of the memoryless property of exponential delays.

Moreover, w.r.t the calculus of IMCs, in the calculus of RIMCs it is important (for "modeling convenience" and for the reasons that we will explain in Sect. 1.4) to also modify prefix and choice by considering operators similar to those of [8]:

- A new prefix operator $\gamma; P$ which is defined: as $\gamma.P$ if γ is τ or λ, as $recX.(\gamma.P + \tilde{\lambda}.X)$, for some $\tilde{\lambda}$, otherwise (where "$recX$" denotes recursion). Such a prefix, which allows visible actions to be delayed as in the calculus of IMCs (hence is suitable for specifying systems), becomes the unique prefix operator in the new calculus, while we will use the "basic" prefix $\gamma.P$ as an auxiliary operator to be used just for building an axiomatization.
- A new choice operator $P \diamond Q$ which, similarly as for the new parallel operator, allows one of P and Q to let time pass only if the other one may let time pass and is defined in such a way that delay execution does not resolve the choice. Such a choice operator, which allows new prefixes $a; P$ (where a is a visible action) to be used without causing the delays λ preceding the execution of the a to solve the choice (hence is suitable for specifying systems), becomes the unique choice operator in the new calculus, while we will use the "basic" choice $P + Q$ as an auxiliary operator to be used just for building an axiomatization.

Finally, we also include in the calculus of RIMC a new symbol "$\underline{1}$" representing a terminated process which may let time elapse (as for "$\underline{0}$" in the calculus of IMCs) defined as $recX.\tilde{\lambda}.X$, for some $\tilde{\lambda}$.

It is worth noting that, from the modeling viewpoint, we can mimic the behavior of the choice operator of the calculus of IMCs, where $\lambda.P + \mu.Q$ represents a choice between P and Q decided by a "race" between the λ and μ delays, by means of term $\lambda; \tau; P \diamond \mu; \tau; Q$ of the calculus of RIMC.

1.4 A New Notion of Markovian Equivalence

As we will see, the calculus of RIMCs, based on the ideas presented in the previous section, also allows for a new notion of Markovian bisimulation equivalence which is coarser than the standard one of [12]. The new version of Markovian equivalence is based on the new idea of "observability" of exponential delays.

As explained in the previous section, the behavior of the new prefix "$\gamma; P$" and of the new symbol "$\underline{1}$" is defined in terms of a generic rate $\tilde{\lambda}$ whose particular value is not important. In particular, $\tilde{\lambda}$ is the rate of an exponentially timed transition leading back to the state in which it is executed (a "selfloop"). Such a definition is consistent from the probabilistic viewpoint because the (transient) behavior of a CTMC (hence also its steady state behavior), defined as the probability of being in a certain state at a certain time, does not depend on the presence of exponentially timed selfloops in states (hence on the particular values for the rate labeling such selfloops). Intuitively, *as long as we consider the firing of exponentially timed transitions to be unobservable* as in CTMCs, it is easy to see that the particular values chosen for selfloop rates in a state of a RIMC do not change its behavior (hence that of derived CTMCs). We distinguish the following two cases. If the state has other outgoing exponentially timed transitions (which are not selfloops) then, the behavior of the RIMC in the state will be just as if selfloops are not present. This because, in the case a selfloop fires before one of the outgoing exponentially timed transitions causes the RIMC to leave the state, when the state is re-entered we can consider, thanks to the memoryless property, outgoing exponentially timed transitions to *continue* from the accomplishment level they reached before such event. Otherwise, if the state does not have other outgoing exponentially timed transitions, the RIMC will stay in the state forever, independently of the particular values of selfloop rates.

Even if in principle considering exponential delays as being "unobservable" could be done for every Markovian specification paradigm, to the best of our knowledge the calculus of RIMCs is the first Markovian process algebra to be compatible with unobservable exponential delays. This because, while all Markovian process algebras previously developed in the literature (see [12,2,18,9] and the references therein) make use of a "$P+Q$" choice operator such that an exponential move of P or Q resolves the choice (hence such a move is indeed "observed by the operator"), all the operators of the calculus of RIMCs (excluding the auxiliary ones to be used in the axiomatization) intuitively do not observe individual exponential firings, but just the global time to the occurrence of the next standard α action.

More precisely, supposing exponential delays are unobservable, we can modify the standard definition of Markovian bisimulation equivalence [12] as follows. Instead of requiring that *every* bisimulation equivalence class must be reached with the same aggregated rate by bisimilar terms, we can just require that this must hold for all equivalence classes *apart from the class including the terms themselves*. We will show that the new notion of Markovian bisimulation equivalence, which preserves the behavior of the underlying CTMC since it just adds insensitivity to rate of selfloops, is a congruence for the calculus of RIMC. On the contrary, such a notion is not a congruence for all existing Markovian process algebras, due to the presence of the "observing" choice operator "$P+Q$".

The notion of observational equivalence that we consider for the calculus of RIMCs is a combination of standard observational congruence and the new notion of Markovian bisimulation equivalence above. In spite of the problem with congruence arising with unobservable delays, the prefix $\theta.P$ and choice "$P+Q$" operators of the calculus of IMC [9] (which are also part of the basic interactive timed calculus for which we have developed a complete axiomatization of observational congruence in [6]) will play a fundamental role in building an axiomatization of such an equivalence. In particular we will build the axiom system by extending the calculus of RIMC with the "observable" exponential delays of [9], denoted by λ^o, and by considering standard Markovian bisimulation equivalence over such delays. In this way, by supposing that θ can be an observable delay λ^o and that "$P+Q$" only works with delays which are observable, we do not break the congruence property.

1.5 Contents of the Paper

In Sect. 2 we define RIMCs and the syntax and semantics of the calculus of RIMCs which contains the "$\gamma; P$", the "$P \diamond Q$", the "$P \parallel_S Q$" and the "P/L" operators and the symbols "$\underline{1}$" and "$\underline{0}$". Moreover we define observational congruence over RIMC terms simply as a combination of our "improved" notion of Markovian bisimulation and the standard notion of observational congruence of [14] and we show that it is indeed a congruence for the new calculus.

