The intensional content of Rice's Theorem

Andrea Asperti

Department of Computer Science, University of Bologna
Mura Anteo Zamboni 7, 40127, Bologna, ITALY
asperti@cs.unibo.it

Content

Content

Content

Content

Content

Content

Content

Outline

Rice's Theorem

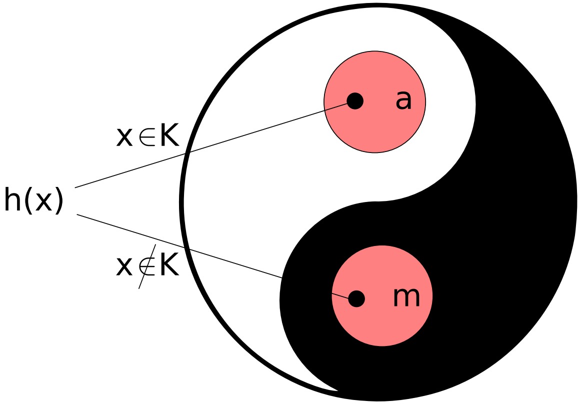
Rice 1953

An estensional property of programs is decidable
if and only if it is trivial.

estensional = closed w.r.t. estensional equivalence

Rice's Yin Yang

$\forall x, \phi_m(x) \uparrow$



the function h

Let $K = dom(\phi_k)$, and define

$$\phi_{h(x)}(y) = \phi_k(x); \phi_a(y)$$

Clearly, if ϕ_m is the everywhere divergent function,

$$\phi_{h(x)} \approx \begin{cases} \phi_a & \text{if } x \in K \\ \phi_m & \text{if } x \notin K \end{cases}$$

Does h preserve any other property, in addition to extensional equivalence? **Yes, complexity!**
Next: investigates the complexity assumptions needed to formalize such result.

the function h

Let $K = dom(\phi_k)$, and define

$$\phi_{h(x)}(y) = \phi_k(x); \phi_a(y)$$

Clearly, if ϕ_m is the everywhere divergent function,

$$\phi_{h(x)} \approx \begin{cases} \phi_a & \text{if } x \in K \\ \phi_m & \text{if } x \notin K \end{cases}$$

Does h preserve any other property, in addition to extensional equivalence? **Yes, complexity!**
Next: investigates the complexity assumptions needed to formalize such result.

the function h

Let $K = dom(\phi_k)$, and define

$$\phi_{h(x)}(y) = \phi_k(x); \phi_a(y)$$

Clearly, if ϕ_m is the everywhere divergent function,

$$\phi_{h(x)} \approx \begin{cases} \phi_a & \text{if } x \in K \\ \phi_m & \text{if } x \notin K \end{cases}$$

Does h preserve any other property, in addition to extensional equivalence? **Yes, complexity!**

Next: investigates the complexity assumptions needed to formalize such result.

the function h

Let $K = dom(\phi_k)$, and define

$$\phi_{h(x)}(y) = \phi_k(x); \phi_a(y)$$

Clearly, if ϕ_m is the everywhere divergent function,

$$\phi_{h(x)} \approx \begin{cases} \phi_a & \text{if } x \in K \\ \phi_m & \text{if } x \notin K \end{cases}$$

Does *h* preserve any other property, in addition to extensional equivalence? **Yes, complexity!**

Next: investigates the complexity assumptions needed to formalize such result.

the function h

Let $K = dom(\phi_k)$, and define

$$\phi_{h(x)}(y) = \phi_k(x); \phi_a(y)$$

Clearly, if ϕ_m is the everywhere divergent function,

$$\phi_{h(x)} \approx \begin{cases} \phi_a & \text{if } x \in K \\ \phi_m & \text{if } x \notin K \end{cases}$$

Does h preserve any other property, in addition to extensional equivalence? **Yes, complexity!**
Next: investigates the complexity assumptions needed to formalize such result.

Outline

Blum's Abstract Complexity

A pair $\langle \phi, \Phi \rangle$ is a *computational complexity measure* if ϕ is a principal effective enumeration of partial recursive functions and Φ satisfies Blum's axioms (Blum 1967):

$$
\begin{array}{ll}
(a) & \phi_i(\vec{n}) \downarrow \leftrightarrow \Phi_i(\vec{n}) \downarrow \\
(b) & \text{the predicate } \Phi_i(\vec{n}) = m \text{ is decidable}
\end{array}
$$

Outline

Big O notation

Big O remind:

1. $f \in O(g)$ if and only if there exist n and c such that for any $m \geq n$, if $g(m) \downarrow$ then $f(m) \leq cg(m)$;
2. $f \in \Theta(g)$ if and only if $f \in O(g)$ and $g \in O(f)$.

