Validity of Inferences[*]

When the systematic study of inferences began with Aristotle, there was in Greek culture already a flourishing argumentative practice with the purpose of supporting or grounding one's assertions or discrediting assertions made by others. Such a practice naturally gives rise to the question what characterizes a *good* argument, one that really justifies its conclusion. In the discussion of this question, the remarkable idea arose that there are deductive arguments or proofs that conclusively establish the truth of their conclusions. Logic started as a reflection over the inferences that such maximally good arguments are built up of. I want to claim that there is still a need of such reflections.

I shall here focus on two features of valid inferences: their power to justify assertions and beliefs and their capability of bringing into being a kind of modality. It is noteworthy that both features were invoked from the very beginning in the study of inferences. In an often quoted passage in *Prior Analytics*, Aristotle says:

> "A syllogism is a form of speech in which, certain things being laid down, something follows of necessity from them, i.e. because of them, i.e. without any further term being needed to justify the conclusion."[1]

Assertions may be made on grounds of various kinds and various strengths. I take it to be a fact, acknowledged by most people at least since the time of Aristotle, that the making of inferences is one way to acquire grounds or justifications and that there is a kind of inference that delivers conclusive grounds, grounds of maximal strength.

---

[*] This is a revised version of the paper I gave at "the 2nd Launer Symposium on the Occasion of the Presentation of the Launer Prize 2006 to Dagfinn Føllesdal". It is not directly related to Føllesdal's works, but shares with them many of them an interest in modality. The main ideas of the paper were also presented in a somewhat different form at talks I gave in the same year at conferences at Pisa and Stockholm and at the Kant lectures at Stanford and, in a form more similar to the present one, in 2007 at talks at Stockholm, Bologna, and Pisa. I thank participants at these occasions for stimulating discussions and professors Cesare Cozzo, Per-Erik Malmnäs, Per Martin-Löf, and Dugald Murdoch for commenting on early versions of this paper.

[1] W. D. Ross, *Aristotle's Prior and Posterior Analytics*, Oxford 1949, p. 287.

Not only may an inference justify an assertion or belief, it may also compel us to hold its conclusion true. The idea that an inference in some way compels us is present already in Parmenides and Plato[2]. It is reflected today in the common practice of inserting the word "must" when announcing an inference. For instance, after having derived a contradiction from the assumption that the square root of 2 is rational, a textbook may continue: "Hence, √2 must be irrational".[3] As one also says, in a more explicit recognition of the compelling force of a valid inference, "by pain of irrationality", one has to accept the conclusion of a valid inference, given that one has accepted the premisses.

To say that there are deductive inferences that give rise to conclusive and even compelling grounds or proofs is of course not to say that there are infallible roads to knowledge. One can never rule out that one is mistaken about what one thinks is a ground or a proof of a sentence. But if the sentence turns out be false, then we say that we did not really have a proof, what we thought was a proof was in fact not a proof. Hence, the possibility of mistakes does not have any bearing on the existence of conclusive proofs. It does not detract from the fact that a valid deductive inference does deliver a conclusive ground for its conclusion given conclusive grounds for the premisses.

However, it is one thing that there is this idea of justification by inference and that we take it to be a conceptual truth that a valid deductive inference delivers a conclusive ground for the conclusion given conclusive grounds for the premisses, and another thing to explain how, at all, there can be such justifications, and in particular, how there can be such things as conclusive grounds. Presumably such an explanation must rest on what it is for an inference to be valid. It is a task for philosophy to account for the validity of an inference in such a way that it follows from the account or is directly included in the account that a valid inference lends itself for justifications – in particular, that a valid deductive inference delivers a conclusive ground for its conclusion, given conclusive grounds for its premisses.

---

[2] For an instructive discussion of some of Plato's views on this, see Cesare Cozzo, "Can a Proof Compel us?" in *Mathematical Reasoning and Heuristics*, ed. by C. Cellucci and D. Gillies, King's College Publications, London 2005, pp. 191-211.

[3] This is an authentic example from a text book of mathematical analysis.