In Sect. 3 we present a sound axiomatization for our notion of observational congruence which is complete for strongly guarded finite-state processes of the new calculus. Such an axiomatization is built by: (*i*) introducing transition systems extending RIMCs with the "observable" exponential delay transitions used in IMCs, (*ii*) by consequently extending our notion of observational congruence so that standard Markovian bisimulation [12] is used over "observable" exponential delays, and (*iii*) by introducing some auxiliary operators: the prefix "$\theta.P$" and the choice "$P+Q$" operators of the calculus of IMCs [9]; the new operator "$\mathcal{H}(P)$" which "hides" exponential delays by turning each "observable" λ^o into an "unobservable" λ; the operator $vis(P)$ that eliminates non-visible behaviors from state P; the operators "$P \,\lfloor\!\lfloor_S\, Q$" and "$P \,|_S\, Q$" that are simple variants of the left merge and synchronization merge operators of [1], and finally the operator "$P \lhd Q$" which is a sort of left merge operator used for axiomatizing choice "$P \diamond Q$". We present the semantics of all auxiliary operators and we show that they preserve the congruence property.

Note that, since, to the best of our knowledge, developing an axiomatization of observational congruence for finite-state processes with unguarded recursion in the presence of "static" operators (like e.g. parallel composition) is an open problem also for standard CCS/CSP, obtaining an axiomatization for strongly guarded finite-state processes is the best that can be done without solving such an open problem. On the other hand in [6] we have already shown how to axiomatize unguarded recursion for interactive timed processes without static operators (the adaptation of the axiomatization of [6] to exponential delays is trivial), similarly to what Milner did for CCS in [16].

2 Calculus of Revisited Interactive Markov Chains

2.1 Revisited Interactive Markov Chains

In the following we present the formal definition of Interactive Markovian Transition System (IMTS). Interactive Markov Chains are IMTSs possessing an initial state. Formally, we denote the set of rates by $Exp = \mathbb{R}^+$, ranged over by λ, μ. Moreover, we denote the set of standard action types used in a IMTS by Act, ranged over by $\alpha, \alpha', \ldots$. As usual Act includes the special type τ denoting internal actions. The set $Act - \{\tau\}$ is ranged over by $a, b, \ldots$. We use $\gamma, \gamma', \ldots$ to range over $Act \cup Exp$, i.e. labels of IMTS transitions. The set of states of an IMTS is denoted by Σ, ranged over by $s, s', \ldots$. In the rest of the paper we will assume the following abbreviations. Let us suppose that $T \subseteq (\Sigma \times \mathit{Labels} \times \Sigma)$ is a transition relation, where *Labels* is a set of transition labels, ranged over by l. In the remainder we use $s \xrightarrow{l} s'$ to stand for $(s, l, s') \in T$; $s \xrightarrow{l}$ to stand for $\exists s' \in \Sigma : s \xrightarrow{l} s'$; and $s \xrightarrow{\mathit{Set}}$, where $\mathit{Set} \subseteq \mathit{Labels}$, to stand for $\exists s' \in \Sigma, l \in \mathit{Set} : s \xrightarrow{l} s'$. $s \not\xrightarrow{l}$ and $s \not\xrightarrow{\mathit{Set}}$, where $\mathit{Set} \subseteq \mathit{Labels}$, denote the negations of $s \xrightarrow{l}$ and $s \xrightarrow{\mathit{Set}}$, respectively.

Definition 2.1 An Interactive Markovian Transition System (IMTS) is a tuple (Σ, Act, T_e, T_a) with

- Σ a set of states,
- Act a set of standard actions,
- $T_e \subseteq (\Sigma \times Exp \times \Sigma)$ and $T_a \subseteq (\Sigma \times Act \times \Sigma)$ two transition relations, containing exponentially timed and action transitions, respectively, such that $\forall s \in \Sigma$:

$$s \xrightarrow{\tau} \text{ implies } s \not\xrightarrow{Exp}$$

■

2.2 Syntax and Semantics of the Calculus of RIMCs

Let Var be a set of process variables ranged over by X, Y, Z. Let $ARFun = \{\varphi : Act \longrightarrow Act \mid \varphi(\tau) = \tau \wedge \varphi(Act - \{\tau\}) \subseteq Act - \{\tau\}\}$ be a set of *action relabeling functions*, ranged over by φ.

Definition 2.2 We define the language *RIMC* as the set of terms generated by the following syntax

$$P ::= \underline{1} \mid \underline{0} \mid X \mid \gamma; P \mid P \diamond P \mid P/L \mid P[\varphi] \mid P \parallel_S P \mid recX.P$$

where $L, S \subseteq Act - \{\tau\}$. A *RIMC* process is a closed term of *RIMC*. We denote by $RIMC_c$ the set of *RIMC* processes and by $RIMC_{cg}$ the set of strongly guarded *RIMC* processes. [3] ■

"$\underline{1}$" denotes a terminated process which allows for the passage of time. "$\underline{0}$" denotes a time deadlock. "$\gamma; P$" is the prefix operator. Similarly as in [8], if γ is τ or a delay λ then it is immediately executed, otherwise (it is a visible action a) it can be arbitrarily delayed. "$P \diamond Q$" is the choice operator. Similarly as in [8], as long as P or Q execute delays λ then they just evolve internally and the choice remains (as if they were in parallel). In particular time is allowed

[3] We consider λ prefixes as being guards in the definition of strong guardedness. Moreover we consider the notion of strong guardedness to account for relabeling and hiding operators: e.g. $(recX.a.X)/\{a\}$ is not strongly guarded (see e.g. [3] Appendix A for a precise definition).

$$\alpha ; P \xrightarrow{\alpha} P$$

$$\frac{P \xrightarrow{\alpha} P'}{P \diamond Q \xrightarrow{\alpha} P'} \qquad \frac{Q \xrightarrow{\alpha} Q'}{P \diamond Q \xrightarrow{\alpha} Q'}$$

$$\frac{P \xrightarrow{\alpha} P'}{P \parallel_S Q \xrightarrow{\alpha} P' \parallel_S Q} \ \alpha \notin S \qquad \frac{Q \xrightarrow{\alpha} Q'}{P \parallel_S Q \xrightarrow{\alpha} P \parallel_S Q'} \ \alpha \notin S$$

$$\frac{P \xrightarrow{a} P' \wedge Q \xrightarrow{a} Q'}{P \parallel_S Q \xrightarrow{a} P' \parallel_S Q'} \ a \in S$$

$$\frac{P \xrightarrow{a} P'}{P/L \xrightarrow{\tau} P'/L} \ a \in L \qquad \frac{P \xrightarrow{\alpha} P'}{P/L \xrightarrow{\alpha} P'/L} \ a \notin L$$

$$\frac{P \xrightarrow{\alpha} P'}{P[\varphi] \xrightarrow{\varphi(\alpha)} P'[\varphi]} \qquad \frac{P\{recX.P/X\} \xrightarrow{\alpha} P'}{recX.P \xrightarrow{\alpha} P'}$$

Table 1
Standard Rules

to advance for one process only if the same holds for the other one. The first between P and Q which executes an action α resolves the choice. "P/L" is the hiding operator which turns the actions in L into τ actions by consequently cutting alternative delay transitions, "$P[\varphi]$" is the relabeling operator which relabels visible actions according to φ. "$P \parallel_S Q$" is the CSP parallel operator, where synchronization over actions in S is required and where, similarly as in [8], time is allowed to advance for one process only if the same holds for the other one. Finally "$recX.P$" denotes recursion in the usual way.

