Split and ST bisimulation semantics *

Roberto Gorrieri† Cosimo Laneve‡

Abstract

In this paper the notion of action atomicity is relaxed by permitting actions to be observed in the middle of their evolution. Non atomic semantic equivalences, based on the notion of bisimulation, are studied over stable event structures. $\mathtt{Split}_n$ bisimulation equivalence (denoted $\overset{n}{\sim}$) considers each event as composed of n phases. `ST` bisimulation equivalence (denoted $\overset{ST}{\sim}$) is a slight refinement of $\overset{2}{\sim}$ where each ending phase is unambiguously associated to a beginning phase. We prove that, by increasing n, we get finer and finer equivalences (i.e. $\overset{n+1}{\sim}\subseteq\overset{n}{\sim}$) and, moreover, that $\overset{n+1}{\sim}$ coincides with $\overset{ST}{\sim}$ over those event structures whose autoconcurrency is at most n. The main consequence of these results is that, for image finite event structures, $\overset{ST}{\sim}$ is the intersection of all the $\overset{n}{\sim}$.

1 Introduction

Most of the behavioural equivalences for concurrent systems are usually based on the assumption that the execution of an action is an *atomic* activity which cannot be observed or tested in the middle of its evolution (see, e.g., [12, 2, 1]). This abstraction step from the actual behaviour of systems has proved its worth in many applications. However, it produces the unrealistic consequence that actions do not take time.

Recently, a considerable amount of research in concurrency has been devoted to the description of the timing aspects of the execution of actions, possibly generalizing the machinery developed for the atomic case. For instance, expressing time duration of actions turns out to be crucial for *action refinement* (see, e.g., [6] and the references therein): an action, having duration in time, can be considered as an abstraction of a whole system, thus introducing the possibility of relating descriptions of the same system at different levels of detail. In this perspective, *non atomic* behavioural equivalences are gaining more and more prominence as possible concurrent semantics.

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†Dip. di Matematica – Università di Bologna, P.zza di P. San Donato 5, I-40127 Bologna, Italy

‡Dip. di Informatica – Università di Pisa, Corso Italia 40, I-56125, Pisa, Italy

Hennessy [8] was probably the first who dropped the atomicity assumption from the standard bisimulation [12], by permitting actions to be observed in the middle of their evolution. In particular, he suggested atomic actions composed of two phases, their *beginnings* and their *endings*. This proposal can be generalized to an arbitrary number of phases, yielding equivalences named $\mathtt{split}_n$.

We investigate whether splitting actions in more and more phases (i.e., refining the *granularity* of actions) produces a chain of finer and finer equivalences and whether there exists a limit of this chain.

We start our investigation by considering *stable event structures*, introducing – for each of them – the *atomic* (or $\mathtt{split}_1$) transition system as the labelled version of its domain of configurations [16]. Over such a model, we define $\mathtt{split}_n$ transition systems in such a way that $\mathtt{split}_n$ equivalences can be characterized as ordinary bisimulations over these systems. The states of a $\mathtt{split}_n$ transition system are particular configurations, called $\mathtt{split}_n$ configurations, which record the set of completely executed events and the phases of the currently active actions, whilst the transitions are labelled by action phases. Subsequently, we prove that, by increasing the number of phases, finer and finer equivalences are obtained and that the limit of this chain is a non atomic equivalence already proposed in the literature under the name of ST *bisimulation equivalence*. ST equivalence was originally defined by van Glabbeek and Vaandrager [4] on Petri Nets, by mimicking split equivalence. A state in the bisimulation relation is modelled by the set of those places (*Stellen*) that contain a token and of the currently firing transitions (*Transitionen*), hence the name ST. According to [3], the states involved in the definition of ST and $\mathtt{split}_2$ bisimulations are the same, namely $\mathtt{split}_2$ configurations. However, the two equivalences are different because ST bisimulation also exploits a bijection f between the events of the two configurations. This bijection must be preserved in the future steps of the bisimulation. Thus, whenever one of the two systems executes the ending of an event e, the other system must complete the event $f(e)$. This implements a mechanism for connecting the ending phases to those beginning phases occurred at the same time. From an observational viewpoint, ST bisimulation assumes the existence of a smart observer which builds up correspondences between events. Hence, it deviates from the standard definition of bisimulation, which exploits only the labelling of the transitions. However, the same mechanism implemented through bijection f could be also provided by coding the needed information in the label of the transition.

Following this intuition, in this paper ST bisimulation is given an alternative definition. Instead of changing the notion of bisimulation, we prefer to enrich slightly the information in the ending phases of the $\mathtt{split}_2$ semantics: every ending phase has a backward pointer towards its own beginning phase (an idea independently suggested also by R. van Glabbeek). To be more precise, a number k is associated to each ending phase, telling that it refers to the *unique* k^{th}-started action. On the model of ST transition systems, ST bisimulation is simply ordinary bisimulation where the labels on the arcs express the set of the possible experiments. The advantage of our approach is twofold: on one hand, all the results that bisimulation theory has produced can be profitably exploited also in the present case; on the other, it is much clearer what is the "essence" of the ST idea.

The main result of the paper is that, when considering event structures having *autoconcurrency* at most n (i.e., in any reachable split configuration there are at most n actions with the same name already started but not yet ended), the identifications induced by $\overset{ST}{\sim}$ are exactly the same induced by $\overset{n+1}{\sim}$. A nice consequence is that, over the class of *image finite* event structures (i.e., for any state s and for any action μ, the set of the states reachable from s with a μ-labelled transition is finite), ST bisimulation equivalence is the intersection of all the $\mathtt{split}_n$ equivalences. This result, we call the *limit theorem*, contrasts with the fact that (interleaving) ST trace semantics is not the limit of the $\mathtt{split}_n$ trace semantics, as shown in [10, 5]. Indeed, our result strongly depends on the power of bisimulation equivalence, which forces to relate in a suitable way those states with the same branching structure.

2 Preliminaries

In this section, after some background notations used throughout the paper, we will briefly recall those notions which will be needed in the following. In particular, *stable event structures* [16] are dealt with in Subsection 2.2. This subsection may be safely skipped by an acquainted reader; on the contrary, the next subsection should be read because it introduces the general ideas behind the proof technique we will exploit for proving the relationships among the various semantics.

2.1 Notations

$\mathbb{N}$ is the set of natural numbers; the set of its nonzero elements is denoted by $\mathbb{N}^+$.

$|S|$ represents the *cardinality* of the set S; $\wp_f(S)$ stands for the set of all the finite subsets of S

$S_0 \times S_1$ denotes the (*cartesian*) *product* of the sets S_0 and S_1, i.e. $\{(s_0, s_1) \mid s_0 \in S_0 \wedge s_1 \in S_1\}$ (generalizable to $S_0 \times S_1 \times \ldots \times S_{n-1}$);

π_0, π_1 are respectively the *first* and *second projection* of the set $S_0 \times S_1$, i.e. $\pi_i(S_0 \times S_1) = S_i$ (generalizable to $\pi_i(S_0 \times S_1 \times \ldots \times S_{n-1})$);

$S_0 + S_1$ stands for the *disjoint union* of the sets S_0 and S_1, i.e. $\{(0, s_0) \mid s_0 \in S_0\} \cup \{(1, s_1) \mid s_1 \in S_1\}$ (generalizable to $S_0 + S_1 + \ldots + S_{n-1}$);

$\imath_0, \imath_1$ are respectively the *first* and *second injection* of the sets S_0 and S_1 into $S_0 + S_1$, i.e. $\imath_i(S_i) = \{(i, s_i) \mid s_i \in S_i\}$ (generalizable to $\imath_i(S_0 + S_1 + \ldots + S_{n-1}$);

$\mathtt{min}\, S$, $\mathtt{max}\, S$ give respectively the *minimum* and *maximum* of a subset S of $\mathbb{N}$, with the agreement that $\mathtt{min}\, \varnothing = \mathtt{max}\, \varnothing = 0$ and $\mathtt{max}\, S = \infty$, when S is infinite;

$f\lceil A$ denotes the restriction of function f to the set A (i.e., $f\lceil A(x) = f(x)$ whenever $x \in A$, undefined otherwise).

2.2 Stable Event Structures and Labelled Transition Systems

Let us recall from [16] the standard material about stable event structures.

Definition 2.1 A *stable event structure* $\mathcal{E}$, labelled on $\mathcal{M}$, (les, in the following) is a quadruple $\langle E, Con, \vdash, \ell \rangle$ where

- E is a countable set of events,
- Con is a nonempty subset of $\wp_f(E)$, called the *consistency predicate*, satisfying

$$X \in Con \textbf{ and } Y \subseteq X \quad \Rightarrow \quad Y \in Con \quad (\textit{downward closure});$$

- $\vdash \subseteq \; Con \times E$ is the *enabling relation* which satisfies:

  i. $X \vdash e$ **and** $X \subseteq Y$ **and** $Y \in Con \quad \Rightarrow \quad Y \vdash e \qquad (\textit{monotonicity})$,

  ii. $X \vdash e$ **and** $Y \vdash e$ **and** $X \cup Y \cup \{e\} \in Con \quad \Rightarrow \quad X \cap Y \vdash e \qquad (\textit{stability})$

- $\ell : E \to \mathcal{M}$ is the *labelling function.*

Let $\mathcal{ES}_{\mathcal{M}}$ be the domain of les's labelled over $\mathcal{M}$. ■

Concurrent systems are described by means of event structures through the following paradigm. An event denotes the occurrence of an action in a particular run; its label gives the action name; the enabling relation $X \vdash e$ means that e can happen after the occurrence of the events in X and the consistency relation is exploited for describing which sets of events can actually be executed in a run.

Les's will be ranged over by $\mathcal{E}, \mathcal{F}, \cdots$ and the components of $\mathcal{E}$ will be denoted by $E_{\mathcal{E}}$, $Con_{\mathcal{E}}$, $\vdash_{\mathcal{E}}$, $\ell_{\mathcal{E}}$, respectively. A configuration of an event structure (or, better, a state of the system represented by the les) is the set of events which occurred before reaching it. The formalization of this concept requires the preliminary notion of proving sequence. A *proving sequence* in $\mathcal{E}$ is a finite sequence $e_1, e_2, \cdots e_n$ of distinct events satisfying:

$$\{e_1, \cdots, e_n\} \in Con \qquad \textbf{and} \qquad \forall i \leq n \,.\, \{e_1, e_2, \cdots e_{i-1}\} \vdash e_i$$

We say that such a sequence is a *proof* of e in X if $e_n = e$ and $\{e_1, e_2, \cdots e_n\} \subseteq X$.

Definition 2.2 A subset $C \subseteq E_{\mathcal{E}}$ is a *configuration* of $\mathcal{E}$, if:

- $\forall X \subseteq_{\mathit{fin}} C \,.\, X \in Con_{\mathcal{E}}$,
- every event e in C has a proof in C.

The set $\mathcal{C}_{\mathcal{E}}$ of configurations of the les $\mathcal{E}$ will be ranged over by C, D, $\cdots$ ∎

Configurations induce a partial ordering relation over events. Let C be a configuration of $\mathcal{E}$. For $e, e' \in C$, we define

$$e' \leq_C e \iff \forall D \in \mathcal{C}_{\mathcal{E}},\, e \in D \textbf{ and } D \subseteq C \;\Rightarrow\; e' \in D.$$

Given the partial ordering $(C, \leq_C)$, e is *maximal* in C if $e \in C$ and $\forall e' \in C.\, e \leq_C e' \Rightarrow e = e'$. We denote with $\mathtt{mm}(C)$ the set of maximal events in the configuration C, whilst $\mathtt{down}_C(e)$ will denote the set of all the events less than or equal to e itself in C; formally:

$$\mathtt{down}_C(e) = \{e' \mid e' \leq_C e\}.$$

Generally, two event structures are identified up to isomorphism, thus inducing a very concrete semantics for concurrent systems modelled by les's. However, this semantics is too informative (too intensional) because it distinguishes systems which should be reasonably identified. More abstract descriptions can be obtained by equating event structures according to some behavioural equivalence. A standard mechanism relies upon the notion of *bisimulation* [12], defined over labelled transition systems, which we rephrase over les's. To this aim we need to define labelled transition systems and how a labelled transition system can be associated to a les.

Definition 2.3 A *labelled transition system* (lts, for short) $\mathcal{G}$ is a quadruple $\langle \mathcal{S}, \mathcal{M}, \mathcal{T}, s_0 \rangle$ where $\mathcal{S}$ is a set of *states*, $\mathcal{M}$ is a set of *labels*, $\mathcal{T} = \{\xrightarrow{\mu} \subseteq \mathcal{S} \times \mathcal{S} \mid \mu \in \mathcal{M}\}$ is the set of *transition relations* and $s_0 \in \mathcal{S}$ is the *initial state*. We will write $s \xrightarrow{\mu} s'$ instead of $\langle s, s' \rangle \in \xrightarrow{\mu}$. ∎

Informally, a lts is a rooted directed graph with labelled arcs where nodes and arcs represent states and transitions, respectively. Every transition $s \xrightarrow{\mu} s'$ specifies that the system in the state s can transit to the state s' by performing the action μ. As usual, the notation $s_1 \overset{\mu_1 \cdot \mu_2 \cdots \mu_{k-1}}{\twoheadrightarrow} s_k$ is a shorthand for the sequence of transitions $s_1 \xrightarrow{\mu_1} s_2 \xrightarrow{\mu_2} \ldots \xrightarrow{\mu_{k-1}} s_k$. When labels are not relevant, $s_1 \overset{\mu_1 \cdot \mu_2 \cdots \mu_{k-1}}{\twoheadrightarrow} s_k$ may be represented more compactly as $s_1 \twoheadrightarrow s_k$; in such a case, we will say that s_k is *reachable* from s_1.