In this paper I shall restrict myself to deductive inferences and conclusive grounds, and henceforth, when I speak of inferences or grounds, I always have in mind deductive inferences or conclusive grounds, respectively. To have a ground for a sentence *A*, as I use the term, will always mean to be justified in holding *A* true and to know that *A*.

1. Logical consequence as defined by Tarski

In logic, as well as in philosophy in general, the dominant view concerning the notions of logical consequence and valid inference has been for long that the essential questions concerning the former notion were settled by Alfred Tarski in his paper "On the Concept of Logical Consequence"[4], and that the latter notion is accounted for by saying that an inference is valid if and only if its conclusion is a logical consequence of its premisses in the sense defined by Tarski.

Tarski himself did not think that he had settled completely how the concept of logical consequence is to be adequately defined. He especially mentioned the open problem of how to draw the line between logical and non-logical terms, a problem which is of course crucial in this context, since the whole idea of Tarski's definition is to vary the content of the non-logical terms, defining a sentence *A* to be a logical consequence of a set Γ of sentences if and only if every variation of the content of the non-logical terms occurring in the sentences involved that makes the sentences of Γ true makes *A* true as well. Furthermore, it should be said that Tarski was not concerned with the validity of inferences but restricted himself to the concept of logical consequence.

Nevertheless, Tarski thought that his definition quite well caught what it means to say that a sentence follows logically from other sentences. He especially stressed that a modal element, namely the one discussed above in connection with valid inferences, was taken care of, saying:

[4] Published 1936, first in Polish and then the same year in German under the title "Über den Begriff der logischen Folgerung" in *Actes du Congrès International de Philosophie Scientifiques* 7, pp 1-11. An English translation became available in the collection of Alfred Tarski's paper that appeared 20 years later: A. Tarski, *Logic, Semantics and Metamathematics*, Oxford 1956.

> "In particular, it can be *proved*, on the basis of [my] definition, that every consequence of true sentences *must* be true"[5] (my italics).

One main claim of my paper is that the concept of valid inference cannot be obtained in this way from the notion of logical consequence as defined by Tarski. This is not an original claim. Critical voices against Tarski's definition of logical consequence, in particular when thought of as also defining the validity of inferences, have been raised before, for instance by John Etchemendy, Göran Sundholm and myself[6]. One aim of this paper is to isolate what I think is the main shortcoming of the dominant view. Focusing on inferences as a means to get grounds for assertions and beliefs, I shall argue that this *raison d´être* of inferences cannot be explained or accounted for when their validity is equated with the relation of logical consequence holding between premisses and conclusion as defined by Tarski. The issue will be kept apart from the dispute between classical and constructive reasoning.

2. Valid and logically valid inferences

I am not denying that the idea of varying the content of the non-logical terms is a natural and fruitful approach even when one is concerned with inferences. The idea of such variations goes back to Aristotle whose standard way of showing that an inference is not valid was exactly to vary the content of the non-logical terms in the involved sentences in order to get a counterexample where the premisses are true and the conclusion is false. It was taken up by Bernard Bolzano and is the main element in his definition of logical consequence[7]. Bolzano's definition differs in some

---

[5] Page 417 in the collection of Tarski's papers referred to in footnote 5.

[6] For instance, John Etchemendy, *The Concept of Logical Consequence*, Harvard University Press 1990, Göran Sundholm, "Inference Versus Consequence", *The LOGICA Yearbook 1998*, Czech Acad. Sc., Praque 1998 (Sundholm is much influenced by Per Martin-Löf, see e.g. his paper "On the Meaning of the Logical Constants and the Justification of the Logical Laws" from 1985, republished in *Nordic Journal of Philosophical Logic*, 1, pp 11-60), Dag Prawitz, "Remarks on some Approaches to the Concept of Logical Consequence", *Synthese*, 62, 1985, pp 153-171, and Dag Prawitz, "Logical Consequence from a Constructivist Point of View", in *The Oxford Handbook of Philosophy of Mathematics and Logic*, Oxford 2005, pp 671-695. I also want to mention a talk by Per Martin-Löf presented at the conference in Pisa mentioned in footnote * on the first page of the paper.