The semantics of RIMC terms is defined as being the $RIMTS$ $(RIMC_c, Act, T_e, T_a)$, where: T_a is the least subset of $RIMC_c \times Act \times RIMC_c$ satisfying the standard operational rules of Table 1 and T_e is obtained from the least multiset over $RIMC_c \times Exp \times RIMC_c$ satisfying the operational rules of Table 2 (similarly to [12,9], we consider a transition to have arity m if and only if it can be derived in m possible ways from the operational rules) by summing the rates of the multiple occurrences of the same transition. As already explained in Sect. 1.4, any value can be chosen for the rate $\tilde{\lambda}$ occurring in Table 2 (different values give rise to equivalent $RIMTSes$).

2.3 Observational Congruence for RIMCs

As explained in Sect. 1.4, the notion of observational congruence over RIMCs: (i) deals with exponentially timed choices according to a coarser variant of Markovian bisimulation [12] which abstracts from selfloops, and (ii) abstracts

$$\lambda;P \xrightarrow{\lambda} P \qquad a;P \xrightarrow{\tilde{\lambda}} a;P \qquad \underline{1} \xrightarrow{\tilde{\lambda}} \underline{1}$$

$$\frac{P \xrightarrow{\lambda} P' \wedge Q \xrightarrow{Exp}}{P \diamond Q \xrightarrow{\lambda} P' \diamond Q} \qquad \frac{Q \xrightarrow{\lambda} Q' \wedge P \xrightarrow{Exp}}{P \diamond Q \xrightarrow{\lambda} P \diamond Q'}$$

$$\frac{P \xrightarrow{\lambda} P' \wedge Q \xrightarrow{Exp}}{P \parallel_S Q \xrightarrow{\lambda} P' \parallel_S Q} \qquad \frac{Q \xrightarrow{\lambda} Q' \wedge P \xrightarrow{Exp}}{P \parallel_S Q \xrightarrow{\lambda} P \parallel_S Q'}$$

$$\frac{P \xrightarrow{\lambda} P' \wedge \nexists a \in L.\, P \xrightarrow{a}}{P/L \xrightarrow{\lambda} P'/L} \qquad \frac{P \xrightarrow{\lambda} P'}{P[\varphi] \xrightarrow{\lambda} P'[\varphi]}$$

$$\frac{P\{recX.P/X\} \xrightarrow{\lambda} P'}{recX.P \xrightarrow{\lambda} P'}$$

Table 2
Rules for Exponentially Timed Moves

from standard τ actions as in observational congruence [14].

Given a RIMTS (Σ, Act, T_e, T_a), a state $s \in \Sigma$ and a set of states $C \subseteq \Sigma$, in the following we denote the total rate of exponentially timed transitions from s to C by $TR(s, C) = \sum\{\!|\, \lambda \mid \exists s' \in C : s \xrightarrow{\lambda} s' \,|\!\}$. [4] Moreover we use $\stackrel{\alpha}{\Longrightarrow}$ to denote $(\xrightarrow{\tau})^* \xrightarrow{\alpha} (\xrightarrow{\tau})^*$, i.e. a sequence of transitions including a single α transition and any number of τ transitions. We also define $\stackrel{\hat{\alpha}}{\Longrightarrow} \; = \; \stackrel{\alpha}{\Longrightarrow}$ if $\alpha \neq \tau$ and $\stackrel{\hat{\tau}}{\Longrightarrow} \; = (\xrightarrow{\tau})^*$, i.e. a possibly empty sequence of τ transitions.

Definition 2.3 Let (Σ, Act, T_e, T_a) be a RIMTS. An equivalence relation β on Σ is a *weak bisimulation* iff $s_1 \;\beta\; s_2$ implies:

(i) for every $\alpha \in Act$ and $s'_1 \in \Sigma$,

$$s_1 \xrightarrow{\alpha} s'_1 \text{ implies } s_2 \stackrel{\hat{\alpha}}{\Longrightarrow} s'_2 \text{ for some } s'_2 \text{ with } s'_1 \;\beta\; s'_2$$

(ii) $s_1 \xrightarrow{Exp}$ implies: $s_2 \stackrel{\hat{\tau}}{\Longrightarrow} s'_2$ for some s'_2 such that $s'_2 \xrightarrow{Exp}$ and for every $C \in \Sigma/\beta$ with $C \neq [s_1]_\beta$,

$$TR(s_1, C) = TR(s'_2, C)$$ [5]

$s_1, s_2 \in \Sigma$ are weakly bisimilar, denoted by $s_1 \approx s_2$, iff (s_1, s_2) is included in some weak bisimulation. ■

Note that for a state s'_2 satisfying condition (ii) it must be that $s_1 \;\beta\; s'_2$ (otherwise it would not be possible that $s_1 \;\beta\; s_2$ since $s_1 \xrightarrow{Exp}$ implies that

[4] We use "$\{\!|$" and "$|\!\}$" as brackets for multisets. Moreover we assume summation over the empty multiset to yield 0.

[5] We use "Σ/β" to denote the set of the equivalence classes of β defined over Σ.

$s_1 \not\xrightarrow{\tau}$), hence $[s_1]_\beta = [s'_2]_\beta$, i.e. for both s_1 and s'_2 we do not consider exponential transitions leading to their own equivalence class. Moreover note that, as shown in [11], trying to "weaken" any further the notion of weak bisimulation above, e.g. by allowing τ transitions to be executed after exponential delays to reach an equivalence class, does not generate a coarser notion of equivalence.

Definition 2.4 Let (Σ, Act, T_e, T_a) be a RIMTS. An equivalence relation β on Σ is an *observational bisimulation* iff $s_1\ \beta\ s_2$ implies:

(i) for every $\alpha \in Act$ and $s'_1 \in \Sigma$,

$$s_1 \xrightarrow{\alpha} s'_1 \text{ implies } s_2 \stackrel{\alpha}{\Longrightarrow} s'_2 \text{ for some } s'_2 \text{ with } s'_1 \approx s'_2$$

(ii) $s_1 \xrightarrow{Exp}$ implies: $s_2 \xrightarrow{Exp}$ and

$$\text{for every } C \in \Sigma/\beta \text{ with } C \neq [s_1]_\beta,$$
$$TR(s_1, C) = TR(s_2, C)$$

$s_1, s_2 \in \Sigma$ are observationally congruent, denoted by $s_1 \simeq s_2$, iff (s_1, s_2) is included in some observational bisimulation. ■

Note that, since $[s_1]_\beta = [s_2]_\beta$, again, in condition (ii) for both s_1 and s_2 we do not consider exponential transitions leading to their own equivalence class.