Definition 2.4 Let $\mathcal{E} \in \mathcal{ES}_{\mathcal{M}}$. The lts $\mathcal{G}(\mathcal{E}) = \langle \mathcal{C}_{\mathcal{E}}, \mathcal{M}, \mathcal{T}_{\mathcal{E}}, \varnothing \rangle$, where $\mathcal{T}_{\mathcal{E}} = \{\xrightarrow{\mu} \subseteq (\mathcal{C}_{\mathcal{E}} \times \mathcal{C}_{\mathcal{E}}) \mid \mu \in \mathcal{M}\}$ is the set of transition relations defined as follows:

$$C \xrightarrow{\mu} C' \quad \Leftrightarrow \quad \exists e\,.\,(\, e \notin C \textbf{ and } C' = C \cup \{e\} \textbf{ and } \ell_{\mathcal{E}}(e) = \mu\,)$$

is the *atomic* (or $\mathtt{split}_1$) lts related to $\mathcal{E}$. ∎

The lts $\mathcal{G}(\mathcal{E})$ has the relevant property of being acyclic, as any other kind of transition system we will associate to a les in the following sections.

Definition 2.5 Let $\mathcal{G} = \langle \mathcal{S}, \mathcal{M}, \mathcal{T}, s_0 \rangle$ and $\mathcal{G}' = \langle \mathcal{S}', \mathcal{M}, \mathcal{T}', s_0' \rangle$ be two lts's. A binary relation $\Re \subseteq \mathcal{S} \times \mathcal{S}'$ is a *bisimulation* if $\langle s_0, s_0' \rangle \in \Re$ and $\langle s, s' \rangle \in \Re$ implies $\forall \mu \in \mathcal{M}$:

- whenever $s \xrightarrow{\mu} s_1$ then $\exists s'_1$ such that $s' \xrightarrow{\mu} s'_1$ and $\langle s_1, s'_1 \rangle \in \Re$;
- whenever $s' \xrightarrow{\mu} s'_1$ then $\exists s_1$ such that $s \xrightarrow{\mu} s_1$ and $\langle s_1, s'_1 \rangle \in \Re$.

$\mathcal{G}$ and $\mathcal{G}'$ are bisimilar, $\mathcal{G} \sim \mathcal{G}'$, if there exists a bisimulation between them. ■

Two les's $\mathcal{E}$ and $\mathcal{F}$ will be atomic bisimilar (notation $\mathcal{E} \overset{1}{\sim} \mathcal{F}$) if and only if $\mathcal{G}(\mathcal{E}) \sim \mathcal{G}(\mathcal{F})$.

Milner originally [11] provided another equivalence whose relationship with $\sim$ will be described by Theorem 2.7 below.

Definition 2.6 Let $\mathcal{G} = \langle \mathcal{S}, \mathcal{M}, \mathcal{T}, s_0 \rangle$ and $\mathcal{G}' = \langle \mathcal{S}', \mathcal{M}, \mathcal{T}', s'_0 \rangle$ be two transition systems. Let $\sim_0 = \mathcal{S} \times \mathcal{S}'$ and

$s \sim_{k+1} s'$ if and only if $\forall \mu \in \mathcal{M}$:

- whenever $s \xrightarrow{\mu} s_1$, $\exists s'_1$ such that $s' \xrightarrow{\mu} s'_1$ and $s_1 \sim_k s'_1$;
- whenever $s' \xrightarrow{\mu} s'_1$, $\exists s_1$ such that $s \xrightarrow{\mu} s_1$ and $s_1 \sim_k s'_1$.

States s and s' are observationally equivalent if $s \sim_k s'$ for every k. ■

A transition system $\langle \mathcal{S}, \mathcal{M}, \mathcal{T}, s_0 \rangle$ is *image finite* if for each $s \in \mathcal{S}$, for each $\mu \in \mathcal{M}$, the set $\{s' \mid s \xrightarrow{\mu} s'\}$ is finite.

Theorem 2.7 [9] For image finite transition systems, $\bigcap_{k \in \mathbb{N}} \sim_k \; = \; \sim$. ■

The notion of image finiteness can be lifted to event structures $\mathcal{E}$ by requiring that the underlying transition systems $\mathcal{G}(\mathcal{E})$ are image finite. Observe the validity of the previous theorem in the class of image finite event structures.

2.3 The Proof Technique

Our original contributions are mostly concerned with establishing relationships between different semantics, which discriminate actions at different levels of granularity. Let us introduce the proof technique we will use for setting such relationships.

Providing a semantics to les's consists of two steps. First we must define a mapping $h_{\mathcal{A}} : \mathcal{ES}_{\mathcal{M}} \rightarrow LTS_{\mathcal{A}}$, where $LTS_{\mathcal{A}}$ is some class of labelled transition systems. Then the right semantics is yielded by quotienting $LTS_{\mathcal{A}}$ through bisimulation, expressed vi a function $Q_{\mathcal{A}} : LTS_{\mathcal{A}} \rightarrow LTS_{\mathcal{A}}|_{\sim}$. Therefore, determining whether some B-semantics is *finer* than some A-semantics means finding a function k such that (see Figure 1.a)

$$Q_{\mathcal{A}} \circ h_{\mathcal{A}} = k \circ Q_{\mathcal{B}} \circ h_{\mathcal{B}}.$$

In general, proving that k is independent of the chosen representative of the bisimulation equivalence class may require some efforts. A way out is shown by the following easy fact.

$$
\begin{tikzcd}
 & LTS_{\mathcal{A}} \arrow[r, "Q_{\mathcal{A}}"] & LTS_{\mathcal{A}}|_{\sim} \\
\mathcal{ES}_{\mathcal{M}} \arrow[ur, "h_{\mathcal{A}}"] \arrow[dr, "h_{\mathcal{B}}"'] & & \\
 & LTS_{\mathcal{B}} \arrow[r, "Q_{\mathcal{B}}"'] & LTS_{\mathcal{B}}|_{\sim} \arrow[uu, "k"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & LTS_{\mathcal{A}} \\
\mathcal{ES}_{\mathcal{M}} \arrow[ur, "h_{\mathcal{A}}"] \arrow[dr, "h_{\mathcal{B}}"'] \arrow[r, phantom, "\sim"] & {} \\
 & LTS_{\mathcal{B}} \arrow[uu, "k_{\sim}"']
\end{tikzcd}
$$

a $\qquad\qquad$ b

Figure 1: The containment relation between semantics $\mathcal{A}$ and $\mathcal{B}$.

Fact 2.8 A function k from $LTS_{\mathcal{B}}|_{\sim}$ to $LTS_{\mathcal{A}}|_{\sim}$, making commutative the diagram of Figure 1.a, does exist if and only if there exists a function $k_{\sim}$ such that

i. $h_{\mathcal{A}}(\mathcal{E}) \sim k_{\sim}(h_{\mathcal{B}}(\mathcal{E}))$, and

ii. $k_{\sim}$ does preserve bisimulation equivalence, i.e., $T \sim T' \Rightarrow k_{\sim}(T) \sim k_{\sim}(T')$, where T and T' are transition systems in $LTS_{\mathcal{B}}$.

Proof: Let $\imath_{\mathcal{A}}$ be a choice function from $LTS_{\mathcal{A}}|_{\sim}$ to $LTS_{\mathcal{A}}$ and $\imath_{\mathcal{B}}$ be a choice function from $LTS_{\mathcal{B}}|_{\sim}$ to $LTS_{\mathcal{B}}$ (these two functions pick a representative for every equivalence class).
(*if*-direction) Given $k_{\sim}$, let k be defined as $Q_{\mathcal{A}} \circ k_{\sim} \circ \imath_{\mathcal{B}}$. By i and ii, it is immediate that $Q_{\mathcal{A}} \circ h_{\mathcal{A}} = k \circ Q_{\mathcal{B}} \circ h_{\mathcal{B}}$.
(*only if*-direction) Take $k_{\sim} = \imath_{\mathcal{A}} \circ k \circ Q_{\mathcal{B}}$. ■

As a consequence of Fact 2.8, we can summarize – in the three steps below – the proof technique we will peruse in the paper:

1. definition of the function $k_{\sim}$ (defined inductively through inference rules);
2. proof that $k_{\sim}$ is *sound* (i.e., the diagram in Figure 1.b commutes modulo bisimulation equivalence);
3. check that $k_{\sim}$ does preserve bisimulation equivalence.

It is worth to remark two relevant facts which will simplify our proofs: (i) function $k_{\sim}$ needs not to be defined over the whole domain $LTS_{\mathcal{B}}$ because, in general, $h_{\mathcal{B}}$ is not surjective; and (ii) function $k_{\sim}$ is not required to map the lts $h_{\mathcal{B}}(\mathcal{E})$ to an lts in the image of $h_{\mathcal{A}}$; the only constraint is that $k_{\sim}(h_{\mathcal{B}}(\mathcal{E}))$ is bisimilar to $h_{\mathcal{A}}(\mathcal{E})$.

3 Split$_n$ bisimulation semantics

We start the investigation about non atomic semantics by removing the assumption that events are instantaneous and indivisible in favour of a more realistic one in which they have duration in time. This view is obtained by considering events as split in a number of phases. As a consequence, the notion of state of a system (or, better, of configuration) must be generalized to express situations in which only a part of an event has been performed. In this section we will formally define the entire spectrum of $\mathtt{split}_n$ semantics for any natural number $n \geq 2$.

Let $\mathcal{M}_n$ be the set $\{\mu_i \mid \mu \in \mathcal{M} \wedge 1 \leq i \leq n\}$. We adopt the convention of distinguishing the various phases of an action by indexing the action itself with the corresponding natural number.

Definition 3.1 A $\mathtt{split}_n$ *configuration* of a stable event structure $\mathcal{E}$ is a pair (C, P), where $C \in \mathcal{C}_\mathcal{E}$ (the *past*) and $P \subseteq (E \times \{i \mid 1 \leq i \leq n-1\})$ (the *present*) are such that

1. $C \cup \pi_0(P) \in \mathcal{C}_\mathcal{E}$ **and** $C \cap \pi_0(P) = \varnothing$ **and** $\pi_0(P) \subseteq \mathtt{mm}(C \cup \pi_0(P))$
2. $|\pi_0(P)| = |P|$

Let $\mathcal{C}^n_\mathcal{E}$ be the set of $\mathtt{split}_n$ configurations of $\mathcal{E}$. ∎

So, configurations become pairs. The first component C describes those events that have been already completed; the other component P specifies the partially executed events. Of course some constraints are needed for a pair (C, P) to be a legal configuration. In particular, condition 1 states that events in P must be maximal, i.e. every event in P is enabled by a set of completely executed events and, furthermore, that no event in the configuration depends on a partially executed event in P [1]. Moreover, condition 2 states that there is only one index for each event, i.e., P cannot contain two pairs involving the same event. Since we are changing configurations and labels, the transition system related to an event structure must be changed correspondingly.

Definition 3.2 Let $\mathcal{G}_n(\mathcal{E}) = \langle \mathcal{C}^n_\mathcal{E}, \mathcal{M}_n, \mathcal{T}^n_\mathcal{E}, (\varnothing, \varnothing)\rangle$ be the $\mathtt{split}_n$ *transition system* related to the les $\mathcal{E}$, where $\mathcal{T}^n_\mathcal{E} = \{\xrightarrow{\mu_i} \subseteq (\mathcal{C}^n_\mathcal{E} \times \mathcal{C}^n_\mathcal{E}) \mid \mu_i \in \mathcal{M}_n\}$ is the set of transition relations defined as follows. $(C, P) \xrightarrow{\mu_i} (C', P')$ if and only if there exists e, $\ell_\mathcal{E}(e) = \mu$, such that one of the following conditions holds:

- $i = 1$ **and** $C = C'$ **and** $e \notin C \cup \pi_0(P)$ **and** $P' = P \cup \{\langle e, 1\rangle\}$
- $1 < i < n$ **and** $C = C'$ **and** $\langle e, i-1\rangle \in P$ **and** $P' = (P \setminus \{\langle e, i-1\rangle\}) \cup \{\langle e, i\rangle\}$
- $i = n$ **and** $C' = C \cup \{e\}$ **and** $\langle e, n-1\rangle \in P$ **and** $P' = P \setminus \{\langle e, n-1\rangle\}$. ∎

[1] Note that the definition of $\mathtt{split}_2$ configuration in [3] is slightly different since, there, C is the set of all the activated events and P is the *past*, hence $P \subseteq C$.

There are three cases, as far as transitions are concerned: a new event is fired, an event already started is progressed and an event in its last phase is terminated. In the first case, assuming to be in a generic configuration (C, P), the event e does not belong to C or to $\pi_0(P)$. Beginning e means firing its first phase, that, in turn, has the effect of adding $\langle e, 1\rangle$ to the set P. This is stated in the first item of the above definition. The event e is in an intermediate phase $i-1$ when its initial $i-1$ phases have been already fired. That is $\langle e, i-1\rangle \in P$. Firing the next phase of e means changing the index $i-1$ into i. The last item of the above definition takes care of the termination of an event e, by moving it from P to C.