[7] Bolzano's term is "Ableitbarkeit". The notion is defined in his book *Wissenschaftslehre*, I-IV, Sulzbach 1837 – published 99 years before Tarski presented his definition. As already his

respects from Tarski's[8], but as far as we are concerned in this discussion, the two definitions are essentially the same in making the non-existence of a counterexample to, not only a necessary condition for logical consequence, but a sufficient one as well.

So far I have not spoken of an inference being logically valid, only of it being valid. When I said that Tarski's analysis of the notion of logical consequence was, and still is considered to answer the question what it is for an inference to be valid, I should have said *logically* valid. But, for obvious reasons, when adopting the view that Bolzano-Tarski's analysis of the notion of logical consequence settles what it is for an inference to be correct, one usually does not distinguish between the two notions.

Yet, from an intuitive point of view, such a distinction is natural to make. There are inferences that one would be inclined pre-theoretically to consider as valid but not as logically valid; a simple natural language example is the inference from the two premisses "Adam is longer than Beatrice" and "Beatrice is longer than Carlo" to the conclusion "Adam is longer than Carlo", a mathematical example is inference by mathematical induction over the natural numbers (as usually formulated in first order logic, where the concept of natural number is not defined impredicatively but is taken as a primitive notion).

What does it then mean to say that an inference is logically valid? An obvious suggestion is that an inference is logically valid if its validity does not depend on the non-logical terms but only on the logical terms that occur in the sentences involved. This may be spelled out by using the very idea of the Bolzano-Tarski definition of logical consequence in this context, saying that the logically valid inferences are those that remain valid regardless of how we vary the content of the non-logical terms.

We may think of a variation of the content of the non-logical terms of a sentence along the lines of Bolzano as being obtained by simply replacing non-logical terms in the sentence by other terms of the same syntactical category. Letting ϕ stand for such substitutions, i.e. for assignments to the

---

choice of words suggests, Bolzano was interested in epistemological aspects of this notion. He explicitly connects it with the correctness of an inference, saying for instance that the relationship of logical consequence is "a remarkable thing since, knowing that it applies, it makes us able to infer immediately from the known truth of the [antecedents] the truth of the [succedent]".

[8] One of the differences is referred to below.

non-logical terms in the sentences $A_1, A_2..., A_n$ and B of other non-logical terms of the same syntactical category, and denoting the result of making such a substitution ϕ in A_i and B by A_i^ϕ and B^ϕ, respectively, we may make the explicit definition:

> An inference from $A_1, A_2, ..., A_n$ to B is *logically valid*, if and only if, for every ϕ, the inference from $A_1^\phi, A_2^\phi, ..., A_n^\phi$ to B^ϕ is *valid*.

Or, better, we may follow the way of Tarski and let ϕ stand for valuations, i.e. assignments of arbitrary values (of appropriate categories) to the non-logical terms, and define:

> An inference from $A_1, A_2, ..., A_n$ to B is *logically valid*, if and only if, for every ϕ, the inference from $A_1, A_2, ..., A_n$ to B is *valid under* ϕ.

What is obtained by following the Bolzano-Tarski approach in this way is thus a reduction of logical validity to validity. The crucial task is then to define what it is for an inference to be valid (or valid under an assignment).

We could in the same way distinguish between *consequence* and *logical consequence*, and reduce the latter notion to the first one by saying that B is a logical consequence of Γ, when B remains a consequence of Γ for every variation of the content of the non-logical terms occurring in A and in the sentences of Γ.[9] But if we just stay within the framework of Bolzano and Tarski, this distinction become pointless, because the notion of consequence will then collapse into that of truth of the corresponding material implication. The same holds of course, if we make the distinction between valid and logically valid inference proposed above, but do not move outside the framework of Bolzano and Tarski when it comes to analyzing the notion of valid inference. It would then only remain to say that an inference is valid if either one of the premisses is false or the conclusion is true, but clearly no one is interested in equating the validity of an inference with such a relation between the truth-values of the involved sentences. Already the inability of the Bolzano-Tarski approach to distinguish in a satisfying way between consequence and logical consequence should make one suspicious of the possibility of analyzing

[9] This is a proposal that I made in my paper "Logical Consequence from a Constructivist Point of View", in *The Oxford Handbook of Philosophy of Mathematics and Logic*, Oxford 2005, pp 671-695.

the concept of (logically) valid inference by using their notion of logical consequence.