We consider $\simeq$ as being defined also on the open terms of *RIMC* by extending observational congruence with the standard approach of [14].

Theorem 2.5 *$\simeq$ is a congruence for the calculus of RIMCs w.r.t. all its operators, including recursion.*

Proof. *Let us start from the choice operator "$P \diamond Q$". It suffices to show that $\beta = \{(P_1 \diamond Q, P_2 \diamond Q) \mid P_1, P_2, Q \in RIMC_c \wedge P_1 \simeq P_2\} \cup ID_{RIMC_c}$ (where ID_{RIMC_c} is the identity relation over $RIMC_c$) is an observational bisimulation. Given $(R_1, R_2) \in \beta$, either $(R_1, R_2) \in ID_{RIMC_c}$ and the proof is trivial, or $R_1 \equiv P_1 \diamond Q$ and $R_2 \equiv P_1 \diamond Q$ for some P_1, P_2 and Q. In the latter case:*

- *If $P_1 \diamond Q$ perfoms a standard action α then $P_2 \diamond Q$ may perform a corresponding move by resorting to standard machinery [14].*
- *If $P_1 \diamond Q \xrightarrow{Exp}$ then $P_1 \xrightarrow{Exp}$ and $Q \xrightarrow{Exp}$. Since $P_1 \simeq P_2$, we have $P_2 \xrightarrow{Exp}$ and for every $C \in RIMC_c/\simeq$ with $C \neq [P_1]_\simeq$, $TR(P_1, C) = TR(P_2, C)$. Therefore $P_2 \diamond Q \xrightarrow{Exp}$ and for every $C \in RIMC_c/\beta$ with $C \neq [P_1 \diamond Q]_\beta$, we have:*
  - *either $C = \{R\}$ for some term R whose outermost operator is not "$\diamond$" and $TR(P_1 \diamond Q, C) = TR(P_2 \diamond Q, C) = 0$,*
  - *or there exists $C' \in RIMC_c/\simeq$ and $Q' \in RIMC_c$ such that $C = \{P \diamond Q' \mid P \in C'\}$. In this case:*
    if $C' \neq [P_1]_\simeq$ and $Q = Q'$ then $TR(P_1 \diamond Q, C) = TR(P_1, C') = TR(P_2, C') = TR(P_2 \diamond Q, C)$;
    if $C' = [P_1]_\simeq$ and $Q \neq Q'$ then $TR(P_1 \diamond Q, C) = TR(Q, \{Q'\}) = TR(P_2 \diamond Q, C)$;

if $C' \neq [P_1]_\simeq$ and $Q \neq Q'$ then $TR(P_1 \diamond Q, C) = 0 = TR(P_2 \diamond Q, C)$;

As far as the parallel operator "$P \,\|_S\, Q$" is concerned, we preliminarily show that "$P \,\|_S\, Q$" is a congruence w.r.t. weak bisimulation, i.e. that, for a given set S, $\beta = \{(P_1 \,\|_S\, Q, P_2 \,\|_S\, Q) \mid P_1, P_2, Q \in RIMC_c \wedge P_1 \simeq P_2\} \cup ID_{RIMC_c}$ is a weak bisimulation. Given $(R_1, R_2) \in \beta$, either $(R_1, R_2) \in ID_{RIMC_c}$ and the proof is trivial, or $R_1 \equiv P_1 \,\|_S\, Q$ and $R_2 \equiv P_1 \,\|_S\, Q$ for some P_1, P_2 and Q. In the latter case:

- *If $P_1 \,\|_S\, Q$ perfoms a standard action α then $P_2 \,\|_S\, Q$ may perform a corresponding move by resorting to standard machinery [14].*
- *If $P_1 \,\|_S\, Q \xrightarrow{Exp}$ then $P_1 \xrightarrow{Exp}$ and $Q \xrightarrow{Exp}$. Since $P_1 \approx P_2$, we have $P_2 \stackrel{\hat{\tau}}{\Longrightarrow} P_2'$ and $P_2' \xrightarrow{Exp}$ and for every $C \in RIMC_c/\approx$ with $C \neq [P_1]_\approx$, $TR(P_1, C) = TR(P_2, C)$. Therefore $P_2 \,\|_S\, Q \stackrel{\hat{\tau}}{\Longrightarrow} P_2' \,\|_S\, Q$ and $P_2' \,\|_S\, Q \xrightarrow{Exp}$ and for every $C \in RIMC_c/\beta$ with $C \neq [P_1 \,\|_S\, Q]_\beta$, we have:*
  - *either $C = \{R\}$ for some term R whose outermost operator is not "$\|_S$" and $TR(P_1 \,\|_S\, Q, C) = TR(P_2 \,\|_S\, Q, C) = 0$,*
  - *or there exists $C' \in RIMC_c/\approx$ and $Q' \in RIMC_c$ such that $C = \{P \,\|_S\, Q' \mid P \in C'\}$. In this case:*

    if $C' \neq [P_1]_\approx$ and $Q = Q'$ then $TR(P_1 \,\|_S\, Q, C) = TR(P_1, C') = TR(P_2', C') = TR(P_2' \,\|_S\, Q, C)$;

    if $C' = [P_1]_\approx$ and $Q \neq Q'$ then $TR(P_1 \,\|_S\, Q, C) = TR(Q, \{Q'\}) = TR(P_2' \,\|_S\, Q, C)$;

    if $C' \neq [P_1]_\approx$ and $Q \neq Q'$ then $TR(P_1 \,\|_S\, Q, C) = 0 = TR(P_2' \,\|_S\, Q, C)$;

Now it suffices to show that, for a given set S, $\beta = \{(P_1 \,\|_S\, Q, P_2 \,\|_S\, Q) \mid P_1, P_2, Q \in RIMC_c \wedge P_1 \simeq P_2\} \cup ID_{RIMC_c}$ is an observational bisimulation. The proof of this fact is identical to the proof above for weak bisimulation (with "$\simeq$" replacing "$\approx$"), apart from the case of a standard action α performed by $P_1 \,\|_S\, Q$. In paricular, we derive $P_1' \,\|_S\, Q \approx P_2' \,\|_S\, Q$, where P_1' and P_2' are the terms reached by P_1 and P_2 respectively, from $P_1' \approx P_2'$ by exploiting the result above about congruence of weak bisimulation w.r.t. parallel.