Definition 3.3 Let $\mathcal{E}, \mathcal{F} \in \mathcal{ES}_{\mathcal{M}}$. $\mathcal{E}$ and $\mathcal{F}$ are $\texttt{split}_n$ *bisimilar* ($\mathcal{E} \overset{n}{\sim} \mathcal{F}$) if and only if $\mathcal{G}_n(\mathcal{E}) \sim \mathcal{G}_n(\mathcal{F})$. ■

$\texttt{Split}_n$ transition systems have a nice property that allows to recognize the event progressed by a transition (or, better, to distinguish it from other events).

Proposition 3.4 Let $(C, P) \in \mathcal{C}^n_{\mathcal{E}}$ and $(C, P) \xrightarrow{\mu_i} (C', P')$, with $i < n$. Furthermore, let $R = \{(D, Q) \mid (C, P) \xrightarrow{\mu_{i+1}} (D, Q)\}$ and $S = \{(D', Q') \mid (C', P') \xrightarrow{\mu_{i+1}} (D', Q')\}$. There exists a unique $(D', Q') \in S$ that is not possible to reach from any configuration in R.

Proof: Easy: the state (D', Q') is that in which the event progressed by $(C, P) \xrightarrow{\mu_i} (C', P')$ is furtherly progressed of one phase. ■

Let (D', Q') be the unique configuration satisfying Proposition 3.4. We will say that

$$(C', P') \xrightarrow{\mu_{i+1}} (D', Q') \quad \texttt{is causally dependent on} \quad (C, P) \xrightarrow{\mu_i} (C', P')$$

because they have the same event progressed. The above proposition will be crucial in defining the transformation of a $\texttt{split}_n$ transition system into an $\texttt{ST}$ transition system (see Definition 6.6).

The first theorem on semantic comparison we prove states that the family of $\texttt{split}_n$ bisimulations (for any $n \geq 1$) forms a spectrum of non increasing equivalences. In the next section we will show that the inclusion is actually strict, by reporting counter-examples.

Theorem 3.5 $\overset{n+1}{\sim} \subseteq \overset{n}{\sim}$.

Proof: 1. By following our proof technique, we abstract a $\texttt{split}_n$-transition system $\langle \mathcal{Q}_{\mathcal{E}}, \mathcal{M}_n, \mathcal{R}_{\mathcal{E}}, (\varnothing, \varnothing)\rangle$ from $\mathcal{G}_{n+1}(\mathcal{E})$ through a suitable function $k_{\sim}$. In particular, $\mathcal{Q}_{\mathcal{E}}$ and $\mathcal{R}_{\mathcal{E}}$ are the least fixpoints determined by the rules a, b and c below.

$$a)\ \frac{}{(\varnothing, \varnothing) \in \mathcal{Q}_{\mathcal{E}}}$$

$$b)\ \frac{(C, P) \in \mathcal{Q}_{\mathcal{E}} \qquad (C, P) \xrightarrow{\mu_i} (C', P') \in \mathcal{T}_{n+1}(\mathcal{E}) \qquad i < n}{(C', P') \in \mathcal{Q}_{\mathcal{E}} \qquad (C, P) \xrightarrow{\mu_i} (C', P') \in \mathcal{R}_{\mathcal{E}}}$$

$$c)\ \frac{(C,P)\in\mathcal{Q}_{\mathcal{E}} \qquad (C,P)\xrightarrow{\mu_n}(C',P')\ \in\ \mathcal{T}_{n+1}(\mathcal{E}) \qquad (C',P')\xrightarrow{\mu_{n+1}}(C'',P'')\ \in\ \mathcal{T}_{n+1}(\mathcal{E})}{(C'',P'')\in\mathcal{Q}_{\mathcal{E}} \qquad (C,P)\xrightarrow{\mu_n}(C'',P'')\ \in\ \mathcal{R}_{\mathcal{E}}}$$

2. To show the correctness of $k_\sim$, note that $(C,P)\in\mathcal{Q}_{\mathcal{E}}$ if and only if $n\notin\pi_1(P)$ and $(C,P)\in\mathcal{C}^{n+1}_{\mathcal{E}}$ and $(\varnothing,\varnothing)\longrightarrow\!\!\!\!\!\rightarrow(C,P)$. This implies that the path which reaches (C,P) from the initial state has no interleaving of the last two phases of different actions. To prove that $\mathcal{G}_n(\mathcal{E})$ and $\langle\mathcal{Q}_{\mathcal{E}},\mathcal{M}_n,\mathcal{R}_{\mathcal{E}},(\varnothing,\varnothing)\rangle$ are bisimilar, consider the following relation $\Re_n\subseteq\mathcal{C}^n_{\mathcal{E}}\times\mathcal{C}^{n+1}_{\mathcal{E}}$:

$$\Re_n\ =\ \{\langle(C,P),(C,P)\rangle\ |\ (C,P)\in\mathcal{Q}_{\mathcal{E}}\}$$

which is a bisimulation between the two lts's.
3. The proof that $k_\sim$ preserves bisimilarity is also straightforward: given a bisimulation $\Re_{n+1}$ between $\mathcal{G}_{n+1}(\mathcal{E})$ and $\mathcal{G}_{n+1}(\mathcal{F})$, the needed bisimulation is its restriction to those pairs which are also $\texttt{split}_n$-configurations. ■

To be precise, the proof above should be slightly adapted in case $n=1$. Indeed, for the definition of $k_\sim$ there is no need of rule b, and the action phase μ_1 in the conclusion of rule c is actually action μ; moreover, for the correctness part, the relation $\Re_1$ is $\{\langle(C,(C,\varnothing)\rangle\ |\ (C,\varnothing)\in\mathcal{Q}_{\mathcal{E}}\}$. Finally, for the third step of the proof technique, the needed bisimulation is the restriction of a $\texttt{split}_2$ bisimulation to those pairs of $\texttt{split}_2$ configurations which have empty present.

4 The Counter-examples

The containment between $\texttt{split}_n$ and $\texttt{split}_{n+1}$ bisimulation is actually strict, as shown in [5]. Let us recall here two counterexamples showing that $\overset{2}{\sim}\subsetneq\overset{1}{\sim}$ and $\overset{3}{\sim}\subsetneq\overset{2}{\sim}$. The latter is introduced here only to help the intuition about the definition of $\texttt{ST}$ bisimulation, that we will provide in the next section.

Example 4.1 The basic example discriminating atomic bisimulation equivalence and $\texttt{split}_2$ bisimulation equivalence is concerned with the two event structures whose domains of configurations are depicted in Figure 2.a and 2.b (for the sake of clarity, the actual naming of states is omitted). It is clear that they are (atomic) bisimilar but the related $\texttt{split}_2$ transition systems in Figure 2.c and 2.d, respectively, are not. Indeed, the transition system in c can manifest the sequence $\alpha_1\cdot\alpha_1\cdot\alpha_2$ while the one in d cannot. ■

Example 4.2 The counter-example showing that $\texttt{split}_3$ bisimulation equivalence is strictly finer than $\texttt{split}_2$ bisimulation equivalence relies on the two transition systems in Figure 3 (the labels of the inner transitions are omitted for improving readability: they can be assigned by looking at the label of the parallel outer transitions). Let us call $\mathcal{E}$ and $\mathcal{F}$ the les's yielding the transition systems in Figure 3.a and 3.b, respectively. For

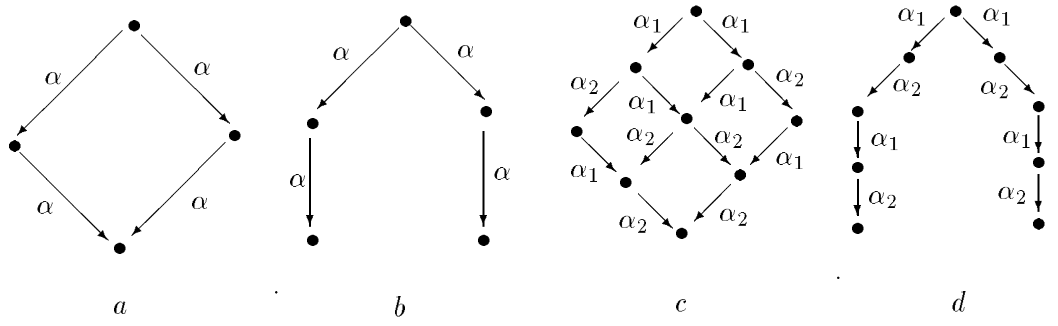


Figure 2: Atomic and $\mathtt{split}_2$ transition systems.

the particular shape of the graph, this example is called in [5] the 'owl' example. It is not difficult to see that the two transition systems are atomic bisimilar. In particular, the state reached after the sequence $\alpha\,\beta$ is very interesting because it shows that bisimulation relates, from now on, 'symmetric' states: if $\mathcal{E}$ performs the γ caused by α, then $\mathcal{F}$ must execute the one caused by β, and vice versa. Note also that whenever the 'wing' is chosen by $\mathcal{E}$, the 'body' is chosen by $\mathcal{F}$, and vice versa.

The $\mathtt{split}_2$ transition systems for $\mathcal{E}$ and $\mathcal{F}$ can be guessed by considering that all the 'diamonds' in the graph are divided into four sub-diamonds, as exemplified in Figure 2.*c*. The two resulting graphs are bisimilar (so $\mathcal{E}$ and $\mathcal{F}$ are $\mathtt{split}_2$ bisimilar) by following an argument similar to the above. The crucial point is reached after the sequence $\alpha_1\,\alpha_2\,\gamma_1\,\beta_1\,\beta_2\,\gamma_1$. In that state (which is the same in both graphs), the γ_2 executed by $\mathcal{E}$ is matched by the 'symmetric' γ_2 in the graph for $\mathcal{F}$. This means that whenever $\mathcal{E}$ completes the γ causally dependent of, let say, α then $\mathcal{F}$ completes the γ caused by β.

The same game cannot be played when splitting actions into three phases; indeed, it is always possible to recognize *observationally* in which of the two directions we are moving, i.e., which of the two γ we are going to complete. This informal consideration is illustrated by the sequence

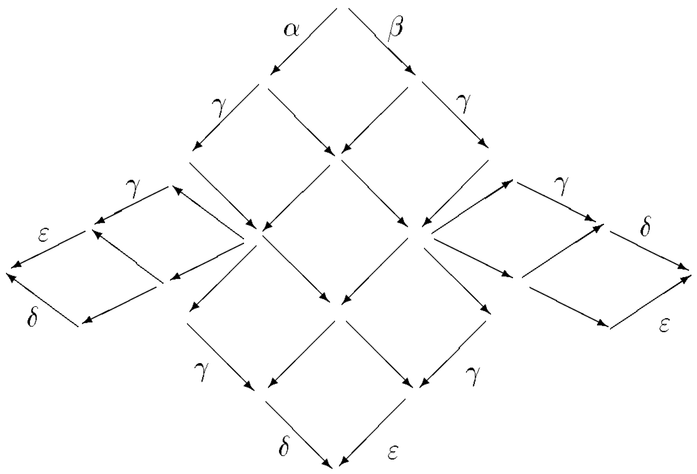
$$\alpha\,\gamma_1\,\beta\,\gamma_2\,\gamma_1\,\gamma_3\,\varepsilon$$

which can be executed by $\mathcal{E}$ but not by $\mathcal{F}$ (unsplit actions stand for the consecutive execution of their three phases). ■

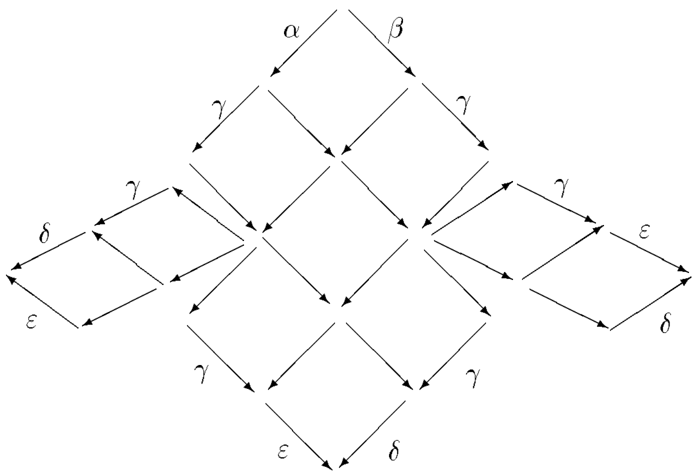
5 ST bisimulation semantics

Our approach to ST bisimulation is a variation of the original one introduced in [3]. Anyway, we will prove that the two definitions are equivalent.

ST bisimulation is, essentially, $\mathtt{split}_2$ bisimulation plus a mechanism to recover the individuality of actions. The drawback of split bisimulation, illustrated in Example 4.2,



a



b

Figure 3: The owl example.

is the inability to properly match up endings of actions with their beginnings. This problem is overcome in ST bisimulation by explicitly observing *causal links* connecting each ending phase of an action to its beginning phase. In this view, the two les's of Example 4.2 are not ST bisimilar. Indeed, consider the following ST trace [3] where a

$$\alpha.\gamma_1.\beta.\gamma_1.\gamma_2.\varepsilon$$

backward pointer stands for causal dependence (as usual, unsplit actions stand for the consecutive execution of their beginning and ending phases). The ST trace above can be performed by $\mathcal{E}$ but not by $\mathcal{F}$. We have coded the causal pointer inside the label, an idea which has analogies with [1]. A beginning phase is denoted by the action itself, while an ending phase is superscripted by a natural number k stating that this phase completes the $k-$th started action. As an example, the ST trace above is coded as follows:

$$\alpha.\alpha^1.\gamma.\beta.\beta^3.\gamma.\gamma^2.\varepsilon.\varepsilon^5$$

Formally, the set of ST *labels* is $\mathcal{M}_{ST} = \mathcal{M} \cup \{\mu^k \mid \mu \in \mathcal{M} \wedge k \in \mathbb{N}\}$, ranged over by η. Let us now define the notions of ST configuration and ST transition system.