Similarity and Complexity Clique

Definition Two programs i and j are *similar* (write $i \approx j$) if and only if

$$\phi_j \cong \phi_i \wedge \Phi_j \in \Theta(\Phi_i)$$

Similarity is an equivalence relation.

Definition Let $\langle \phi, \Phi \rangle$ be an abstract complexity measure. A set P of natural numbers is a *Complexity Clique*, if and only if for all i and j

$$i \in P \wedge j \approx i \rightarrow j \in P$$

Examples of Complexity Cliques

1. $\emptyset$ and ω;
2. for any index i, $[i]_{\approx}$;
3. for any index i, $\{j | \Phi_j \in O(\Phi_i)\}$.
4. all programs with polynomial (exponential, ...) complexity.

Warning: not every Complexity Class is a Complexity Cliques.

Complexity Cliques are closed w.r.t to union, intersection, and complementation.

Complexity Assumptions: s-m-n

Definition A pair $\langle \phi, \Phi \rangle$ has the *s-m-n property* if for all m and n there exists a recursive function s^n_m such that, for any i and all $x_1, \ldots, x_m$

(a) $\phi_{s^n_m(i,x_1,\ldots,x_m)} \cong \lambda y_1, \ldots, y_n.\phi_i(x_1, \ldots, x_m, y_1, \ldots, y_n)$

(b) $\Phi_{s^n_m(i,x_1,\ldots,x_m)} \in \Theta(\lambda y_1, \ldots, y_n.\Phi_i(x_1, \ldots, x_m, y_1, \ldots, y_n))$

Complexity Assumptions: s-m-n

Definition A pair $\langle \phi, \Phi \rangle$ has the *s-m-n property* if for all m and n there exists a recursive function s^n_m such that, for any i and all $x_1, \ldots, x_m$

$$(a)\ \phi_{s^n_m(i, x_1, \ldots, x_m)} \cong \lambda y_1, \ldots, y_n . \phi_i(x_1, \ldots, x_m, y_1, \ldots, y_n)$$

$$(b)\ \Phi_{s^n_m(i, x_1, \ldots, x_m)} \in \Theta(\lambda y_1, \ldots, y_n . \Phi_i(x_1, \ldots, x_m, y_1, \ldots, y_n))$$

Complexity Assumptions: composition

Definition A pair $\langle \phi, \Phi \rangle$ has the composition property if there exists a total computable function h such that

$$(a)\ \phi_{h(i,j)} \cong \phi_i \circ \phi_j$$
$$(b)\ \Phi_{h(i,j)} \in \Theta(max\{\Phi_i \circ \phi_j, \Phi_j\})$$

we only ask that **there exists** a way of composing functions with the above complexity.

Complexity Assumptions: composition

Definition A pair $\langle \phi, \Phi \rangle$ has the composition property if there exists a total computable function h such that

$$(a)\ \phi_{h(i,j)} \cong \phi_i \circ \phi_j$$
$$(b)\ \Phi_{h(i,j)} \in \Theta(max\{\Phi_i \circ \phi_j, \Phi_j\})$$

we only ask that **there exists** a way of composing functions with the above complexity.

Complexity Assumptions: composition

Definition A pair $\langle \phi, \Phi \rangle$ has the composition property if there exists a total computable function h such that

$$(a)\ \phi_{h(i,j)} \cong \phi_i \circ \phi_j$$
$$(b)\ \Phi_{h(i,j)} \in \Theta(max\{\Phi_i \circ \phi_j, \Phi_j\})$$

we only ask that **there exists** a way of composing functions with the above complexity.

Generalized Rice's Theorem

Asperti 2008

Under the s-m-n and the composition assumptions, a Complexity Clique P is recursive if and only if it is trivial, i.e. $P = \emptyset \vee P = \omega$.