But the weakness of such an analysis is seen most clearly when we ask why an inference whose conclusion is a logical consequence of the premisses in the Bolzano-Tarski sense should have the power to justify a belief in the truth of the conclusion. One can hardly expect that there is a satisfactory answer to that question if one bears in mind the following facts. To say that the conclusion of an inference is a logical consequence of its premisses in the sense of Bolzano-Tarski amounts just to saying two things: 1) that the inference preserves truth (which only means here that if the premisses are true then so is the conclusion) and 2) that all inferences of the same logical form do the same. Now, although the first property is certainly *relevant* for the question whether the inference has the power to justify a belief in the conclusion (being a necessary condition for that), it is obviously not sufficient for the inference to have this power. The second property seems not even relevant to this question; it concerns other inferences of the same logical form as the first one, and is relevant to the question whether *they* have the power to justify their conclusions. Property 2 is admittedly a necessary condition for the inference to be *logically* valid but is not sufficient for the simple reason that it only amounts to the requirement that all inferences of the same logical form satisfy property 1, and, as already said and as anyone agrees, property 1 is not sufficient for an inference to have the power to justify beliefs or to be valid.

Nevertheless, the view that an inference is (logically) valid if and only if its conclusion is a logical consequence of the premisses in the sense of Bolzano and Tarski is, as noted, the prevailing one. In the sequel, I shall for the sake of brevity sometimes refer to it as the Bolzano-Tarski view or notion of valid inference (even if Tarski was primarily concerned with the notion of logical consequence and did not explicitly speak about valid inferences). In consideration of the dominance of this view, I shall devote part of the following sections to showing in more detail why it does not seem possible to account for the use of inferences to justify beliefs from that point of view.

Admittedly, it is not immediately obvious what such an account is to consist of. The requirement that some such account should flow from an analysis of the concept of valid inference is certainly a fair one, but we must discuss how it is to be formulated more precisely. This is an important issue for any concept of valid inference and will also be a topic of the next sections.

3. How valid inferences do not justify their conclusions

What conditions have to be satisfied in order that a person is to be justified in holding the conclusion of an inference true? This question is more difficult to answer than is sometimes realized. Shall we say simply:

(1) *If an inference* J *from a sentence* A *to a sentence* B *is valid, then a person who is justified in holding* A *true is also justified in holding* B *true?*

This is sometimes affirmed. It is even said that if *B* is a logical consequence of *A*, then without further investigation we may conclude that *B* is true given that we know that *A* is true.

However, it is enough to consider an inference with a great distance between the premiss and the conclusion to see that implication (1) does not hold in general. Let *B* be a difficult mathematical theorem, for instance Fermat's last theorem, and let *A* be the conjunction of axioms or starting points for a proof of the theorem. We may grant that Andrew Wiles was justified in holding true all the facts from which he started when proving Fermat's last theorem and that the one step inference from these starting points to Fermat's last theorem is valid. But these two assumptions are clearly not enough to make Wiles or anybody else justified in holding Fermat's last theorem true; they were, we may assume, satisfied long before Wiles gave his proof.

It is instructive to reflect upon the fact that Wiles first had to withdraw a purported proof because of a discovered gap. Let us suppose that when he later filled the gap, there was no expansion of the starting points and that he was justified in holding them true when he gave his first incomplete proof. Then, on these assumptions, the one step inference that we are considering, that is the one inferring Fermat's last theorem from the conjunction of the starting points of this withdrawn proof, was indeed valid, although, as long as the gap remained, one did not see the validity. Hence, according to implication (1), Wiles would have been justified in asserting Fermat's last theorem as soon as he gave his first incomplete proof, contrary to the fact that in reality he soon afterwards had to withdraw it because of the discovered gap.

This clearly shows that it is not the *validity* of the inferences of a proof *per se* that makes one entitled to assert or believe in the conclusion of the proof. What makes one so entitled is a sufficiently detailed proof, in other words, a proof without gaps.