Definition 5.1 Let $\mathcal{E}$ be a les. An ST *configuration* of $\mathcal{E}$ is a pair (C, P) where $C \in \mathcal{C}_{\mathcal{E}}$ and $P \subseteq E \times \mathbb{N}^+$ such that

1. $C \cup \pi_0(P) \in \mathcal{C}_{\mathcal{E}}$ **and** $C \cap \pi_0(P) = \varnothing$ **and** $\pi_0(P) \subseteq \mathtt{mm}(C \cup \pi_0(P))$;
2. $|\pi_i(P)| = |P|$ for $i = 0, 1$;
3. $\forall \langle e, z\rangle \in P \,.\, |\bigcup_{e' \in R(z)} \mathtt{down}_Q(e')| \leq z \leq |Q|$
   where $R(z) = \{e' \mid \langle e', z'\rangle \in P \wedge z' \leq z\}$ and $Q = C \cup \pi_0(P)$.

The set of ST configurations of $\mathcal{E}$ will be denoted by $\mathcal{C}_{\mathcal{E}}^{ST}$. ■

The above definition requires some comments. Our aim is to characterize *exactly* all the configurations that can be yielded by firing ST labels. Similarly to the case of $\mathtt{split}_n$ configurations, ST configurations are pairs (C, P) where C denotes the past and P the presently active events. The first constraint makes explicit the analogies with $\mathtt{split}_n$ configurations. In particular, every event under execution must be maximal. The second proviso – when $i = 0$ – ensures that there is only one index associated with each event, whilst – when $i = 1$ – it prevents partially executed events to be labelled by the same natural number; so, the indexes characterize uniquely the events in P. The third condition states a lower and an upper bound to the index z to be associated to a partially executed event e: z must be greater than (or equal to) the cardinality of the set of all the events causing e and the sets of events causing any other partially executed event e' which started before e; z must be smaller than (or equal to) the number of the events in C and P. Notice that this latter condition is essential in order to describe exactly ST configurations. For instance, the pair $(\varnothing, \{\langle e, 2\rangle\})$ is excluded to be an ST configuration by condition 3, because the index 2 means that at least one event different from e has already started.

The following simple fact guarantees the correctness of the definition of ST transition system, given below.

Fact 5.2 Let $(C, P) \in \mathcal{C}_{\mathcal{E}}^{ST}$ and $e \notin C \cup \pi_0(P)$. If $C \cup \pi_0(P) \cup \{e\} \in \mathcal{C}_{\mathcal{E}}$ then

$$(C, P \cup \{\langle e, |C \cup P| + 1\rangle\}) \in \mathcal{C}_{\mathcal{E}}^{ST}$$

Moreover, if $\langle e, k\rangle \in P$ then $(C \cup \{e\}, P \setminus \{\langle e, k\rangle\}) \in \mathcal{C}_{\mathcal{E}}^{ST}$. ■

Definition 5.3 Let $\mathcal{G}_{ST}(\mathcal{E}) = \langle \mathcal{C}_{\mathcal{E}}^{ST}, \mathcal{M}_{ST}, \mathcal{T}_{\mathcal{E}}^{ST}, (\varnothing, \varnothing)\rangle$ be the labelled ST transition system related to the les $\mathcal{E}$, where $\mathcal{T}_{\mathcal{E}}^{ST} = \{\xrightarrow{\eta} \subseteq (\mathcal{C}_{\mathcal{E}}^{ST} \times \mathcal{C}_{\mathcal{E}}^{ST}) \mid \eta \in \mathcal{M}_{ST}\}$ is the set of transition relations. $(C, P) \xrightarrow{\eta} (C', P')$ if and only if there exists e, $\ell_{\mathcal{E}}(e) = \mu$, such that one of the following holds:

- $C' = C$ **and** $e \notin C \cup \pi_0(P)$ **and** $P' = P \cup \{\langle e, k\rangle\}$ **and** $k = |C \cup P| + 1$ **and** $\eta = \mu$
- $C' = C \cup \{e\}$ **and** $\langle e, k\rangle \in P$ **and** $P' = P \setminus \{\langle e, k\rangle\}$ **and** $\eta = \mu^k$. ■

A few comments are necessary. There are two cases: the firing of the initial part of an event and of its end. Firing a new event means giving it an index equal to the number of already started events plus one. We have found rather intuitive to think about a clock value which elapses each time a new event starts. Then the index is the value of the clock when the event begins. The so-built pair is added to the present P. The second condition describes the termination of an event, which has the effect of moving it from the present P to the past C; the label of the transition shows the clock value when the event started. Finally, notice that Fact 5.2 ensures that the pairs (C, P) reached by the transition relation are actually ST configurations.

Example 5.4 In Figure 4, we illustrate the ST transition system of the event structure whose domain of configurations has been described in Figure 2.*a*. The actual naming of states is omitted for the sake of readability. It is interesting to compare it with the $\mathtt{split}_2$ lts in Figure 2.*c*. ■

Definition 5.5 Let $\mathcal{E}, \mathcal{F} \in \mathcal{ES}_{\mathcal{M}}$. $\mathcal{E}$ and $\mathcal{F}$ are ST *bisimilar* ($\mathcal{E} \overset{ST}{\sim} \mathcal{F}$) if and only if $\mathcal{G}_{ST}(\mathcal{E}) \sim \mathcal{G}_{ST}(\mathcal{F})$. ■

Now we compare the above definition of ST bisimulation with the original one proposed in [3], that we will recall below (giving it the name of *link preserving* bisimulation). The next proposition states that both definitions make the same identifications over $\mathcal{ES}_{\mathcal{M}}$. For notational convenience, we will assume that configurations $(C, P) \in \mathcal{C}_{\mathcal{E}}^2$ are such that P is a set of events instead of pairs in $E_{\mathcal{E}} \times \{1\}$, thus forgetting no relevant information.

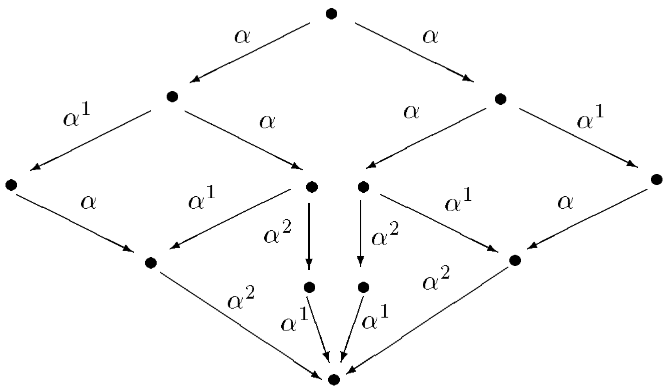


Figure 4: ST semantics for the les of Example 4.1.

Definition 5.6 Let $\mathcal{E}$ and $\mathcal{F}$ be two les's. A relation $\Re_{lp} \subseteq \mathcal{C}^2_{\mathcal{E}} \times \mathcal{C}^2_{\mathcal{F}} \times \wp(E_{\mathcal{E}} \times E_{\mathcal{F}})$ is called a *link preserving* bisimulation if:

1. $\langle(\emptyset, \emptyset), (\emptyset, \emptyset), \emptyset\rangle \in \Re_{lp}$
2. if $\langle(C, P), (D, Q), f\rangle \in \Re_{lp}$, then for every $\mu_i \in \mathcal{M}_2$:
   a.) $f : C \cup P \to D \cup Q$ is a bijection satisfying $\ell_{\mathcal{F}}(f(e)) = \ell_{\mathcal{E}}(e)$ and $f(P) = Q$;
   b.) whenever $(C, P) \xrightarrow{\mu_i} (C', P')$ then $\exists(D', Q'), f'$ such that $(D, Q) \xrightarrow{\mu_i} (D', Q')$, $\langle(C', P'), (D', Q'), f'\rangle \in \Re_{lp}$ and $f'\lceil(C \cup P) = f$;
   c.) whenever $(D, Q) \xrightarrow{\mu_i} (D', Q')$ then $\exists(C', P'), f'$ such that $(C, P) \xrightarrow{\mu_i} (C', P')$, $\langle(C', P'), (D', Q'), f'\rangle \in \Re_{lp}$ and $f'\lceil(C \cup P) = f$.

$\mathcal{E}$ and $\mathcal{F}$ are link equivalent ($\mathcal{E} \overset{lp}{\sim} \mathcal{F}$) if there exists a link preserving bisimulation between them. ■

Proposition 5.7 Given two les's $\mathcal{E}$ and $\mathcal{F}$, $\mathcal{E} \overset{lp}{\sim} \mathcal{F}$ if and only if $\mathcal{E} \overset{ST}{\sim} \mathcal{F}$.

Proof: Given a bisimulation, we will define the other one, leaving to the reader to prove that the relations we provide are indeed bisimulations.

($\mathcal{E} \overset{lp}{\sim} \mathcal{F} \Rightarrow \mathcal{E} \overset{ST}{\sim} \mathcal{F}$). Let $\Re_{lp}$ be a link preserving bisimulation between $\mathcal{E}$ and $\mathcal{F}$. Then the relation $\mathcal{H} \in \mathcal{C}^{ST}_{\mathcal{E}} \times \mathcal{C}^{ST}_{\mathcal{F}}$

$$((C, P), (D, Q)) \in \mathcal{H} \;\; \textit{if} \;\; \text{there exists } f \text{ such that} \quad \begin{array}{l} 1.\ \langle(C, \pi_0(P)), (D, \pi_0(Q)), f\rangle \in \Re_{lp} \\ 2.\ Q = \{\langle f(e), z\rangle \mid \langle e, z\rangle \in P\} \end{array}$$

is a bisimulation between $\mathcal{G}_{ST}(\mathcal{E})$ and $\mathcal{G}_{ST}(\mathcal{F})$.

($\mathcal{E} \overset{ST}{\sim} \mathcal{F} \Rightarrow \mathcal{E} \overset{lp}{\sim} \mathcal{F}$). Let $\mathcal{H}$ be a bisimulation between $\mathcal{G}_{ST}(\mathcal{E})$ and $\mathcal{G}_{ST}(\mathcal{F})$. Take the restriction $\mathcal{H}'$ of $\mathcal{H}$ to the finite configurations only. Also $\mathcal{H}'$ is a bisimulation between the two ST transition systems. Then the following relation $\Re_{lp} \in \mathcal{C}^2_{\mathcal{E}} \times \mathcal{C}^2_{\mathcal{F}} \times \wp(E_{\mathcal{E}} \times E_{\mathcal{F}})$:

a. $\langle (\varnothing, \varnothing), (\varnothing, \varnothing), \varnothing \rangle \in \Re_{lp}$;

b. $\langle (C, P), (D, Q), f \rangle \in \Re_{lp}$ whenever the following conditions are satisfied:

1. there exists $((C, P^+), (D, Q^+)) \in \mathcal{H}'$ such that $\pi_0(P^+) = P$ and $\pi_0(Q^+) = Q$;
2. if $f(e) = e'$ then $\exists z. \langle e, z \rangle \in P^+ \Leftrightarrow \langle e', z \rangle \in Q^+$;
3. for every C', P' with $C' \cup P' = C$ and $(C', P') \in \mathcal{C}^2_{\mathcal{E}}$, $\langle (C', P'), (f(C'), f(Q')), f\lceil C \rangle \in \Re_{lp}$.

is a link preserving bisimulation between $\mathcal{E}$ and $\mathcal{F}$. ■

6 On comparing split and ST semantics

We can start our programme of fixing the correspondence between `ST` bisimulation equivalence and the spectrum of $\mathtt{split}_n$ bisimulation equivalences. The proofs mainly will follow the strategy introduced in Section 2.3.

6.1 The containment $\overset{ST}{\sim} \subseteq \overset{n}{\sim}$

This is the easy direction to establish. The transformation from an `ST` transition system to a $\mathtt{split}_n$ one is provided by the function $\bigtriangledown^{ST}_n$ below. We will comment it afterwards.