Outline

Rice-Shapiro's Theorem

Shapiro 1956

If P is a r.e extensional property of programs then

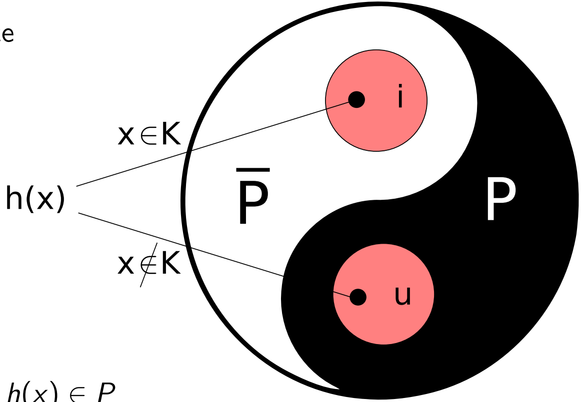
$$i \in P \Leftrightarrow \exists u \in P \ \phi_u \text{ is finite} \wedge \phi_u \leq \phi_i$$

$\Leftarrow$ monotonicity

$\Rightarrow$ compactness

Rice-Shapiro's Yin Yang (monotonicity)

$\phi_u \leq \phi_i$
ϕ_u is finite



$x \notin K \Leftrightarrow h(x) \in P$

the function h

$$\phi_i(x)|\phi_j(x) = \begin{cases} \phi_i(x) & \text{if } \Phi_i(x) \leq \Phi_j(x) \\ \phi_j(x) & \text{otherwise} \end{cases}$$

Let $K = dom(\phi_k)$. Then

$$\phi_{h(x)}(y) = \phi_u(y)|\phi_k(x);\phi_i(y)$$

Clearly,

$$\phi_{h(x)} \approx \begin{cases} \phi_u & \text{if } x \notin K \\ \phi_i & \text{if } x \in K \end{cases}$$

the function h

$$\phi_i(x)|\phi_j(x) = \begin{cases} \phi_i(x) & \text{if } \Phi_i(x) \leq \Phi_j(x) \\ \phi_j(x) & \text{otherwise} \end{cases}$$

Let $K = dom(\phi_k)$. Then

$$\phi_{h(x)}(y) = \phi_u(y)|\phi_k(x); \phi_i(y)$$

Clearly,

$$\phi_{h(x)} \approx \begin{cases} \phi_u & \text{if } x \notin K \\ \phi_i & \text{if } x \in K \end{cases}$$

the function h

$$\phi_i(x)|\phi_j(x) = \begin{cases} \phi_i(x) & \text{if } \Phi_i(x) \leq \Phi_j(x) \\ \phi_j(x) & \text{otherwise} \end{cases}$$

Let $K = dom(\phi_k)$. Then

$$\phi_{h(x)}(y) = \phi_u(y)|\phi_k(x); \phi_i(y)$$

Clearly,

$$\phi_{h(x)} \approx \begin{cases} \phi_u & \text{if } x \notin K \\ \phi_i & \text{if } x \in K \end{cases}$$

parallel computation property

Definition (Landweber and and Robertson, 1972)
A pair $\langle \phi, \Phi \rangle$ has the *parallel computation* property if there exists a total computable function h such that

$$(a) \quad \phi_{h(i,j)}(x) = \begin{cases} \phi_i(x) & \text{if } \Phi_i(x) \leq \Phi_j(x) \\ \phi_j(x) & \text{otherwise} \end{cases}$$

$$(b) \quad \Phi_{h(i,j)} \in \Theta(\lambda x.min\{\Phi_i(x), \Phi_j(x)\})$$

Assuming the parallel computation property we may generalize monotonicity to r.e. Complexity Cliques.

parallel computation property

Definition (Landweber and and Robertson, 1972)
A pair $\langle \phi, \Phi \rangle$ has the *parallel computation* property if there exists a total computable function h such that

$$
\begin{array}{ll}
(a) & \phi_{h(i,j)}(x) = \begin{cases} \phi_i(x) & \text{if } \Phi_i(x) \leq \Phi_j(x) \\ \phi_j(x) & \text{otherwise} \end{cases} \\
(b) & \Phi_{h(i,j)} \in \Theta(\lambda x.min\{\Phi_i(x), \Phi_j(x)\})
\end{array}
$$