But what is a gap-free proof? Should we say that each inference of the proof is to be seen to be valid, and what would that mean? Perhaps one should conclude from what is said above that the antecedent of implication (1) has to be strengthened to say not only that the inference is valid but also that the person in question knows it to be valid. Let us consider this proposal:

(2) *If a person knows the inference* J *from a sentence* A *to a sentence* B *to be valid, and is justified in holding* A *true, then she is also justified in holding* B *true.*

From an intuitive understanding of what it is for an inference to be valid, we certainly expect this implication to hold. However, we should also ask whether the antecedent of this implication describes a way to acquire new knowledge; otherwise, the fact that the implication holds is of little interest. In other words, we should ask whether the following holds:

(3) *We can become justified in holding a sentence* B *true (and hence get to know that* B *is true) by first getting to know two things: 1) that an inference* J *from a sentence* A *to the sentence* B *is valid, and 2) that* A *is true.*

If we think that this is an adequate description of how we acquire new knowledge by using inferences, we may test a proposed definition of the concept of valid inference by investigating whether it is able to support (3), i.e. whether the statement comes out true when we replace “valid” with what it is to be valid according to that definition. Such a test on the Bolzano-Tarski notion of valid inference is in effect suggested by John Etchemendy.[10] According to him the test comes out negatively, because in

[10] Ibid, p. 93, where he writes: ”A logically valid argument must, at the very least, be capable of justifying its conclusion. It must be possible to come to know that the conclusion is true on basis of the knowledge that the argument is valid and that its premisses are true.”

general we are not able to know that *J* is valid in the proposed sense without first knowing that the conclusion *B* is true.[11]

This is an interesting argument, but I think that it is not at all obvious how it can be supported.[12] However, this is not something that I shall discuss here, because I think that the statements (2) and (3) are on the wrong track when we are to describe how we normally use inferences to justify beliefs. Implication (2) was formulated as a possible response to the problem that arose when noting that the mere validity of an inference whose premisses are known to be true does not justify a person in asserting the conclusion, it then being suggested that it is only when a person sees the validity of the inference that she is so justified. Identifying 'seeing the validity of an inference' with 'knowing the inference to be valid', such knowledge was suggested as a necessary condition for an inference to justify a belief, which we may formulate as follows:

(4) *It is only when a person knows an inference to be valid and its premisses to be true that the inference justifies her in holding the conclusion true.*

But further reflection shows that it is doubtful that we could ever use inferences to acquire knowledge, if we had first to establish their validity. Furthermore, as far as the Bolzano-Tarski view of valid inference is

[11] Etchemendy's argues that the Bolzano-Tarski view of valid arguments "leaves such arguments impotent as a means of justifying their conclusions" saying: "Tarski's account equates validity with the joint truth preservation of a collection of arguments. … In general, it will be impossible to know whether an argument is a member of such a collection, hence whether it is "valid", without antecedently knowing the specific truth-values of its constituent sentences".

[12] Of course, Etchemendy does not want to deny that in many cases we can come to know that the conclusion of an inference is a logical consequence of the premisses in the sense of Bolzano and Tarski without knowing antecedently the truth-values of the involved sentences. What he is saying is that this is not possible in general, in other words that there are cases where the validity of an inference *J* in the proposed sense cannot be established without first establishing the truth of the conclusion *B* or the falsity of *A*. To show that this is so, we have thus to find sentences *A* of *B* of that kind and then develop a theory about the order in which things can be demonstrated, showing that the fact that *B* is a logical consequence of *A* in Bolzano-Tarski's sense cannot be demonstrated before having demonstrated that *B* is true or *A* is false. We would then have shown that there are valid inferences in the proposed sense that cannot be used to gain new knowledge.

concerned, we need not bother about (3),[13] because not even implication (2) seems possible to support when one assumes that view and (4).

4. **Some regress arguments**

There are several regress arguments that seem to show that we would never be able to justify beliefs by making inferences if we had first to establish the validity of the inferences. The most well known regress of this kind is the one arising in the tale told by Lewis Carroll about Achilles and the Tortoise[14], but a regress of the same structure was already described by Bernard Bolzano[15].