Definition 6.1 Let $\mathcal{E}$ be a les. The function $\bigtriangledown^{ST}_n$ transforms the `ST` transition system $\mathcal{G}_{ST}(\mathcal{E})$ into the $\mathtt{split}_n$ transition system $\langle \mathcal{Q}_{\mathcal{E}}, \mathcal{M}_n, \mathcal{R}_{\mathcal{E}}, \langle \varnothing, (\varnothing, \varnothing) \rangle \rangle$ where $\mathcal{Q}_{\mathcal{E}}$ and $\mathcal{R}_{\mathcal{E}}$ are the least fixpoints of the following rules:

1. $$\frac{}{\langle \varnothing, (\varnothing, \varnothing) \rangle \in \mathcal{Q}_{\mathcal{E}}}$$

2. $$\frac{\langle s, (C, P) \rangle \in \mathcal{Q}_{\mathcal{E}} \qquad (C, P) \xrightarrow{\mu} (C', P') \in \mathcal{T}^{ST}_{\mathcal{E}}}{\langle s \cup \{\mu^z_1\}, (C', P') \rangle \in \mathcal{Q}_{\mathcal{E}} \qquad \langle s, (C, P) \rangle \xrightarrow{\mu_1} \langle s \cup \{\mu^z_1\}, (C', P') \rangle \in \mathcal{R}_{\mathcal{E}}}$$
where $z = |C' \cup P'|$;

3. $$\frac{\langle s, (C, P) \rangle \in \mathcal{Q}_{\mathcal{E}} \qquad \mu^k_i \in s \qquad i < n-1}{\langle s', (C, P) \rangle \in \mathcal{Q}_{\mathcal{E}} \qquad \langle s, (C, P) \rangle \xrightarrow{\mu_{i+1}} \langle s', (C, P) \rangle \in \mathcal{R}_{\mathcal{E}}}$$
where $s' = (s \setminus \{\mu^k_i\}) \cup \{\mu^k_{i+1}\}$

4. $$\frac{\langle s, (C, P) \rangle \in \mathcal{Q}_{\mathcal{E}} \qquad \mu^k_{n-1} \in s \qquad (C, P) \xrightarrow{\mu^k} (C', P') \in \mathcal{T}^{ST}_{\mathcal{E}}}{\langle (s - \{\mu^k_{n-1}\}), (C', P') \rangle \in \mathcal{Q}_{\mathcal{E}} \qquad \langle s, (C, P) \rangle \xrightarrow{\mu_n} \langle (s - \{\mu^k_{n-1}\}), (C', P') \rangle \in \mathcal{R}_{\mathcal{E}}}$$ ■

States of $\bigtriangledown_n^{ST}(\mathcal{G}_{ST}(\mathcal{E}))$ are pairs: the first component records the progressing of a given event (unambiguously represented as) μ^z, whilst the second component is the ST configuration we have reached. Then, rule 1 says that the root of the ST transition system is turned into the root of the $\mathtt{split}_n$ one. Rule 2 ensures that, whenever an initial phase is performed by the ST transition system, an initial phase is also possible by the corresponding $\mathtt{split}_n$ state. Moreover, μ_1^z is added to the first component of the state, in order to record that the event μ^z has been started up to the completion of its first phase. Rule 3 accounts for the execution of intermediate phases, by progressing those actions in the first component. The last rule models the closure of an action.

Let us run $\bigtriangledown_n^{ST}$ over a simple example.

Example 6.2 Take the les consisting of two α-labelled, independent events e_1 and e_2 such that any subset of them is consistent (the domain of configurations is depicted in Figure 2.a). Some iterations of $\bigtriangledown_3^{ST}(\mathcal{G}_{ST}(\mathcal{E}))$ are reported in Figure 5. In a we apply the transformation $\bigtriangledown_3^{ST}$ to the initial state $(\varnothing, \varnothing)$, in b to $(\varnothing, \{\langle e_1, 1\rangle\})$ and in c to $(\varnothing, \{\langle e_2, 1\rangle\})$. It is instructive to compare the $\mathtt{split}_3$ transition system obtained by the above procedure with $\mathcal{G}_3(\mathcal{E})$, which can be guessed by looking at Figure 2.a and c. Note that the two states

$$\langle\{\alpha_1^1, \alpha_1^2\}, (\varnothing, \{\langle e_1, 1\rangle, \langle e_2, 2\rangle\})\rangle \;\; \textit{and} \;\; \langle\{\alpha_1^1, \alpha_1^2\}, (\varnothing, \{\langle e_1, 2\rangle, \langle e_2, 1\rangle\})\rangle\ ,$$

are distinguished here, while they are identified by the split semantics (both corresponds to the $\mathtt{split}_3$ state $(\varnothing, \{\langle e_1, 1\rangle, \langle e_2, 1\rangle\})$). Nonetheless, we prove below that $\bigtriangledown_n^{ST}(\mathcal{G}_{ST}(\mathcal{E}))$ and $\mathcal{G}_n(\mathcal{E})$ are bisimilar. ■

The following proposition guarantees the correctness of $\bigtriangledown_n^{ST}$ and that it preserves bisimulation equivalence.

Proposition 6.3 (1) For every les $\mathcal{E}$, $\bigtriangledown_n^{ST}(\mathcal{G}_{ST}(\mathcal{E})) \sim \mathcal{G}_n(\mathcal{E})$;

(2) $\bigtriangledown_n^{ST}$ preserves bisimulation equivalence.

Proof: We will show how to transform a bisimulation into another, leaving to the reader to check that the output relation is actually a bisimulation.

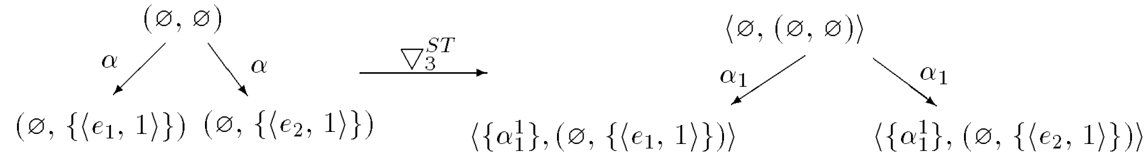
(1) Let $\langle \mathcal{Q}_{\mathcal{E}}, \mathcal{M}_n, \mathcal{R}_{\mathcal{E}}, \langle\varnothing, (\varnothing, \varnothing)\rangle\rangle$ be the $\mathtt{split}_n$ transition system $\bigtriangledown_n^{ST}(\mathcal{G}_{ST}(\mathcal{E}))$ and let $\sigma : \mathcal{Q}_{\mathcal{E}} \to \mathcal{C}_{\mathcal{E}}^n$ be the function:

$$\sigma(\langle s, (C, P)\rangle) = (C, P^\star)$$

where $P^\star = \{\langle e, i\rangle \mid \exists\langle e, k\rangle \in P.\,(\ell_{\mathcal{E}}(e) = \mu \;\; \textbf{and} \;\; \mu_i^k \in s)\}$. By definitions of ST configuration, $\mathtt{split}_n$ configuration and $\mathcal{Q}_{\mathcal{E}}$, function σ is surjective. Then the relation $\Re_n \subseteq \mathcal{Q}_{\mathcal{E}} \times \mathcal{C}_{\mathcal{E}}^n$ defined as $\{(q, \sigma(q)) \mid q \in \mathcal{Q}_{\mathcal{E}}\}$ is a bisimulation between $\bigtriangledown_n^{ST}(\mathcal{G}_{ST}(\mathcal{E}))$ and $\mathcal{G}_n(\mathcal{E})$.

(2) Let $\Re$ be an ST bisimulation between $\mathcal{G}_{ST}(\mathcal{E})$ and $\mathcal{G}_{ST}(\mathcal{F})$. The following relation $\Re_n$

$$\Re_n = \{\langle\langle s, (C, P)\rangle, \langle s', (C', P')\rangle\rangle \mid s = s' \wedge \langle (C, P), (C', P')\rangle \in \Re\}.$$



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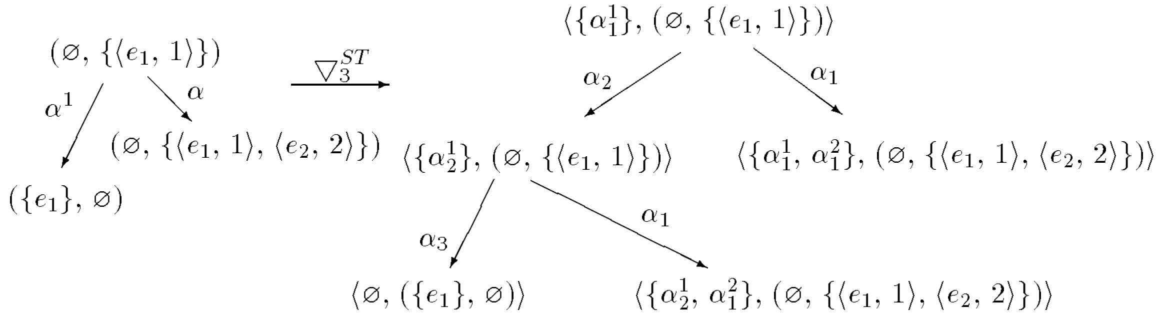

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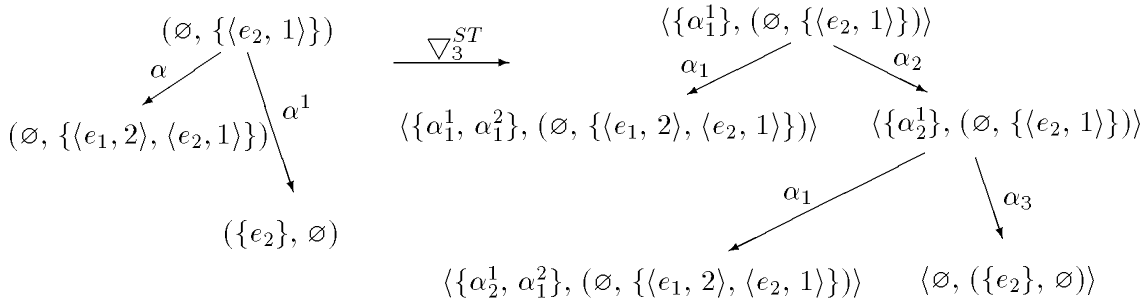

c

Figure 5: Application of the transformation $\bigtriangledown_3^{ST}$.

is a $\mathtt{split}_n$ bisimulation between $\bigtriangledown_n^{ST}(\mathcal{G}_{ST}(\mathcal{E}))$ and $\bigtriangledown_n^{ST}(\mathcal{G}_{ST}(\mathcal{F}))$. ■

According to our proof procedure, the containment $\stackrel{ST}{\sim} \subseteq \stackrel{n}{\sim}$ is a direct consequence of Proposition 6.3.

Theorem 6.4 For every pair $\mathcal{E}$, $\mathcal{F}$ of les's, $\mathcal{E} \stackrel{ST}{\sim} \mathcal{F}$ implies $\mathcal{E} \stackrel{n}{\sim} \mathcal{F}$. ■

6.2 The transformation $\triangle_n^{ST}$

As anticipated in the Introduction, the converse of Theorem 6.4 does not hold for every event structure. It is worth to recall that the difference between ST and split semantics is that the former *always* keeps distinguished the actions that are currently in execution. That is, if in a given state, we have n μ-labelled actions "in progress", the index in the ST label gives identity to each of them. This is not the case for split labels, since in the same situation we could have, for example, two μ-labelled events progressed to the same i-th phase. But, if we knew the maximal amount of actions (with the same label) that, at every moment, can be in execution, this situation of potential "confusion" could be avoided: it is enough to take a large number of splittings and to keep distinguished all these actions by progressing each of them to a different phase. So, we need an upper bound to the number of actions with the same label that at any moment can be active. This upper bound is formalized through the notion of autoconcurrency.

Definition 6.5 Given a les $\mathcal{E}$, let $(C, P) \in \mathcal{C}^2_{\mathcal{E}}$. Configuration (C, P) has *autoconcurrency* $\mathtt{auto}_{(C,P)}$ where

$$\mathtt{auto}_{(C,P)} \;=\; \mathtt{max}\,\{\mathtt{auto}^{\mu}_{(C,P)} \mid \mu \in \mathcal{M}\}$$

and $\mathtt{auto}^{\mu}_{(C,P)} = |\{(C',\,P') \mid (C,\,P) \xrightarrow{\mu_2} (C',\,P')\}|$. The les $\mathcal{E}$ has autoconcurrency m if

$$m \;=\; \mathtt{max}\,\{\,\mathtt{auto}_{(C,P)} \mid (C,\,P) \in \mathcal{C}^2_{\mathcal{E}}\}$$

Let $\mathcal{ES}^m_{\mathcal{M}}$ be the set of les's having autoconcurrency less than or equal to m. ■

The definition of autoconcurrency can be given also over $\mathtt{split}_n$ transition systems by changing the definition of $\mathtt{auto}^{\mu}_{(C,P)}$ as follows:

$$\mathtt{auto}^{\mu}_{(C,P)} = |\{(C',\,P') \mid (C,\,P) \xrightarrow{\mu_n} (C',\,P')\}|.$$

Similarly, we can provide an ST version of autoconcurrency as follows:

$$\mathtt{auto}^{\mu}_{(C,P)} = |\{(C',\,P') \mid \exists k.\,(C,\,P) \xrightarrow{\mu^k} (C',\,P')\}|.$$

It is possible to show that all these definitions associate the same number m to a les $\mathcal{E}$ (the proof consists in picking in one transition system the configuration having maximal autoconcurrency and mapping it into the other transition system).

Observe that autoconcurrency induces a hierarchy on the class of event structures such that $\mathcal{ES}^n_{\mathcal{M}} \subsetneq \mathcal{ES}^{n+1}_{\mathcal{M}}$. We are going to prove that, fixing the autoconcurrency, there exists a way of splitting actions such that it is possible to simulate ST semantics. This transformation is provided by function $\triangle^{ST}_n$, we will comment after the formal definition.