As far as the hiding operator "P/L" is concerned, we preliminarily show that "P/L" is a congruence w.r.t. weak bisimulation, i.e. that, for a given set L, $\beta = \{(P_1/L, P_2/L) \mid P_1, P_2 \in RIMC_c \wedge P_1 \simeq P_2\} \cup ID_{RIMC_c}$ is a weak bisimulation. Given $(R_1, R_2) \in \beta$, either $(R_1, R_2) \in ID_{RIMC_c}$ and the proof is trivial, or $R_1 \equiv P_1/L$ and $R_2 \equiv P_1/L$ for some P_1, P_2. In the latter case:

- *If P_1/L perfoms a standard action α then P_2/L may perform a corresponding move by resorting to standard machinery [14].*
- *If $P_1/L \xrightarrow{Exp}$ then $P_1 \xrightarrow{Exp}$ and $\nexists a \in L : P_1 \xrightarrow{a}$. Since $P_1 \approx P_2$, we have $P_2 \stackrel{\hat{\tau}}{\Longrightarrow} P_2'$ and $P_2' \xrightarrow{Exp}$ and for every $C \in RIMC_c/\approx$ with $C \neq [P_1]_\approx$, $TR(P_1, C) = TR(P_2, C)$. Therefore $P_2/L \stackrel{\hat{\tau}}{\Longrightarrow} P_2'/L$ and (since $P_2' \approx P_1$, hence $\nexists a \in L : P_2' \xrightarrow{a}$) $P_2'/L \xrightarrow{Exp}$ and for every $C \in RIMC_c/\beta$ with $C \neq [P_1/L]_\beta$, we have:*

- *either* $C = \{R\}$ *for some term* R *whose outermost operator is not "*$/L$*" and* $TR(P_1/L, C) = TR(P_2/L, C) = 0$*,*
- *or there exists* $C' \in RIMC_c/\approx$*, with* $C' \neq [P_1]_\approx$*, such that* $C = \{P/L \mid P \in C'\}$*. In this case* $TR(P_1/L, C) = TR(P_1, C') = TR(P_2', C') = TR(P_2'/L, C)$*.*

Now it suffices to show that, for a given set L*,* $\beta = \{(P_1/L, P_2/L) \mid P_1, P_2 \in RIMC_c \wedge P_1 \simeq P_2\} \cup ID_{RIMC_c}$ *is an observational bisimulation. The proof of this fact is identical to the proof above for weak bisimulation (with "*$\simeq$*" replacing "*$\approx$*"), apart from the case of a standard action* α *performed by* P_1/L*. In paricular, we derive* $P_1'/L \approx P_2'/L$*, where* P_1' *and* P_2' *are the terms reached by* P_1 *and* P_2 *respectively, from* $P_1' \approx P_2'$ *by exploiting the result above about congruence of weak bisimulation w.r.t. hiding.*

The proof of congruence w.r.t. prefix "$\gamma; P$*" and relabeling "*$P[\varphi]$*" is trivial.*

As far as recursion "$recX.P$*" is concerned, we apply the technique we introduced in [4]. We have to show that, for all* $P_1, P_2 \in RIMC_c$ *containing at most the variable* X *free, we have that* $P_1 \simeq P_2$ *implies* $recX.P_1 \simeq recX.P_2$*. We do this by showing that the relation* $\beta = \{(Q\{recX.P_1/X\}, Q\{recX.P_2/X\}) \mid Q \in RIMC_c\}$ *is such that, given* $\beta' = \beta \cup \beta^{-1}$*, whenever* $R_1 \; \beta \; R_2$ *we have:*

(i) *for every* $\alpha \in Act$ *and* $R_1' \in RIMC_c$*,*
$$R_1 \xrightarrow{\alpha} R_1' \text{ implies } R_2 \overset{\alpha}{\Longrightarrow} R_2' \text{ for some } R_2' \text{ with } R_1' \approx \cup \beta \; R_2'$$

(ii) $R_1 \xrightarrow{Exp}$ *implies:* $R_2 \xrightarrow{Exp}$ *and*
$$\text{for every } C \in RIMC_c/(\simeq \cup \beta')^+ \text{ with } C \neq [R_1]_{(\simeq \cup \beta')^+},$$
$$TR(R_1, C) = TR(R_2, C)$$

In particular we induce on the maximum depth of the inference of the transitions leaving term R_1 *and we show* $TR(R_1, C) \leq TR(R_2, C)$ *only. The converse is obtained by a symmetrical argument on the moves of* R_2*. In such an induction, the only significant novelty w.r.t. the proof of [4] is the exclusion of selfloops when evaluating total rates. However such an exclusion is "compatible" with the proof because when an equivalence class* C *considered at maximum depth* d *is expressed in terms of the corresponding ones* $C_i, i \in I$ *considered at maximum depth* $d - 1$*, we have that if* C *does not constitute a selfloop none of the classes* $C_i, i \in I$ *constitutes a selfloop. Intuitively a recursion* $recX.P$ *cannot unfold a selfloop already present in* P *(thus making the total rate of the selfloop "observable"). Note that from the statement above it is immediate to conclude that* $(\approx \cup \beta')^+$ *is a weak bisimulation and, then, that* $(\simeq \cup \beta')^+$ *is an observational bisimulation. Therefore, by taking* $Q \equiv X$ *in* β*, we are done.* □

3 Axiomatizing Revisited Interactive Markov Chains

In this section we present an axiom system which is complete for $\simeq$ on strongly guarded finite-state *RIMC* processes.

In order to build the axiomatization we need to extend RIMTSes and our notion of observational equivalence with the "observable" exponential delays of [9] and to introduce some auxiliary operators. Formally, we denote the set of rates of observable delays by $Exp^o = \mathbb{R}^+$, ranged over by $\lambda^o, \mu^o, \ldots$. We use $\theta, \theta', \ldots$ to range over $Act \cup Exp \cup Exp^o$.