Noting that when we infer a sentence *B* from a sentence *A* the validity of that inference, call it *J*, is a prerequisite if *A* is to be a ground for *B*, Bolzano asks whether besides *A*, we should not count the validity of *J* as an additional ground for *B* in order to get the complete ground for *B*. To do so, Bolzano argues, is to say that *B* is true because *A* is true and *J* is valid, which is really to replace the original inference of *B* from *A*, by a new inference to the truth of *B* from two premisses: (a) *A* is true and (b) if *A* is true, then *B* is true.

By the same reasoning, Bolzano continues, we should now say that to get the complete ground for *B*, we should also count the validity of the new inference to the truth of *B* from (a) and (b) as part of the grounds for *B*, thereby giving us yet another inference with three premisses: besides (a) and (b), we have now also the premiss

(c) if (a) *A* is true and
(b) if *A* is true, then *B* is true,
then *B* is true.

The regress that this reasoning leads to is now obvious.

Carroll's regress arises instead from the question that was put in the introduction of this paper: why are we compelled to hold true the conclusion of a valid inference whose premisses we accept? In Carroll's tale, construed as a race between Achilles and the Tortoise, the Tortoise

[13] and the connected problems mentioned in footnote 12.
[14] Lewis Carroll, "What the Tortoise said to Achilles", *Mind* IV, 1895, pp. 278-280.
[15] Ibid. My attention to this was drawn by Göran Sundholm, "When, and why, did Frege read Bolzano?", *LOGICA Yearbook 1999*, Filosofia Publishers, Czech Academy of Science, Prague, 2000, pp. 164-174.

has agreed to the truth of a sentence *A* from which a sentence *B* immediately follows (actually being one of the first steps in Euclid), and Achilles is given the task of compelling him to accept the conclusion *B* as well. Achilles starts by asking the Tortoise if he is not willing to accept that if *A* is true then *B* must be true. The Tortoise does not object to that. He thus accepts not only that *A* is true but also that if *A* is true then *B* must be true, but still does not see why he must then accept *B*. Achilles argues that *B* follows logically from what the Tortoise has already accepted, and that therefore the implication (c) above holds (with "must" inserted). The Tortoise accepts this implication too, still not seeing why he must accept *B*, and all that Achilles has to offer him as argument for doing so is again that if all the premisses, now (a) - (c), are true, then *B* must be true. And so the regress continues almost exactly as in Bolzano, the only differences being that Carroll strengthens "*B* is true" to "*B* must be true" and that the two regresses start from different questions.

By slightly changing Carroll's story, letting Achilles' task be instead to show why the Tortoise is justified in holding *B* true, we can make it directly relevant to the issue with which we are now concerned: the regress will now question that we can establish the truth of the implication (2) above. Assume that a person, call her *P*, knows a sentence *A* to be true and let *J* be an inference from *A* to a sentence *B*. Assume further not only that *J* is valid but that *P* knows this, as required in (4), if she is to use the inference *J* to justify her in believing that *B*. Why should *P* now be justified in holding *B* true? Suppose that we argue that, given *P*'s knowledge of the validity of *J*, *P* knows that if *A* is true then *B* is true (an implication that Achilles gets *P* to accept), and that therefore *P* may just apply modus ponens and conclude that *B* is true. But to be an argument showing that *P* is justified in holding *B* true, it must also be assumed because of (4) that *P* knows modus ponens to be a valid inference. Assuming that *P* has that knowledge, we may infer in the same way as before that *P* knows (a), (b), and (c) as stated above in Bolzano's and Carroll's regress, and given all this, we may try to argue that *P* is justified in holding *B* true. But, clearly, we have got involved in very same regress as Bolzano and Carroll.

To stop this regress we must be able to infer directly that *P* is justified in holding *B* true given that *P* knows the inference *J* to be valid and *A* to be true. In other words, it must be right to say that *P* is entitled to infer *B* from *A*, and is thus justified in holding *B* true, just because *P* knows *J* to be valid and is justified in holding *A* true, without having to assume the additional premiss that *P* knows the validity of the inference from the two premisses (a) and (b) to the conclusion that *B* is true.