Definition 6.6 Let $\mathcal{E}$ be a les, $\triangle^{ST}_n(\mathcal{G}_n(\mathcal{E})) = \langle \mathcal{S}_\mathcal{E}, \mathcal{M}_{ST}, \mathcal{V}_\mathcal{E}, \langle \varnothing, (\varnothing, \varnothing)\rangle\rangle$ is the ST transition system where $\mathcal{S}_\mathcal{E}$ and $\mathcal{V}_\mathcal{E}$ are the least fixpoints of the following rules:

1. $$\frac{}{\langle \varnothing, (\varnothing, \varnothing)\rangle \in \mathcal{S}_\mathcal{E}}$$

2. $$\frac{\langle s, (C, P)\rangle \in \mathcal{S}_\mathcal{E} \qquad (C, P) \overset{\mu_{j+1}\cdot\mu_j\cdot\ldots\cdot\mu_1}{\longrightarrow\!\!\!\!\rightarrow} (C', P') \qquad j = \mathtt{min}\,\{i \mid \mu^k_{i+1} \notin s\}}{\langle s', (C', P')\rangle \in \mathcal{S}_\mathcal{E} \qquad \langle s, (C, P)\rangle \xrightarrow{\mu} \langle s', (C', P')\rangle \in \mathcal{V}_\mathcal{E}}$$

   where $s' = s\,\theta \cup \{\mu^z_1\}$, $\theta = \{{}^{\mu^k_{x+1}}/_{\mu^k_x} \mid 1 \le x \le j\}$ and $z = |C' \cup P'|$.

3. $$\frac{\langle s, (C_1, P_1)\rangle \in \mathcal{S}_\mathcal{E} \quad \mu^k_i \in s \quad \langle (C_1, P_1), (C_2, P_2), \ldots, (C_{n-i+1}, P_{n-i+1})\rangle \in \Im_{\mu^k_i}((C_1, P_1))}{\langle s', (C_{n-i+1}, P_{n-i+1})\rangle \in \mathcal{S}_\mathcal{E} \qquad \langle s, (C_1, P_1)\rangle \xrightarrow{\mu^k} \langle s', (C_{n-i+1}, P_{n-i+1})\rangle \in \mathcal{V}_\mathcal{E}}$$

   where $s' = s \setminus \{\mu^k_i\}$ and $\Im_{\mu^k_i}((C_1, P_1))$ is the set of sequences $\langle (C_1, P_1), (C_2, P_2), \ldots, (C_{n-i+1}, P_{n-i+1})\rangle$ such that the following hold:

   - for all $1 \le j \le n - i.\, (C_j, P_j) \xrightarrow{\mu_{i+j}} (C_{j+1}, P_{j+1}) \in \mathcal{T}^n_\mathcal{E}$;
   - for all $1 \le j \le n - i - 1$. $(C_{j+1}, P_{j+1}) \xrightarrow{\mu_{i+j+1}} (C_{j+2}, P_{j+2})$ `is causally dependent on` $(C_j, P_j) \xrightarrow{\mu_{i+j}} (C_{j+1}, P_{j+1})$. ■

Notice that the states in $\triangle^{ST}_n(\mathcal{G}_n(\mathcal{E}))$ are pairs, as for $\bigtriangledown^{ST}_n$. Again the first component records the events in execution; on the contrary, the second component is now a $\mathtt{split}_n$ configuration.

As far as the transitions are concerned, two cases are to be analyzed: the beginning and the ending of each event. Rule 2 takes care of the beginnings. Assume to start an action labelled μ. It will have a superscript that encodes the moment when the action began (namely, the number of actions performed till then). This action, since it is just begun, should have phase index 1. Of course, it is possible that another μ-labelled action can be started; hence, the $\mathtt{split}_n$ state may have two autoconcurrent actions progressed to the same phase. States showing this phenomenon, called *confusion states*, must be avoided since it will be not possible to recognize which of the two actions is responsible for the execution of the successive phase. Rule 2 avoids such configurations by increasing the phases of the already started autoconcurrent actions (*opening path*). When $\triangle^{ST}_n$ is applied to $\mathtt{split}_n$ transition systems with autoconcurrency $n-1$, the upper bound on autoconcurrency guarantees the consistency of such increments. Indeed, in this case, none of the started actions may be completed when a new action begins.

The case of the endings is more tricky. In a state where several actions μ have already started we must be able to close each of them. Of course, indiscriminate closures may make individuality lost: we have to perform a *closure path* where the same autoconcurrent event is actually progressed till to completion.

More in detail, since all the autoconcurrent actions have been progressed to a different phase, the first transition can be unambiguously determined. Inductively, let us assume that a prefix of the closure path has been already determined. Then, the next transition in the path must be selected on the basis of what suggested by Proposition 3.4: the selected transition is the one which `is causally dependent on` the previous one. In this way, the whole closure path can be singled out, even if confusion states of the $\texttt{split}_n$ transition system may be passed through while reaching the final state (C_{n-i+1}, P_{n-i+1}) of the path, which, by construction, is confusion free.

We first illustrate the transformation $\triangle_n^{ST}$ through an example; next, we will discuss some properties of those $\texttt{split}_n$ configurations that are exploited for simulating `ST` configurations.

Example 6.7 Let us apply $\triangle_3^{ST}$ to the les $\mathcal{E}$ in Example 6.2. Some iterations of the procedure are reported in Figure 6.*a* and *b*, where the transformation is applied to the initial state $(\varnothing, \varnothing)$ and to $(\varnothing, \{\langle e_1, 1\rangle\})$.

Note that, in *b*, the `ST` transition labelled α corresponds to a $\texttt{split}_3$ two-step computation, where the already started autoconcurrent action is progressed to a greater phase. Similarly, the `ST` transition labelled α^1 corresponds to a $\texttt{split}_3$ two-step computation, where the same event is progressed. Starting from the $\texttt{split}_3$ state $(\varnothing, \{\langle e_1, 2\rangle, \langle e_2, 1\rangle\})$, we can complete the execution of either e_1 or e_2. In case we choose to finish e_2, we have to select a proper path inside the $\texttt{split}_3$ transition system. The situation is depicted in Figure 6.*c* where, by Proposition 3.4, $(\varnothing, \{\langle e_1, 2\rangle, \langle e_2, 2\rangle\}) \xrightarrow{\alpha_3} (\{e_2\}, \{\langle e_1, 2\rangle\})$ `is causally dependent on` $(\varnothing, \{\langle e_1, 2\rangle, \langle e_2, 1\rangle\}) \xrightarrow{\alpha_2} (\varnothing, \{\langle e_1, 2\rangle, \langle e_2, 2\rangle\})$. ∎

Propositions 6.9 and 6.10 below are intended to guarantee the existence of a unique way of beginning a new event or ending an old one. In other words, there exists a unique path $(C, P) \overset{\mu_{j+1}\cdot\mu_j\cdot\ldots\cdot\mu_1}{\longrightarrow\!\!\!\!\!\rightarrow} (C', P')$ in rule 2 and the set $\Im_{\mu_i^k}$ in rule 3 is always a singleton.

Definition 6.8 Let $\mathcal{E}$ be a les in $\mathcal{ES}_{\mathcal{M}}^{n-1}$, $(C, P) \in \mathcal{C}_{\mathcal{E}}^n$ and $P_\mu = \{\langle e, i\rangle \mid \langle e, i\rangle \in P \wedge \ell_{\mathcal{E}}(e) = \mu\}$. Then P is called *μ-confusion free* if

$$\forall \langle e, i\rangle, \langle e', j\rangle \in P_\mu \,.\, e \neq e' \Rightarrow i \neq j.$$

P is called *confusion free* if it is μ-confusion free for every $\mu \in \mathcal{M}$. A configuration (C, P) is confusion free if P is so. ∎

Proposition 6.9 Let $\mathcal{E}$ be a les in $\mathcal{ES}_{\mathcal{M}}^{n-1}$ and $(C, P) \in \mathcal{C}_{\mathcal{E}}^n$ such that P is μ-confusion free. Let also $r = \mathtt{min}\{i \mid i+1 \notin \pi_1(P_\mu)\}$. If $(C, P) \xrightarrow{\mu_1} (C, P') \in \mathcal{T}_{\mathcal{E}}^n$ then there exists a unique path of the form:

$$(C, P) = (C, P_1) \xrightarrow{\mu_{r+1}} (C, P_2) \xrightarrow{\mu_r} \ldots \xrightarrow{\mu_2} (C, P_{r+1}) \xrightarrow{\mu_1} (C, P_{r+2})$$

$$
\begin{tikzcd}[column sep=small]
 & (\varnothing,\, \varnothing) \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_1"] & & & & \langle \varnothing,\, (\varnothing,\, \varnothing)\rangle \arrow[dl, "\alpha"'] \arrow[dr, "\alpha"] & \\
(\varnothing,\, \{\langle e_2,\, 1\rangle\}) & & (\varnothing,\, \{\langle e_1,\, 1\rangle\}) \arrow[r, phantom, "\xrightarrow{\triangle_3^{ST}}"] & & \langle\{\alpha_1^1\}, (\varnothing,\, \{\langle e_1,\, 1\rangle\})\rangle & & \langle\{\alpha_1^1\},\, (\varnothing,\, \{\langle e_2,\, 1\rangle\})\rangle
\end{tikzcd}
$$

a

$$
\begin{tikzcd}[column sep=small]
 & (\varnothing,\, \{\langle e_1,\, 1\rangle\}) \arrow[d, "\alpha_2"] & & & & \\
 & (\varnothing,\, \{\langle e_1,\, 2\rangle\}) \arrow[dl, "\alpha_3"'] \arrow[dr, "\alpha_1"] & \arrow[r, "\triangle_3^{ST}"] & {} & \langle\{\alpha_1^1\},\, (\varnothing,\, \{\langle e_1,\, 1\rangle\})\rangle \arrow[dl, "\alpha^1"'] \arrow[dr, "\alpha"] & \\
(\{e_1\},\, \varnothing) & & (\varnothing,\, \{\langle e_1,\, 2\rangle,\, \langle e_2,\, 1\rangle\}) & \langle \varnothing,\, (\{e_1\},\, \varnothing)\rangle & & \langle\{\alpha_3^1,\, \alpha_2^2\},\, (\varnothing,\, \{\langle e_1,\, 2\rangle,\, \langle e_2,\, 1\rangle\}\}\rangle
\end{tikzcd}
$$

b

$$
\begin{tikzcd}[column sep=small]
 & (\varnothing,\, \{\langle e_1,\, 2\rangle,\, \langle e_2,\, 1\rangle\}) \arrow[dl, "\alpha_3"'] \arrow[dr, "\alpha_2"] & & \\
(\{e_1\},\, \{\langle e_2,\, 1\rangle\}) \arrow[dr, "\alpha_2"] & & (\varnothing,\, \{\langle e_1,\, 2\rangle,\, \langle e_2,\, 2\rangle\}) \arrow[dl, "\alpha_3"'] \arrow[dr, "\alpha_3"] & \\
 & (\{e_1\},\, \{\langle e_2,\, 2\rangle\}) & & (\{e_2\},\, \{\langle e_1,\, 2\rangle\})
\end{tikzcd}
$$

c

Figure 6: Application of the transformation $\triangle_3^{ST}$.

Moreover, $(C,\, P_{r+2})$ is confusion free.

Proof: There are two cases: when $r = 0$ and when $0 < r \leq n-2$.
1) If $r = 0$ then there is no $\langle e,\, 1\rangle \in P_\mu$. Therefore, the required path is the transition

$$(C,\, P) = (C,\, P_1) \xrightarrow{\mu_{r+1}} (C,\, P_2) = (C,\, P')$$

2) If $0 < r \leq n-2$ then, by hypothesis $\neg\exists\langle e,\, r+1\rangle \in P$ and there exists exactly one pair $\langle e_1,\, r\rangle$ in P. Thus the first transition

$$(C,\, P) = (C,\, P_1) \xrightarrow{\mu_{r+1}} (C,\, P_2)$$

is uniquely determined. Now, if $r-1 = 0$ then the proof reduces to the case 1), otherwise let us consider the event $\langle e_2,\, r-1\rangle \in P_1$, which still belongs to P_2. By the hypothesis of μ-confusion free, there exists a unique transition

$$(C,\, P_2) \xrightarrow{\mu_r} (C,\, P_3)$$

where, in P_3, e_2 is the unique event progressed till to phase r. As a consequence, P_3 is μ-confusion free, too. So, the further transitions can be derived by iterating 1 or 2, according to the value of r.

Finally, the reader can easily realize that the opening path in the statement satisfies the following property:

a. $P_{i+1} = (P_i \setminus \{\langle e_i, r-i+1\rangle\}) \cup \{\langle e_i, r-i+2\rangle\},\ 1 \leq i \leq r$ and $\langle e_i, r-i+1\rangle \in P$;

b. $P_{r+2} = P_{r+1} \cup (P' \setminus P)$.

where each one of the involved events takes part to one transition only. Hence, the final state $(C,\, P_{r+2})$ is μ-confusion free. ■

Notice that, in the above proposition, the hypothesis $\mathcal{E} \in \mathcal{ES}^{n-1}_{\mathcal{M}}$ ensures that $r+1 \leq n-1$, i.e., no event is completed in an opening path.

Proposition 6.10 Let $\mathcal{E}$ be a les and $(C,\, P) \in \mathcal{C}^n_{\mathcal{E}}$ such that P is μ-confusion free. Let $\langle e,\, i\rangle \in P_\mu$. Then there exists a unique path $(C,\, P) = (C,\, P_1) \xrightarrow{\mu_{i+1}} (C,\, P_2) \xrightarrow{\mu_{i+2}} \ldots (C,\, P_{n-i}) \xrightarrow{\mu_n} (C',\, P_{n-i+1})$ where each transition causally depends on the previous one; moreover, $(C',\, P_{n-i+1})$ is μ-confusion free.