Definition 3.1 An Extended Interactive Markovian Transition System (EIMTS) is a tuple $(\Sigma, Act, T_o, T_e, T_a)$ with

- Σ a set of states,
- Act a set of standard actions,
- $T_o \subseteq (\Sigma \times Exp^o \times \Sigma)$, $T_e \subseteq (\Sigma \times Exp \times \Sigma)$, and $T_a \subseteq (\Sigma \times Act \times \Sigma)$ three transition relations, containing observable exponentially timed, unobservable exponentially timed, and action transitions, respectively, such that $\forall\, s \in \Sigma$:

(i) $s \xrightarrow{Exp^o}$ implies $s \overset{Exp}{\not\longrightarrow}$ (or equivalently $s \xrightarrow{Exp}$ implies $s \overset{Exp^o}{\not\longrightarrow}$)

(ii) $s \xrightarrow{\tau}$ implies: $s \overset{Exp^o}{\not\longrightarrow}$ and $s \overset{Exp}{\not\longrightarrow}$ ■

Now we formally introduce the auxiliary operators needed to build the axiomatization, whose semantics is presented in Table 3. The operators "$\theta.P$" and "$P+Q$" are those of the calculus of IMCs [9]; in particular "$P+Q$" works on observable delays only. The new operator "$\mathcal{H}(P)$", which "hides" exponential delays by turning "observable" λ^o immediately executable by P into "unobservable" λ (and "restricts" unobservable delays previously executable by P), will play a fundamental role in the axiomatization. In particular it will allow us to express the operators of the calculus of RIMCs in terms of the prefix and choice operators of the calculus of IMCs [9]. The operators "$P \lfloor\!\lfloor_S Q$" and "$P \,|_S\, Q$" are simple variants of the left merge and synchronization merge operators of [1], while "$P \triangleleft Q$" is a sort of left merge operator used for axiomatizing choice "$P \diamond Q$". Note that "$P \lfloor\!\lfloor_S Q$" and "$P \triangleleft Q$" are defined in such a way that: (i) since they have to be used as arguments of a "$P+Q$" operator, they require delays immediately executable (by P) to be observable; and (ii) they can execute exponential delays of P also in the case Q can execute τ transitions (and, e.g., not delay transitions), so that the axioms $(LC3)$ and $(LM4)$ of Table 1 are sound. Moreover, the definition of the operational rule for "$P \,|_S\, Q$" allows for actions "τ" to be skipped so to get a congruence [1]. Finally, the operator "$vis(P)$" eliminates non-visible behaviors (those immediately starting in P with an observable or unobservable exponential delay or a weak τ transition). Such an operator will play a important role in the axiomatization of parallel composition and choice "$P \diamond Q$" in that it allows us to check for the absence of executable exponential delays and/or τ actions

$$\theta.P \xrightarrow{\;\theta\;} P$$

$$\frac{P \xrightarrow{\alpha} P'}{P + Q \xrightarrow{\alpha} P'} \qquad \frac{Q \xrightarrow{\alpha} Q'}{P + Q \xrightarrow{\alpha} Q'}$$

$$\frac{P \xrightarrow{\lambda^o} P' \wedge Q \stackrel{\tau}{\not\longrightarrow}}{P + Q \xrightarrow{\lambda^o} P'} \qquad \frac{Q \xrightarrow{\lambda^o} Q' \wedge P \stackrel{\tau}{\not\longrightarrow}}{P + Q \xrightarrow{\lambda^o} Q'}$$

$$\frac{P \xrightarrow{\alpha} P'}{\mathcal{H}(P) \xrightarrow{\alpha} P'} \qquad \frac{P \xrightarrow{\lambda^o} P'}{\mathcal{H}(P) \xrightarrow{\lambda} P'}$$

$$\frac{P \xrightarrow{\alpha} P'}{P \lhd Q \xrightarrow{\alpha} P'} \qquad \frac{P \xrightarrow{\lambda^o} P' \wedge (Q \stackrel{Exp}{\longrightarrow} \vee Q \stackrel{\tau}{\longrightarrow})}{P \lhd Q \xrightarrow{\lambda^o} P' \diamond Q}$$

$$\frac{P \xrightarrow{\alpha} P'}{P \,\lfloor\!\lfloor_S\, Q \xrightarrow{\alpha} P' \,\|_S\, Q}\ \alpha \notin S \qquad \frac{P \xrightarrow{\lambda^o} P' \wedge (Q \stackrel{Exp}{\longrightarrow} \vee Q \stackrel{\tau}{\longrightarrow})}{P \,\lfloor\!\lfloor_S\, Q \xrightarrow{\lambda^o} P' \,\|_S\, Q}$$

$$\frac{P \stackrel{a}{\Longrightarrow} P' \wedge Q \stackrel{a}{\Longrightarrow} Q'}{P \,|_S\, Q \xrightarrow{a} P' \,\|_S\, Q'}\ a \in S \qquad \frac{P \stackrel{a}{\Longrightarrow} P'}{vis(P) \xrightarrow{a} P'}$$

Table 3
Rules for Auxiliary Operators

(see axioms $(LC4)$ and $(LM5)$ of Table 1).

We define the language EIMC to be the set of terms obtained by extending the calculus of RIMCs with the auxiliary operators above (we denote the set of closed EIMC terms by EIMC_c). Moreover we define the semantics of EIMC terms to be the EIMTS $(EIMC_c, Act, T_o, T_e, T_a)$ obtained from the operational rules of Table 1, Table 2 and Table 3 plus an additional rule for both the hiding "P/L" and the relabeling "$P[\varphi]$" operators which is obtained from that of Table 2 by replacing λ^o transitions for λ transitions. Note that "P/L" and "$P[\varphi]$" are conservatively extended.

The notions of weak bisimulation and observational congruence for EIMC are conservative extensions of those for RIMC. In the following, we denote the total rate of observable exponentially timed transitions from s to I by $TR^o(s, I)$, which is defined similarly as for "unobservable" delays.

Definition 3.2 Let $(\Sigma, Act, T_o, T_e, T_a)$ be an EIMTS. An equivalence relation β on Σ is a *weak bisimulation* iff $s_1 \ \beta \ s_2$ implies: the 2 conditions of Definition 2.3 and the additional condition

(iii) $s_1 \stackrel{Exp^o}{\longrightarrow}$ implies: $s_2 \stackrel{\hat{\tau}}{\Longrightarrow} s_2'$ for some s_2' such that $s_2' \stackrel{Exp^o}{\longrightarrow}$ and

$$\text{for every } C \in \Sigma/\beta, \quad TR^o(s_1, C) = TR^o(s_2', C)$$

$s_1, s_2 \in \Sigma$ are weakly bisimilar, denoted by $s_1 \approx s_2$, iff (s_1, s_2) is included in some weak bisimulation. ∎

Definition 3.3 Let $(\Sigma, Act, T_o, T_e, T_a)$ be an EIMTS. An equivalence relation β on Σ is an *observational bisimulation* iff $s_1 \; \beta \; s_2$ implies: the 2 conditions of Definition 2.4 and the additional condition

(iii) $s_1 \xrightarrow{Exp^o}$ implies: $s_2 \xrightarrow{Exp^o}$ and
$$\text{for every } C \in \Sigma/\beta, \quad TR^o(s_1, C) = TR^o(s_2, C)$$

$s_1, s_2 \in \Sigma$ are observationally congruent, denoted by $s_1 \simeq s_2$, iff (s_1, s_2) is included in some observational bisimulation. ∎

The extension of the calculus of *RIMCs* preserves the congruence property.

