Optimality and intrinsic computational complexity

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Content

Optimal sharing [Levy 78]

The problem

- optimize reduction of lambda terms by maximazing sharing
- provide an intrinsic measure of the computational cost of lambda terms

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We need to reduce under lambdas:

$$n = \lambda xy.(x\ (x \ldots (x\ y)))$$
$$2 = \lambda xy.(x\ (x\ y))$$
$$I = \lambda x.x$$

$$\begin{array}{lcl} n\ 2\ I\ I & \rightarrow & (2 \ldots (2\ (2\ I)) \ldots)I \\ & \rightarrow & (2 \ldots (2\ (\lambda y.I(I\ y))) \ldots)I \\ & \rightarrow & (2 \ldots (\lambda y.(\lambda y.I(I\ y))(\lambda y.I(I\ y))\ y) \ldots)I \end{array}$$

In Haskell or Ocaml the reduction of $n\ 2\ I\ I$ is exponential in n (while innermost reduction is obviously linear)

time to go beyond weak reduction?

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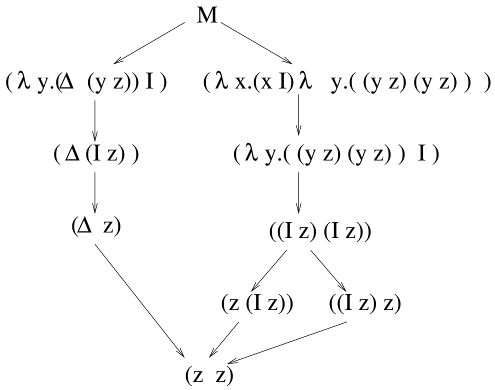
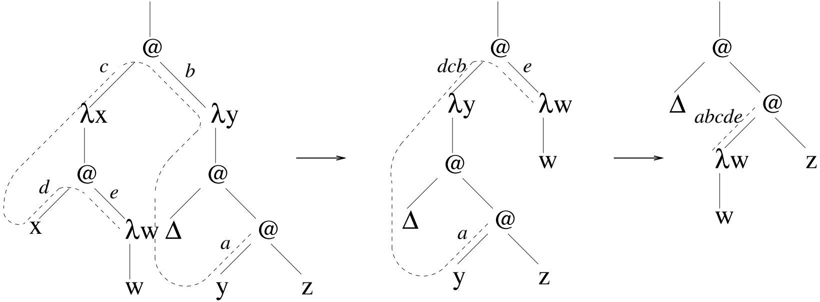
Innermost reduction of needed redexes is not optimal

$\Delta = \lambda x.(x\ x) \qquad M = \lambda x.(x\ I)\ \lambda y.(\Delta(y\ z))$



If M contains multiple occurrences of x we have a tension between the outermost and the innermost strategy.

Virtual redexes and paths

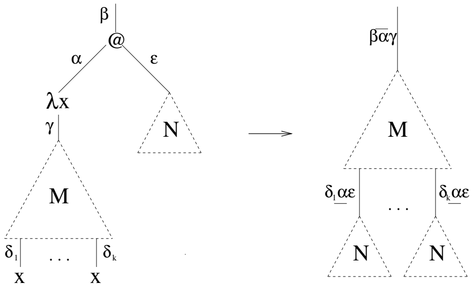


Labeling [Levy 76]

Every subterm gets a label

$$((\lambda x.M)^{\alpha}\ N)^{\beta} \rightarrow \beta \cdot \overline{\alpha} \cdot M[\underline{\alpha} \cdot N/x]$$

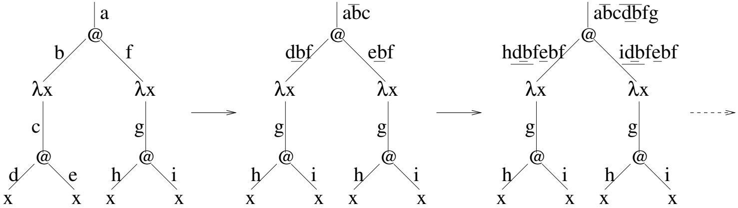
where $\alpha \cdot M^{\beta} = M^{\alpha\beta}$



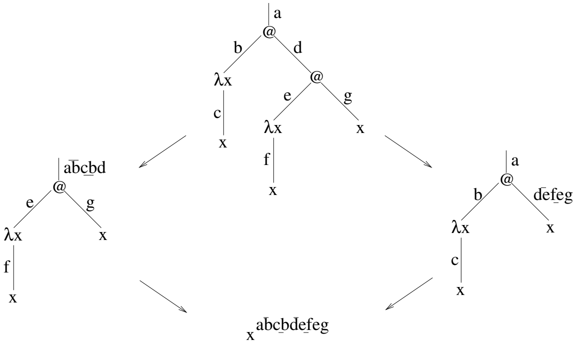
The label keeps a trace of the creation history of each edge (in fact, a path in the original term)

Two redexes belong to the same family if they have the same label.

Example 1

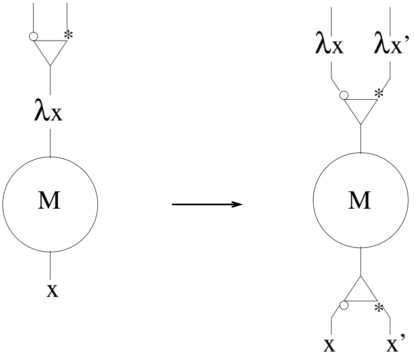


Example 2



Sharing and Unsharing [Lamping 90]

Optimal sharing is feasible!



At what price?