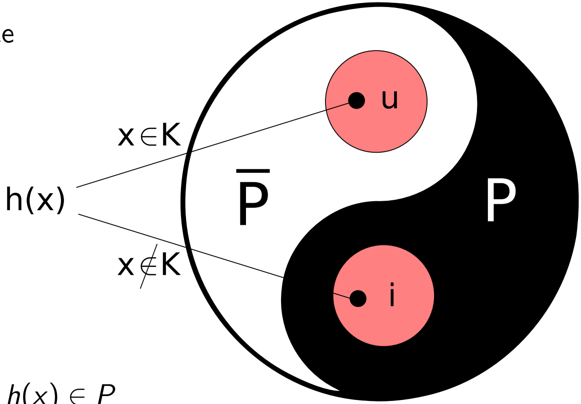
Assuming the parallel computation property we may generalize monotonicity to r.e. Complexity Cliques.

Rice-Shapiro's Yin Yang (compactness)

For *some u*

$\phi_u \leq \phi_i$

ϕ_u is finite



$x \notin K \Leftrightarrow h(x) \in P$

Let $K = dom(\phi_k)$.

$$\begin{array}{rl} \phi_{h(x)}(y) = & match\ FST(\phi_k(x))|SND(\phi_i(y))\ with \\ & |FST\_ \Rightarrow\uparrow \\ & |SND(a) \Rightarrow a \end{array}$$

If $\Phi_i \notin O(1)$, and $\phi_k(x)\downarrow$, $\Phi_i(y) > \Phi_k(x)$ almost everywhere. Hence

$$\phi_{h(x)} \approx \begin{cases} \phi_i & \text{if } x \notin K \\ \text{some finite subfunction of } \phi_i & \text{if } x \in K \end{cases}$$

generalized parallel computation

Definition A pair $\langle \phi, \Phi \rangle$ has the *generalized parallel computation* property if there exists a total computable function p such that for all i, i', j, j'

$$(a) \quad \phi_{p(i,i',j,j')}(x) = \begin{cases} \phi_{i'}(\phi_i(x)) & \text{if } \Phi_i(x) \leq \Phi_j(x) \\ \phi_{j'}(\phi_j(x)) & \text{otherwise} \end{cases}$$

$$(b) \quad \Phi_{p(i,i',j,j')} \in \Theta \left(\lambda x. \begin{cases} \Phi_{h(i',i)}(x) & \text{if } \Phi_i(x) \leq \Phi_j(x) \\ \Phi_{h(j',j)}(x) & \text{otherwise} \end{cases} \right)$$

Assuming the parallel computation property we may prove that for any r.e. Complexity Cliques P, if $i \in P$ and $\Phi_i \notin O(1)$ then there exists $u \in P$ such that ϕ_u is finite and $\phi_u < \phi_i$.

generalized parallel computation

Definition A pair $\langle \phi, \Phi \rangle$ has the *generalized parallel computation* property if there exists a total computable function p such that for all i, i', j, j'

$$
\begin{array}{ll}
(a) & \phi_{p(i,i',j,j')}(x) = \begin{cases} \phi_{i'}(\phi_i(x)) & \text{if } \Phi_i(x) \leq \Phi_j(x) \\ \phi_{j'}(\phi_j(x)) & \text{otherwise} \end{cases} \\
(b) & \Phi_{p(i,i',j,j')} \in \Theta\left(\lambda x. \begin{cases} \Phi_{h(i',i)}(x) & \text{if } \Phi_i(x) \leq \Phi_j(x) \\ \Phi_{h(j',j)}(x) & \text{otherwise} \end{cases}\right)
\end{array}
$$

Assuming the parallel computation property we may prove that for any r.e. Complexity Cliques P, if $i \in P$ and $\Phi_i \notin O(1)$ then there exists $u \in P$ such that ϕ_u is finite and $\phi_u < \phi_i$.

Outline

Corollaries

Corollary Let P be a r.e. Complexity Clique. If $i \in P$ and $\Phi_i \notin O(1)$ then for every j such that $\phi_j \cong \phi_i$ we have $j \in P$.

Proof By compactness, there exists a finite sub-function $\phi_r \leq \phi_i$ such that $r \in P$, and by monotonicity, any j such that $\phi_r \leq \phi_j$, *independently from its complexity* Φ_j, must belong to P.

Corollary No Complexity Clique of total functions and containing (indices of) programs with non constant complexity can be r.e.

Proof By compactness.