Theorem 3.4 *$\simeq$ is a congruence for the calculus of EIMCs w.r.t. all its operators, including recursion.*

Proof. *Given $P \simeq Q$, for each operator "op" it just sufficies to show that the (symmetric and transitive closure of the) relation obtained by adding $(op(P), op(Q))$ to $\simeq$ is an observational bisimulation, by exploiting the congruence property of observational congruence w.r.t. both parallel "$\|_S$" and choice "$\diamond$", and the congruence property of weak bisimulation w.r.t. parallel "$\|_S$". Note that for obtaining congruence w.r.t. the operator "$\lhd$" it is essential that in item 3 of Definition 3.3 we consider equivalence classes w.r.t. relation β instead of just considering relation $\approx$.* □

We are now in a position to present the axiom system $\mathcal{A}_{EIMC}$ for $\simeq$ on *EIMC* terms, which is formed by the axioms presented in Fig. 1. The axioms $(Ter) - (SM6)$, with the help of axioms $(Vis1), (Vis2), (A4)$ and $(Rec1) - (Rec3)$, are used to transform $RIMC_{cg}$ processes into normal form.

Definition 3.5 A process $P \in RIMC_c$ is in normal form if it is either of the form "$\mathcal{H}(\sum_{i \in I} \theta_i.P_i)$" [6] or "$recX.\mathcal{H}(\sum_{i \in I} \theta_i.P_i)$" or "$X$", where:

- for each $i \in I$, $\theta_i \in Exp^o \cup Act$,
- if there exists $i \in I$ such that $\theta_i = \tau$ then there is no $i \in I$ such that $\theta_i \in Exp^o$.

∎

The standard axioms $(A1) - (A4), (Tau1) - (Tau4)$ and $(Rec1) - (Rec3)$ (the axiom $(Tau4)$, which is new w.r.t. the standard axiomatization of [14], reflects the fact that our notion of observational congruence requires an action transition, as opposed to a delay transition, to be performed before weak bisimulation is considered), the axiom $(ExpT)$, which captures additivity of exponential delays, the axiom $(MProg)$, which captures the maximal progress assumption, and the totally new axiom $(ExpRec)$ which captures the insensitivity to selfloops of exponential delays, are used to equate normal forms

[6] We assume $\sum_{i \in I} \theta_i.P_i$ to be "$\underline{0}$" when $I = \emptyset$.

$(A1)$	$P + Q = Q + P$		$(A2)$	$(P + Q) + R = P + (Q + R)$	
$(A3)$	$\alpha.P + \alpha.P = \alpha.P$		$(A4)$	$\mathcal{H}(P + \underline{0}) = \mathcal{H}(P)$	
$(Tau1)$	$\alpha.\mathcal{H}(\tau.P) = \alpha.P$				
$(Tau2)$	$\mathcal{H}(P + \tau.\mathcal{H}(P) + Q) = \mathcal{H}(\tau.\mathcal{H}(P) + Q)$				
$(Tau3)$	$\alpha.\mathcal{H}(P + \tau.Q) + \alpha.Q = \alpha.\mathcal{H}(P + \tau.Q)$				
$(Tau4)$	$\alpha.P\{\lambda.\mathcal{H}(\tau.Q)/X\} = \alpha.P\{\lambda.Q/X\}$				
$(ExpT)$	$\lambda^o.P + \mu^o.P = (\lambda^o + \mu^o).P$				
$(MProg)$	$\tau.P + \lambda^o.Q = \tau.P$				
$(Vis1)$	$vis(a.P) = a.P$		$(Vis2)$	$vis(P + Q) = vis(P) + vis(Q)$	
(Ter)	$\underline{1} = recX.\mathcal{H}(\lambda^o.X)$				
$(Dead)$	$\underline{0} = \mathcal{H}(\underline{0})$				
$(Pre1)$	$a;P = recX.\mathcal{H}(a.P + \lambda^o.X)$		$(Pre2)$	$\tau;P = \mathcal{H}(\tau.P)$	
$(Pre3)$	$\lambda;P = \mathcal{H}(\lambda^o.P)$				
$(Hi1)$	$\mathcal{H}(P)/L = \mathcal{H}(P/L)$		$(Hi2)$	$\underline{0}/L = \underline{0}$	
$(Hi3)$	$(\theta.P)/L = \theta.(P/L)$	$\theta \notin L$	$(Hi4)$	$(a.P)/L = \tau.(P/L)$	$a \in L$
$(Hi5)$	$(P + Q)/L = P/L + Q/L$				
$(Rel1)$	$\mathcal{H}(P)[\varphi] = \mathcal{H}(P[\varphi])$		$(Rel2)$	$\underline{0}[\varphi] = \underline{0}$	
$(Rel3)$	$(\alpha.P)[\varphi] = \varphi(\alpha).(P[\varphi])$		$(Rel4)$	$(\theta.P)[\varphi] = \theta.(P[\varphi])$	
$(Rel5)$	$(P + Q)[\varphi] = P[\varphi] + Q[\varphi]$				
(Ch)	$\mathcal{H}(P) \diamond \mathcal{H}(Q) = \mathcal{H}(P \triangleleft \mathcal{H}(Q) + Q \triangleleft \mathcal{H}(P))$				
$(LC1)$	$\underline{0} \triangleleft P = \underline{0}$				
$(LC2)$	$(\alpha.P) \triangleleft Q = \alpha.P$				
$(LC3)$	$(\lambda^o.P) \triangleleft \mathcal{H}(\mu^o.Q + R) = \lambda^o.(P \diamond \mathcal{H}(\mu^o.Q + R))$				
$(LC4)$	$(\lambda^o.P) \triangleleft \mathcal{H}(vis(Q)) = \underline{0}$				
$(LC5)$	$(P + Q) \triangleleft R = P \triangleleft R + Q \triangleleft R$				
(Par)	$\mathcal{H}(P) \parallel_S \mathcal{H}(Q) = \mathcal{H}(P \lfloor\!\lfloor_S \mathcal{H}(Q) + Q \lfloor\!\lfloor_S \mathcal{H}(P) + P \mid_S Q)$				
$(LM1)$	$\underline{0} \lfloor\!\lfloor_S P = \underline{0}$				
$(LM2)$	$(a.P) \lfloor\!\lfloor_S Q = \underline{0}$	$a \in S$			
$(LM3)$	$(\alpha.P) \lfloor\!\lfloor_S Q = \alpha.(P \parallel_S Q)$	$\alpha \notin S$			
$(LM4)$	$(\lambda^o.P) \lfloor\!\lfloor_S \mathcal{H}(\mu^o.Q + R) = \lambda^o.(P \parallel_S \mathcal{H}(\mu^o.Q + R))$				
$(LM5)$	$(\lambda^o.P) \lfloor\!\lfloor_S \mathcal{H}(vis(Q)) = \underline{0}$				
$(LM6)$	$(P + Q) \lfloor\!\lfloor_S R = P \lfloor\!\lfloor_S R + Q \lfloor\!\lfloor_S R$				
$(SM1)$	$P \mid_S Q = Q \mid_S P$				
$(SM2)$	$\underline{0} \mid_S P = \underline{0}$				
$(SM3)$	$(\theta.P) \mid_S (\theta'.Q) = \underline{0}$	$(\theta \notin S \ \vee \ \theta \neq \theta') \wedge \tau \notin \{\theta, \theta'\}$			
$(SM4)$	$(\tau.P) \mid_S Q = P \mid_S Q$				
$(SM5)$	$(a.P) \mid_S (a.Q) = a.(P \parallel_S Q)$	$a \in S$			
$(SM6)$	$(P + Q) \mid_S R = P \mid_S R + Q \mid_S R$				
$(Rec1)$	$recX.P = recY.(P\{Y/X\})$	provided that Y is not free in $recX.P$			
$(Rec2)$	$recX.P = P\{recX.P/X\}$				
$(Rec3)$	$P = Q\{P/X\}$ implies $P = recX.Q$	if X is serial and strongly guarded in Q			
$(ExpRec)$	$recX.\mathcal{H}(\lambda^o.X + \mu^o.P + Q) = recX.\mathcal{H}(\mu^o.P + Q)$				