Proof: We must show the following two items:

a. $P_{j+1} = (P_j \setminus \{\langle e,\, i+j-1\rangle\}) \cup \{\langle e,\, i+j\rangle\},\ 1 \leq j \leq n-i-1$;

b. $C' = C \cup \{e\}$ and $P_{n-i+1} = P_1 \setminus \{\langle e,\, i\rangle\}$.

There are two cases: when i is the maximum index such that $\langle e, i\rangle \in P_\mu$ and when it is not.
1) If $\langle e, i\rangle \in P_\mu$ and e is the most progressed event in P_μ, then, by Definition 3.2, there is a unique transition

$$(C, P) = (C, P_1) \xrightarrow{\mu_{i+1}} (C', P_2).$$

If $i + 1 = n$ then $C' = C \cup \{e\}$ and $P_2 = P_1 \setminus \{\langle e, n-1\rangle\}$; thus condition b. is satisfied and a. vacuously holds. Otherwise, $C' = C$ and $P_2 = (P_1 \setminus \{\langle e, i\rangle\}) \cup \{\langle e, i+1\rangle\}$; thus satisfying condition a. Note that P_2 has, like P_1, the property that e is the most progressed μ-labelled event. Therefore, we can reiterate the procedure.
2) If $\langle e, i\rangle \in P_\mu$ and e is not the most progressed event in P_μ, then, since P is μ-confusion free, we are sure that there is a unique transition

$$(C, P) = (C, P_1) \xrightarrow{\mu_{i+1}} (C, P_2)$$

satisfying condition a. In order to show condition b, let us consider P_2. Two subcases are in order:
2.1) P_2 is still μ-confusion free. Then we have to iterate step 2.
2.2) $\exists\langle e', i+1\rangle \in P_2 . e \neq e'$. In this case we can apply Proposition 3.4, that ensures the existence of a unique transition $(C, P_2) \xrightarrow{\mu_{i+2}} (C, P_3)$ which causally depends on $(C, P_1) \xrightarrow{\mu_{i+1}} (C', P_2)$. That is, in both transitions the same event has been progressed. If $i + 2 = n$, then $C' = C \cup \{e\}$ and $P_3 = P_2 \setminus \{\langle e, i+1\rangle\} = P_1 \setminus \{\langle e, i\rangle\}$, hence condition b holds. Note that P_3 is still confusion free. If $i + 2 < n$, then we can repeat case 2). ■

The reader can easily realize that also for the unique "causal" closure path of the Proposition above, the final state (C', P_{n-i+1}) is μ-confusion free. Hence, the (C, P) component of any state in $\triangle_n^{ST}(\mathcal{G}_n(\mathcal{E}))$ is always confusion-free, because this holds for the initial state, and the property is preserved by the states reached through opening and (causal) closure paths.

Proposition 6.11 For every $\mathcal{E} \in \mathcal{ES}_{\mathcal{M}}^{n-1}$, every state $\langle s, (C, P)\rangle$ of $\triangle_n^{ST}(\mathcal{G}_n(\mathcal{E}))$ is such that (C, P) is confusion free. ■

6.3 The containment $\overset{n}{\sim} \subseteq \overset{ST}{\sim}$

Now let us show our main result: $\mathtt{split}_n$ semantics can simulate the $\mathtt{ST}$ one, provided that autoconcurrency is bounded to $n-1$. Lemma 6.12 below gives half of the above result, namely the correctness of the transformation $\triangle_n^{ST}$.

Lemma 6.12 Let $\mathcal{E}$ be a les such that $\mathcal{E} \in \mathcal{ES}_{\mathcal{M}}^{n-1}$, then $\triangle_n^{ST}(\mathcal{G}_n(\mathcal{E})) \sim \mathcal{G}_{ST}(\mathcal{E})$.

Proof: Let $\triangle_n^{ST}(\mathcal{G}_n(\mathcal{E})) = \langle \mathcal{S}_\mathcal{E}, \mathcal{M}_{ST}, \mathcal{V}_\mathcal{E}, \langle \varnothing, (\varnothing, \varnothing)\rangle\rangle$ and let $\sigma : \mathcal{S}_\mathcal{E} \to \mathcal{C}_\mathcal{E}^{ST}$ be such that

$$\sigma(\langle s, (C, P)\rangle) \;=\; (C, P^\star)$$

where $P^\star = \{\langle e, k\rangle \mid \langle e, i\rangle \in P \ \wedge \ \ell_\mathcal{E}(e) = \mu \ \wedge \ \mu_i^k \in s\}$. Now, consider the following relation $\Re_{ST} \subseteq \mathcal{S}_\mathcal{E} \times \mathcal{C}_\mathcal{E}^{ST}$:

$$\Re_{ST} \ = \ \{(q, \sigma(q)) \mid q \in \mathcal{S}_\mathcal{E}\}$$

We will prove that $\Re_{ST}$ is a bisimulation and, at the same time, that $\sigma(q)$ is an ST configuration.

The initial states of $\triangle_n^{ST}(\mathcal{G}_n(\mathcal{E}))$ and $\mathcal{G}_{ST}(\mathcal{E})$ are related by $\Re_{ST}$, since $\sigma(\langle\varnothing, (\varnothing, \varnothing)\rangle) = (\varnothing, \varnothing)$. Let $\langle\langle s, (C, P)\rangle, \ (C, P^\star)\rangle \in \Re_{ST}$, where $(C, P^\star) = \sigma(\langle s, (C, P)\rangle)$. Two cases are possible, depending on the nature of the executed phase.

(**beginning phase**) Let us suppose that $\langle s, (C, P)\rangle \xrightarrow{\mu} \langle s', (C', P')\rangle$ is due to the opening path $(C, P) \overset{\mu_{j+1}\cdot\mu_j\cdot\ldots\cdot\mu_1}{\twoheadrightarrow} (C', P')$ in $\mathcal{G}_n(\mathcal{E})$, where $j = \mathtt{min}\{i \mid \mu_{i+1}^k \notin s\}$. Since P is confusion free, let $P_\mu = \{\langle e_i, i\rangle \in P \mid 1 \leq i \leq j \wedge \ell_\mathcal{E}(e_i) = \mu\}$. Hence the (confusion free) configuration (C', P') is such that $C' = C$ and

$$P' = (P \setminus P_\mu) \cup \{\langle e_i, i+1\rangle \mid \langle e_i, i\rangle \in P_\mu\} \cup \{\langle e, 1\rangle\}$$

where e is the event which corresponds to the last transition of the path. Moreover $s' = s\theta \cup \{\mu_1^z\}$, where $\theta = \{{}^{\mu_{x+1}^k}/_{\mu_x^k} \mid 1 \leq x \leq j\}$. It is easy to see that $\sigma(\langle s', (C, P')\rangle) = (C, P^\star \cup \{\langle e, z\rangle\})$ $(z = |C \cup P'| + 1)$ and that $(C, P^\star) \xrightarrow{\mu} (C, P^\star \cup \{\langle e, z\rangle\})$ is an ST transition (therefore $(C, P^\star \cup \{\langle e, z\rangle\})$ is an ST configuration).
For the vice versa, let $(C, P^\star) \xrightarrow{\mu} (C, P^+) \in \mathcal{T}_\mathcal{E}^{ST}$ and $P^\star = P^+ \cup \{\langle e, z\rangle\}$, where $z = |C \cup P^\star| + 1$. Now, by hypothesis, $\sigma(\langle s, (C, P)\rangle) = (C, P^\star)$, where

$$P = \{\langle e, i\rangle \mid \langle e, k\rangle \in P^\star \ \wedge \ \ell_\mathcal{E}(e) = \mu \ \wedge \ \mu_i^k \in s\}.$$

Since (C, P) is confusion free and autoconcurrency is bounded to $n-1$, by Proposition 6.9, there exists a unique path:

$$(C, P) \overset{\mu_{j+1}\cdot\mu_j\cdot\ldots\cdot\mu_1}{\twoheadrightarrow} (C', P')$$

in $\mathcal{G}_n(\mathcal{E})$ where $j = \mathtt{min}\{i \mid \mu_{i+1}^k \notin s\}$ and $P' = (P \setminus P_\mu) \cup \{\langle e_i, i+1\rangle \mid 1 \leq i \leq j \wedge \langle e_i, i\rangle \in P_\mu\} \cup \{\langle e, 1\rangle\}$. Therefore, by Definition 6.6:

$$\langle s, (C, P)\rangle \xrightarrow{\mu} \langle s', (C', P')\rangle \in \mathcal{V}_\mathcal{E}$$

where $s' = s\theta \cup \{\mu_1^z\}$. Moreover, note that $\sigma(\langle s', (C', P')\rangle) = (C, P^+)$, hence $\langle\langle s', (C', P')\rangle, (C, P^+)\rangle \in \Re_{ST}$.

(**ending phase**) Let $\sigma(\langle s, (C, P)\rangle) = (C, P^\star)$ and let $\mu_i^k \in s$. Suppose to close the action $\langle e, i\rangle \in P$. By Proposition 6.10 a closure path $(C, P) = (C_1, P_1) \xrightarrow{\mu_{i+1}} (C_2, P_2) \xrightarrow{\mu_{i+2}} \ldots \xrightarrow{\mu_n} (C_{n-i+1}, P_{n-i+1}) = (C', P')$ does exist such that the progressed event is always e. Hence, $C' = C \cup \{e\}$ and $P' = P \setminus \{\langle e, i\rangle\}$. By rule 3 in Definition 6.6, we obtain the ST transition:

$$\langle s, (C, P)\rangle \xrightarrow{\mu^k} \langle s \setminus \{\mu_i^k\}, (C', P')\rangle.$$

Notice that $\sigma(\langle s \setminus \{\mu_i^k\}, (C',\, P')\rangle) = (C',\, P^\star \setminus \{\langle e,\, k\rangle\})$, which is an ST configuration since $(C,\, P^\star)$ is so. Moreover, by definition of ST transition, we also get

$$(C, P^\star) \xrightarrow{\mu^k} (C',\, P^\star \setminus \{\langle e,\, k\rangle\}) \in \mathcal{C}_{\mathcal{E}}^{ST}.$$

By a symmetric argument, the other 'half' of the bisimulation definition can be proved, too. ■

The last step for proving $\overset{n}{\sim} \subseteq \overset{ST}{\sim}$ is provided by the following lemma.

Lemma 6.13 For every pair of les's $\mathcal{E}, \mathcal{F} \in \mathcal{ES}_{\mathcal{M}}^{n-1}$, $\mathcal{E} \overset{n}{\sim} \mathcal{F}$ implies $\triangle_n^{ST}(\mathcal{G}_n(\mathcal{E})) \sim \triangle_n^{ST}(\mathcal{G}_n(\mathcal{F}))$.

Proof: Let $\Re_n$ be the largest bisimulation between $\mathcal{G}_n(\mathcal{E})$ and $\mathcal{G}_n(\mathcal{F})$. Let $\Re_{ST}$ be the relation:

$$\{\langle\langle s, (C,\, P)\rangle, \langle s', (C',\, P')\rangle\rangle \mid s = s' \wedge \langle (C,\, P), (C',\, P')\rangle \in \Re_n\}$$

We prove that $\Re_{ST}$ is a bisimulation between the transformed, ST labelled, transition systems. The proof proceeds by induction on the reachability relation. The initial states of the transformed lts's are related by $\Re_{ST}$ because they are the image of the initial states, which are in $\Re_n$. Let $\langle\langle s, (C,\, P)\rangle, \langle s', (C',\, P')\rangle\rangle \in \Re_{ST}$: two cases are possible, depending on the nature of the executed phase.

A **beginning phase** $\langle s, (C,\, P)\rangle \xrightarrow{\mu} \langle s^+, (C,\, P^+)\rangle$ is a transition of $\triangle_n^{ST}(\mathcal{G}_n(\mathcal{E}))$ only if there is the opening path $(C,\, P) \overset{\mu_{j+1}\cdot\mu_j\cdot\ldots\cdot\mu_1}{\twoheadrightarrow} (C,\, P^+)$, where $j = \mathtt{min}\{i \mid \mu_{i+1}^k \notin s\}$ and $s^+ = s\theta \cup \{\mu_1^{z_1}\}$, with $\theta = \{{}^{\mu_{x+1}^k}/_{\mu_x^k} \mid 1 \le x \le j\}$. Now, since $\langle (C,\, P), (C',\, P')\rangle \in \Re_n$, there exists $(C',\, P^\star)$ such that $(C',\, P') \overset{\mu_{j+1}\cdot\mu_j\cdot\ldots\cdot\mu_1}{\twoheadrightarrow} (C',\, P^\star)$ is the corresponding opening path in $\mathcal{G}_n(\mathcal{F})$. Since $s' = s$, index j satisfies the minimality condition required by the premise of rule 2. Applying this rule, we get

$$\langle s', (C',\, P')\rangle \xrightarrow{\mu} \langle s^\star, (C',\, P^\star)\rangle \in \triangle_n^{ST}(\mathcal{G}_n(\mathcal{F}))$$

where $s^\star = s'\theta \cup \{\mu_1^{z_1'}\}$. It is immediate to verify that $s^+ = s^*$; furthermore, $\langle (C,\, P^+), (C',\, P^\star)\rangle \in \Re_n$ because the opening path is unique; hence, $\langle\langle s^+, (C,\, P^+)\rangle\ ,\ \langle s^\star, (C',\, P^\star)\rangle\rangle \in \Re_{ST}$. A symmetric argument can be applied if the transition starts from $\langle s', (C',\, P')\rangle$.