Corollaries

Corollary Let P be a r.e. Complexity Clique. If $i \in P$ and $\Phi_i \notin O(1)$ then for every j such that $\phi_j \cong \phi_i$ we have $j \in P$.

Proof By compactness, there exists a finite sub-function $\phi_r \leq \phi_i$ such that $r \in P$, and by monotonicity, any j such that $\phi_r \leq \phi_j$, *independently from its complexity* Φ_j, must belong to P.

Corollary No Complexity Clique of total functions and containing (indices of) programs with non constant complexity can be r.e.

Proof By compactness.

Corollaries

Corollary Let P be a r.e. Complexity Clique. If $i \in P$ and $\Phi_i \notin O(1)$ then for every j such that $\phi_j \cong \phi_i$ we have $j \in P$.

Proof By compactness, there exists a finite sub-function $\phi_r \leq \phi_i$ such that $r \in P$, and by monotonicity, any j such that $\phi_r \leq \phi_j$, *independently from its complexity* Φ_j, must belong to P.

Corollary No Complexity Clique of total functions and containing (indices of) programs with non constant complexity can be r.e.

Proof By compactness.

Corollaries

Corollary Let P be a r.e. Complexity Clique. If $i \in P$ and $\Phi_i \notin O(1)$ then for every j such that $\phi_j \cong \phi_i$ we have $j \in P$.

Proof By compactness, there exists a finite sub-function $\phi_r \leq \phi_i$ such that $r \in P$, and by monotonicity, any j such that $\phi_r \leq \phi_j$, *independently from its complexity* Φ_j, must belong to P.

Corollary No Complexity Clique of total functions and containing (indices of) programs with non constant complexity can be r.e.

Proof By compactness.

Outline

Kleene's Fixed Point Theorem

Kleene 1952

For any total recursive function f, there exists a such that

$$\phi_a \cong \phi_{f(a)}$$

Can we always choose a such that $\Phi_a \in \Theta(\Phi_{f(a)})$?

Kleene's Fixed Point Theorem

Kleene 1952

For any total recursive function f, there exists a such that

$$\phi_a \cong \phi_{f(a)}$$

Can we always choose a such that $\Phi_a \in \Theta(\Phi_{f(a)})$?

Complexity Theoretic version of Kleene's Theorem

Theorem Let $\langle \phi, \Phi \rangle$ be an abstract complexity measure with the s-m-n property, and let u be an index for the universal function. Then for any total recursive function ϕ_i there exists an index m such that, for any x,

$$\begin{array}{ll} (1) & \phi_m \cong \phi_{\phi_i(m)} \\ (2) & \Phi_m \in \Theta(\lambda y.\Phi_u(\phi_i(m), y) \end{array}$$

But what about the complexity of the interpreter u?

Complexity Theoretic version of Kleene's Theorem

Theorem Let $\langle \phi, \Phi \rangle$ be an abstract complexity measure with the s-m-n property, and let u be an index for the universal function. Then for any total recursive function ϕ_i there exists an index m such that, for any x,

$$\begin{array}{ll} (1) & \phi_m \cong \phi_{\phi_i(m)} \\ (2) & \Phi_m \in \Theta(\lambda y.\Phi_u(\phi_i(m), y)) \end{array}$$

But what about the complexity of the interpreter u?

Fair Interpreters

Definition We say that a universal function ϕ_u is *fair* if for any x

$$\lambda y.\Phi_u(x,y) \in \Theta(\Phi_x)$$

Corollary Let $\langle \phi, \Phi \rangle$ be an abstract complexity measure with the s-m-n property. If it admits a fair universal function u then for any total recursive function ϕ_i there exists an index m such that, for any x,

$$\begin{array}{ll} (1) & \phi_m \cong \phi_{\phi_i(m)} \\ (2) & \Phi_m \in \Theta(\Phi_{\phi_i(m)}) \end{array}$$

Outline

- Complexity Cliques generalize estensional sets
- Complexity Cliques in Δ_1^0 are trivial
- Complexity Cliques in Σ_1^0 and Π_1^0 have trivial complexities.

- Complexity Cliques vs. Complexity Classes
- Complexity-theoretic revisitation of Recursion Theory
- Complexity-theoretic aspects of the metatheory of programming languages
- Old Quest for a Machine Independent Theory of Complexity