Fig. 1. Axiomatization for RIMC

which are equivalent according to $\simeq$. Concerning axiom $(Rec3)$, we define X to be serial in a term if each free occurrence of X in that term is in the scope of $\theta.P$, $P+Q$, $\mathcal{H}(P)$ and $recX.P$ only. Moreover we assume the standard definition of [14] for strong guardedness of serial variables (λ and λ^o prefixes are considered as being guards) and of terms in normal form.

Theorem 3.6 *The axioms of $\mathcal{A}_{EIMC}$ are sound for $\simeq$ over EIMC terms.*

Proof. *For each pair of equated terms it is sufficient to show that there exists an observational bisimulation which includes such a pair.* □

Lemma 3.7 *If a process $P \in RIMC_{cg}$ is finite state, then $\exists P' \in EIMC_c$: $\mathcal{A}_{EIMC} \vdash P = P'$ with P' strongly guarded term in normal form.*

Proof. *Let $P_1 \dots P_n$ be the states of the RIMC derived from the semantics of P, $P_n \equiv P$. Since P is strongly guarded, each state P_i of the semantics of P is finitely branching. It can be easily seen that (thanks to an inductive usage of axioms $(Ter) - (SM6)$, with the help of axioms $(Vis1), (Vis2), (A4)$ and $(Rec1) - (Rec2)$, on the syntactic structure of states) for each $i \in \{1 \dots n\}$, there exist $m_i \in \mathbb{N}$, $\{\theta^i_j\}_{j \le m_i}$, $\{k^i_j\}_{j \le m_i}$ s.t. $\mathcal{A}_{EIMC} \vdash P_i = \mathcal{H}(\sum_{j \le m_i} \theta^i_j.P_{k^i_j})$ where:*

- *for each $j \le m_i$, $\theta^i_j \in Exp^o \cup Act$,*
- *if there exists $j \le m_i$ such that $\theta^i_j = \tau$ then there is no $j \le m_i$ such that $\theta^i_j \in Exp^o$.*

Hence we can characterize the behavior of P by means of a set of equations similarly to [15]. Moreover, similarly to the unique solution of equations theorem of [15], we have that there is a (strongly guarded) term P in normal form such that $\mathcal{A}_{EIMC} \vdash P = P_n \equiv P$. This can be shown as follows. For each i, from 1 to n, we do the following. If i is such that $\exists j \le m_i : k^i_j = i$ we have, by applying $(Rec3)$, that $P_i = recX.\mathcal{H}(\sum_{j \le m_i : k^i_j \ne i} \theta^i_j.P_{k^i_j} + \sum_{j \le m_i : k^i_j = i} \theta^i_j.X)$. Then we replace each subterm P_i occurring in the equations for $P_{i+1} \dots P_n$ with its equivalent term. When, in the equation for $P_n \equiv P$, we have replaced P_{n-1}, we are done. □

Lemma 3.8 *If $P, Q \in EIMC_c$ are strongly guarded terms in normal form such that $P \simeq Q$ then $\mathcal{A}_{EIMC} \vdash P = Q$*

Proof. *The proof is carried out similary to [6] and [9] by using the standard technique based on "guarded equation sets" [16]. In particular, when applying such a technique, we take "standard guarded equation sets" to be guarded equation sets whose structure follows exactly our definition of normal forms for terms. Given that, it is quite simple to verify that each strongly guarded term in normal form satisfies some standard guarded equation set and that, by using axioms $(A1) - (A4)$, (Seq), $(Tau1) - (Tau4)$, $(ExpT)$, $(MProg)$, $(Rec1) - (Rec3)$ and $(ExpRec)$, it is possible to build a common standard guarded equation set which is satisfied by both P and Q, thus obtain-*

ing $\mathcal{A}_{EIMC} \vdash P = Q$ *(see the explanation above of the role of these axioms in proving equality of equivalent normal forms).* □

Theorem 3.9 $\mathcal{A}_{EIMC}$ *is complete for* $\simeq$ *over finite state processes of* $RIMC_{cg}$*.*

Proof. *A direct consequence of Lemmas 3.7 and 3.8* □

4 Conclusion

We would like to observe that our τ-divergence insensitive notion of observational congruence (simplified so that "ticks" replace exponential delays) is a congruence also for the timed algebra of [8] and in this context a much simpler and suitably varied version of the axiom system that we have presented (where the operator "$\mathcal{H}(P)$" is not used and delays synchronize instead of being interleaved) can be used to obtain an axiomatization that is complete over strongly guarded finite-state processes. Moreover we believe that the same "transformation" we performed on the calculus of IMC [9] can be applied also to other interactive timed calculi, as, e.g., the calculus of IWMC (see [3] Chapter 4) and the calculus of IGSMP (see [5] or [3] Chapters 6 and 7) which are basically extensions of the calculus of IMC [9] with probabilistic choices and generally distributed delays.

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