An **ending phase** $\langle s, (C,\, P)\rangle \xrightarrow{\mu^k} \langle s \setminus \{\mu_i^k\}, (C^+, P_{n-i+1})\rangle$ is a transition of $\triangle_n^{ST}(\mathcal{G}_n(\mathcal{E}))$ provided that $\mu_i^k \in s$ and there exists the closure path:

$$(C,\, P) \;=\; (C,\, P_1) \xrightarrow{\mu_{i+1}} (C,\, P_2) \xrightarrow{\mu_{i+2}} \ldots \xrightarrow{\mu_n} (C^+,\, P_{n-i+1}) \qquad (1)$$

By Proposition 6.10, there also exists the (unique causal) closure path for $(C',\, P')$:

$$(C',\, P') \;=\; (C',\, P_1') \xrightarrow{\mu_{i+1}} (C',\, P_2') \xrightarrow{\mu_{i+2}} \ldots \xrightarrow{\mu_n} (C^\star,\, P_{n-i+1}') \qquad (2)$$

Furthermore, since $\langle\langle s, (C, P)\rangle, \langle s', (C', P')\rangle\rangle \in \Re_{\mathtt{ST}}$, we know that $s = s'$, hence $\mu_i^k \in s'$, too. Therefore, rule 3 can be applied producing the transition:

$$\langle s', (C', P')\rangle \xrightarrow{\mu} \langle s' \setminus \{\mu_i^k\}, (C^\star, P'_{n-i-1})\rangle \in \triangle_n^{ST}(\mathcal{G}_n(\mathcal{F}))$$

Now, it remains to prove that $\langle\langle s \setminus \{\mu_i^k\}, (C^+, P_{n-i+1})\rangle, \langle s' \setminus \{\mu_i^k\}, (C^\star, P'_{n-i-1})\rangle\rangle \in \Re_{ST}$, i.e., that $\langle (C^+, P_{n-i-1}), (C^\star, P'_{n-i-1})\rangle \in \Re_n$.

We already know that $\langle (C, P), (C', P')\rangle \in \Re_n$. Since there is just one μ_{i+1}-outgoing transition from the confusion free configurations (C, P_1) and (C', P'_1), also the pair $\langle (C, P_2), (C', P'_2)\rangle$ belongs to $\Re_n$. So it remains to prove that $\langle (C, P_j), (C', P'_j)\rangle \in \Re_n$ for $j = 3, \ldots, n-i-1$. This is, however, a direct consequence of the fact that $\Re_n$ is the largest bisimulation. Let us see why.

$$\begin{tikzcd}
& (C, P_1) \arrow[d, "\mu_{i+1}"] & \\
& (C, P_2) \arrow[dl, "\mu_{i+2}"'] \arrow[dr, "\mu_{i+2}"] & \\
(C, P_3) & & (C, P_a)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& (C', P'_1) \arrow[d, "\mu_{i+1}"] & \\
& (C', P'_2) \arrow[dl, "\mu_{i+2}"'] \arrow[dr, "\mu_{i+2}"] & \\
(C', P'_3) & & (C', P_b)
\end{tikzcd}$$

In the above picture, the configurations (C, P_a) and (C, P_b) are reached by progressing different autoconcurrent events, i.e., each one is reached with a transition not causally dependent on the preceding one. If $\Re_n$ relates (C, P_3) with (C', P_b) but not the former with (C', P'_3), it could be possible that the final states (C^+, P_{n-i+1}) of (1) and $(C^\star, P'_{n-i+1})$ of (2) are not in $\Re_n$; hence, $\Re_{ST}$ could fail to be a bisimulation.

By contradiction, assume that $\langle (C, P_3), (C', P'_3)\rangle \notin \Re_n$. As $\Re_n$ is a bisimulation, it must contain at least the following pairs: $\langle (C, P_3), (C', P_b)\rangle$ and $\langle (C, P_a), (C', P'_3)\rangle$. However, observe that we are in the following situation:

$$\begin{tikzcd}
& (C, P_1) \arrow[d, "\mu_{i+1}"] \arrow[drr, "\mu_{i+2}"] & & \\
& (C, P_2) \arrow[dl, "\mu_{i+2}"'] \arrow[dr, "\mu_{i+2}"] & & (C, P_c) \arrow[dl, "\mu_{i+1}"] \\
(C, P_3) & & (C, P_a) &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& (C', P'_1) \arrow[d, "\mu_{i+1}"] \arrow[drr, "\mu_{i+2}"] & & \\
& (C', P'_2) \arrow[dl, "\mu_{i+2}"'] \arrow[dr, "\mu_{i+2}"] & & (C, P_d) \arrow[dl, "\mu_{i+1}"] \\
(C', P'_3) & & (C', P_b) &
\end{tikzcd}$$

Since $\langle (C, P_j), (C', P'_j)\rangle \in \Re_n$ $(j = 1, 2)$, $\Re_n$ must contain the pair $\langle (C, P_c), (C, P_d)\rangle$, due to the confusion-freeness of (C, P_1) and (C', P'_1). Moreover, there exists exactly one μ_{i+1}-successor of (C, P_c) and (C, P_d). So, also the pair $\langle (C, P_a), (C', P_b)\rangle$ must be part of $\Re_n$. A direct consequence is the contradiction that $\Re_n$ is not the largest bisimulation as it is not transitive (remember that the largest bisimulation is an equivalence relation): indeed, by transitivity, also the pair $\langle (C', P'_3), (C', P'_3)\rangle$ belongs to the largest bisimulation.

Iterating this argument, we can prove that, for any $1 \leq j \leq n-i-1$, the j-th pairs in the two causal closure paths are related by the largest bisimulation. In particular, $\langle (C^+, P_{n-i-1}), (C^\star, P'_{n-i-1})\rangle \in \Re_n$. ■

We want to remark that in the above proof we need to exploit the largest bisimulation between $\mathcal{G}_n(\mathcal{E})$ and $\mathcal{G}_n(\mathcal{F})$, rather than a generic bisimulation, as done in the proof of Theorems 3.5 and 6.4. Given any bisimulation there is no certainty that, starting from two bisimilar $\mathtt{split}_n$ configurations and closing one event in $\mathcal{E}$ and one in $\mathcal{F}$, the corresponding states of the two causal closure paths are related. Actually a generic bisimulation may map a causal closure path of the first transition system to a "degenerate" closure path, i.e., a path not satisfying the causality conditions of Proposition 6.10. (We thank W. Vogler for pointing this out, suggesting us to use the largest bisimulation.)

The nice consequence of Lemma 6.12 and Lemma 6.13 is stated in the following theorem.

Theorem 6.14 For every pair of les's $\mathcal{E}$, $\mathcal{F}$, such that $\mathcal{E}, \mathcal{F} \in \mathcal{ES}^{n-1}_{\mathcal{M}}$, $\mathcal{E} \overset{n}{\sim} \mathcal{F}$ implies $\mathcal{E} \overset{ST}{\sim} \mathcal{F}$. ■

The last result we provide is the coincidence between the ST semantics and the limit of $\mathtt{split}_n$ semantics when image finite event structures are taken. Roughly this is a conjunction of Theorems 6.4, 6.14 and 2.7. Let's see the details.

In the class of image finite event structures, ST bisimulation can be defined iteratively by rephrasing Definition 2.6. That is, if $\mathcal{E}$ and $\mathcal{F}$ are two les's, $\mathcal{E} \overset{ST}{\sim}_k \mathcal{F}$ if and only if $\mathcal{G}_{ST}(\mathcal{E}) \sim_k \mathcal{G}_{ST}(\mathcal{F})$. Similarly for $\overset{n}{\sim}$. Given an acyclic lts $\mathcal{G}$, we define $\mathtt{trunc}_k(\mathcal{G})$ as the truncation of $\mathcal{G}$ to states at depth k (only states reached through computations of length at most k are considered). Observe that, as all the transition systems we consider are (image finite and) acyclic, we have that $\mathtt{trunc}_k(\mathcal{G}) \sim \mathtt{trunc}_k(\mathcal{G}')$ if and only if $\mathcal{G} \sim_k \mathcal{G}'$.

Corollary 6.15 Let $\mathcal{E}$ and $\mathcal{F}$ be two image finite les's. Then $\mathcal{E} \overset{k+1}{\sim}_{k\times(k+1)} \mathcal{F}$ implies $\mathcal{E} \overset{ST}{\sim}_k \mathcal{F}$.

Proof: The autoconcurrency $\mathtt{trunc}_k(\mathcal{G}_{ST}(\mathcal{E}))$ can manifest is bounded by k itself, because paths of length k cannot have more than k started actions. As k autoconcurrent actions can be performed in at most $k \times (k+1)$ transitions of a $\mathtt{split}_{k+1}$ transition system, the k-truncation of an ST transition system can be simulated by the $k \times (k+1)$-truncation of the associated $\mathtt{split}_{k+1}$ transition system, following the proof of Theorem 6.14. Therefore, $\mathtt{trunc}_{k\times k+1}(\mathcal{G}_{k+1}(\mathcal{E})) \sim \mathtt{trunc}_{k\times k+1}(\mathcal{G}_{k+1}(\mathcal{F}))$ implies $\mathtt{trunc}_k(\mathcal{G}_{ST}(\mathcal{E})) \sim \mathtt{trunc}_k(\mathcal{G}_{ST}(\mathcal{F}))$, from which the thesis follows immediately. ■

Theorem 6.16 (**The Limit Theorem**) Let $\mathcal{E}$ and $\mathcal{F}$ be two image finite les's. Then $\mathcal{E} \overset{ST}{\sim} \mathcal{F}$ if and only if $\mathcal{E} \overset{n}{\sim} \mathcal{F}$, for every n.

Proof: We have only to prove the *if* direction because the other one is given by Theorem 6.4. Since $\mathcal{G}_{ST}$ is image finite then, by Theorem 2.7, $\mathcal{G}_{ST}(\mathcal{E}) \sim \mathcal{G}_{ST}(\mathcal{F})$ if and only if, for every n, $\mathcal{G}_{ST}(\mathcal{E}) \sim_n \mathcal{G}_{ST}(\mathcal{F})$. Therefore, if $\mathcal{E} \overset{ST}{\not\sim} \mathcal{F}$ then there exists m such that $\mathcal{E} \overset{ST}{\not\sim}_m \mathcal{F}$.

By contradiction, assume that $\mathcal{E} \overset{ST}{\not\sim} \mathcal{F}$, and thus $\mathcal{G}_{ST}(\mathcal{E}) \not\sim_m \mathcal{G}_{ST}(\mathcal{F})$. If this is the case, since paths of length m can not have more than m autoconcurrent actions, by

Corollary 6.15, $\mathtt{split}_{m+1}$ semantics is powerful enough to distinguish $\mathcal{E}$ and $\mathcal{F}$. Therefore, $\mathcal{E} \overset{m+1}{\not\sim}_{m \times (m+1)} \mathcal{F}$ holds and, by Theorem 3.5, for every $n > m+1$, $\mathcal{E} \overset{n}{\not\sim} \mathcal{F}$ holds, too. ∎

7 Conclusions

$\mathtt{Split}_n$ and $\mathtt{ST}$ semantics of stable event structures have been investigated in full detail, showing that in the class of image finite event structures, $\mathtt{ST}$ bisimulation coincide with the limit of the $\mathtt{split}_n$ bisimulations. Moreover, we have provided a new formulation of $\mathtt{ST}$ bisimulation where the observable effect of a transition is completely represented through its label. This is in contrast with earlier definitions of $\mathtt{ST}$ semantics, based on more elaborate observability notions making use of external information, such as suitable bijections on the involved events. This improvement might appear rather slight, or even irrelevant. On the contrary, it is fundamental for the practical purposes of the present paper and related ones.

1. In order to prove the semantic relationships among the infinity of equivalences we have investigated, we have exploited a proof technique which is based on the definition of bisimulation-equivalence-preserving graph transformations. Our capability of defining and exploiting such transformations is strongly based on the nature of our proposal of $\mathtt{ST}$ bisimulation semantics.

2. Operational semantics for Process Description Languages may be defined rather easily in a purely inductive way following the approach of Plotkin [13]. This is due to the fact that the labels over the transitions directly express the $\mathtt{ST}$ information of the causal link (see, e.g., [7] for $\mathtt{split}$ and $\mathtt{ST}$ operational semantics for CCS). Furthermore, the fact that $\mathtt{ST}$ bisimulation equivalence is a congruence over the language may be easily proved resorting to standard techniques (see, e.g., [14]).

Splitting actions into phases may be seen as a restricted operation of *action refinement*, where an action is refined to the sequence of its phases. Action refinement has been widely studied over prime event structures in [3, 15], where the structure substituted for an event is conflict free. In particular, [3] proves that $\mathtt{ST}$ bisimulation equivalence is a *congruence*, while [15] shows that it is the *coarsest* congruence contained into bisimulation equivalence for this operation (together with some more results illustrating the relationship between various $\mathtt{ST}$ semantics).

In this perspective, another way of looking at the limit theorem described here is that $\mathtt{ST}$ bisimulation is the coarsest congruence contained in bisimulation w.r.t. an operation of refinement which randomly splits actions into any number of phases. We plan to extend the definition of action refinement to the more general class of stable event structures, where the refining structures need not to be deadlock-free and conflict-free. As expected, $\mathtt{ST}$ bisimulation equivalence is the coarsest congruence also in this case. The proof of this "optimality" result is strongly based on the limit theorem, presented here.

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