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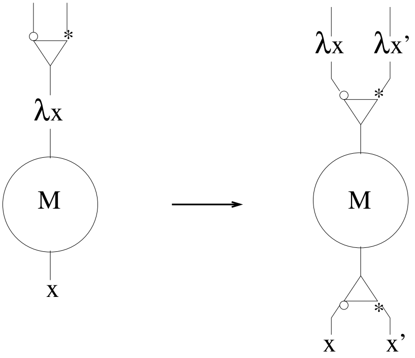


At what price?

The cost of family reduction

Asperti, Mairson [AM98]. **Parallel beta reduction is not elementary recursive**. Inf. & Comp. 2001 (short version at POPL'98)
the cost of reducing a redex family cannot be bounded by any elementary function.

The result is composed of two different parts:

- a technique (eta-expansion technique) aimed to prove that, with a bit of preprocessing (few eta and beta expansion steps) each simply typed lambda terms can be reduced in a linear number of family reduction steps.
- a clever exploitation of an old result by Meyer stating that the reduction of simply typed lambda terms cannot be bounded by any elementary function.

Ten years of oblivion

- Negative Perception of the result in [AM98], **but why**?
  Like to say that Meyer's result is a negative result for the simply typed lambda calculus . . .

In particular [AM98]

- does not say that pursuing optimal sharing is a bad idea
- says **nothing at all** (and hence nothing bad) about Lamping's algorithm.

It says indeed that the mere **number** of redex families is not a good measure of the "intrinsic complexity" of lambda terms. But this is **intriguing**: it means that the problem is still open.

We must **understand what's wrong** with the notion of family, **improve it** or **propose an alternative notion of cost**.

The cost of redexes

creation the redex is created by firing other redexes: the cost should be attributed to them

firing we create a large number of new connections (recall we are firing a whole family in a single step). The connections are only relevant as fragments of new redexes: they will be taken into account when these redexes will be fired.

So, what's so drastically wrong in considering the firing of a family of redexes as a finitistic operation?

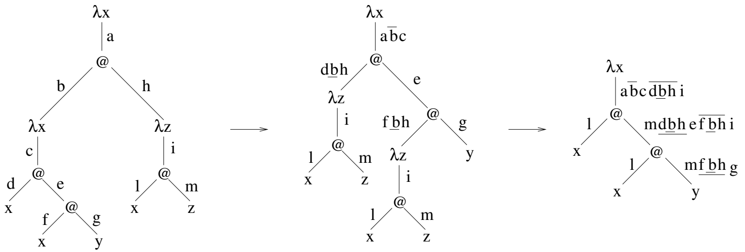
Why can we reduce a simply typed lambda terms in such a small number of family steps? Let's have a closer at the eta expansion technique.

The eta-expansion technique

Consider the term

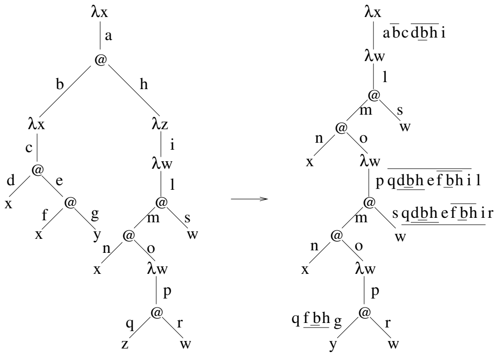
$$\lambda x.x\ (x\ y)$$

When x is instantiated we create two different redexes. But x is a function, and can be eta-expanded into $\lambda z.x\ z$: if instead of sharing x we *share* its eta-expansion (with a let-in or a beta expansion), we shall be able to share the redex created by the instantiation of x.



The eta-expansion technique (2)

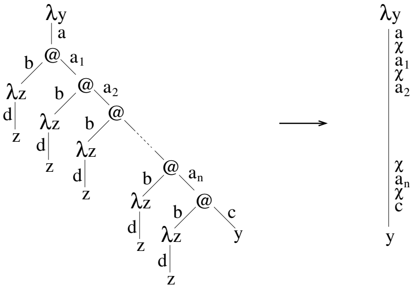
We can adapt the technique to the type of x.
For instance, if $x : (o \to o) \to (o \to o)$ we obtain



We not only share the redex at n, but also all redexes created by its firing.

Firing a family is not a finitistic operation

Consider a church integer $n = \lambda x.\lambda y.x(\ldots(x\ y))$ where all x have a same label b (as can be obtained by eta-expansion). Suppose now to instantiate x with an identity an to fire the unique redex family.



We fired a single redex, but the cost of connecting all edges a_i (all different!) is clearly linear in n.

So little to do, so much time

Although it improves optimal sharing, the eta-expansion technique is, clearly, **not a real optimization**.

It is a **trick** explicitly conceived to scoff at the notion of family:

- redexes have simple labels: the actual cost is not part of their creation history
- all work is "postponed" in new connections that are never going to be part of a full redex
- a huge number of connections are required to generate a single link

The optimal cost

Labeling is OK.

We should just take it more seriously. Instead of counting the number of different labels for redexes, we should count

the whole number of different labels

generated along a standard reduction sequence.
(already suggested by Lawall and Mairson in '96).

Let us call this quantity the optimal cost $o(t)$ of a λ-term t.

A lot of interesting problems

- compute $o(t)$ for a large number of terms
- try to characterize the class of terms for which traditional implementation techniques are indeed optimal
- try to understand what can be the performance loss of traditional (weak) techniques (worse than exponential?)
- try to understand what can be the performance loss w.r.t. (say) the best reduction strategy using explicit substitutions.
- compare $o(t)$ with the number of duplication steps in Lamping's algorithm
- ...

A child of five would understand this.
Send someone to fetch a child of five.
– Groucho Marx

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