But with what right can we stop the regress in that way? Presumably that must depend on what it means that the inference *J* is valid. If the validity of the inference *J* only means that the implication "if *A*, then *B*" is true for all variations of the non-logical terms in *A* and *B*, then it is difficult to see that knowing *J* to be valid supplies one with some information relevant for being justified in holding *B* true, except just the information that if *A* is true then *B* is true, which as we have seen does not stop the regress.

As a lesson from the Bolzano-Carroll regress[16] we have to conclude, it seems, that if we assume that one must know an inference to be valid in order to use it to justify a belief, then the implication (2) cannot be derived from a meaning of valid inference that takes it in the sense of Bolzano-Tarski.

The failure to support (2) does not depend on the assumed fact that the Bolzano-Tarski notion of logical consequence lacks a genuine modal ingredient,[17] because the same regress seems to arise if we take the validity of the inference to mean that the conclusion is a necessary consequence of the premisses; at least, it is not easy to see what kind of necessity would help the situation. As we saw, in Carroll's regress, it is assumed that the Tortoise accepts, not simply that it 'if *A* is true, then *B* is true', but that 'if *A* is true, then *B* *must* be true', without it being possible to see that this

[16] The conclusion that Bolzano draws from the regress is that his initial question must be answered negatively, that is the validity of the inference is not to be counted as a part of the ground for the conclusion. But this is just a matter of terminology, which leads to a distinction between grounds and justifications, contrary to the terminology that I declared in the introduction to this paper. The validity of the inference remains a condition (a necessary but not a sufficient one if (4) is assumed, as we have seen) to be justified in holding the conclusion true. As Bolzano puts it, the validity of the inference is a prerequisite for the premiss to be a ground for the conclusion.

Carroll does not draw any conclusion from the regress but leaves it as a riddle why the Tortoise is compelled to accept the conclusion of an inference whose premisses and whose validity he is willing to accept.

[17] The view of Etchemendy (ibid) seems to be that the failure does depend on such an absence.

stronger assumption gives rise to some more information relevant to the question of being justified in holding *B* true.

To avoid misunderstanding, I should repeat that implication (2) certainly holds from a pre-theoretical stance. Any reasonable explication of what it is to be justified in a belief and what it is for an inference to be valid should support implication (2), I think. If we take the validity of an inference in the Bolzano-Tarski sense, then the implication (2) should still hold on a reasonable understanding of what it is to be justified in holding the conclusion *B* true. What has been shown so far is only that a regress arises when we combine the Bolzano-Tarski notion of validity with the idea expressed in (4), that knowledge of the validity of an inference is not only a sufficient but also a necessary condition of being able to use an inference to justify a belief. This is of no immediate concern for the Bolzano-Tarski notion of validity since one does not need to subscribe to (4), and should not do that, I think. However, in the next section it will be seen that the same regress lurks even when we impose a less stringent condition than knowing the inference to be valid, as long as we clinch to the Bolzano-Tarski notion of validity and do not make the condition so weak that the problem encountered in connection with implication (1) reappears.

There are other objections, not connected with any particular view on the validity of inferences, against the requirement expressed in (4) that one has to know the validity of an inference if one is to be justified by the inference in holding the conclusion true. Such a requirement seems directly to give rise to a circle or a new regress, perhaps a more straightforward one than the regress noted by Bolzano and Carroll. It appears as soon as we assume that the knowledge of the validity of the inference has to be explicit and ask how we know the validity. If the validity is not immediately evident, it must come about by a demonstration, whose inferences must again be known to be valid according to the requirement expressed in (4). These inferences must either be of the same kind as the inference whose validity we try to demonstrate or be of another kind, and so we get either into a circle or regress. It hardly seems reasonable to think that this circle or regress can be avoided by saying that, for sufficiently many inferences, their validity is immediately evident. On the contrary, one must expect that generally the validity has to be shown by arguments, perhaps short ones but requiring some inference steps, based on some definition or explication of what it is for an inference to be valid.[18]

As a matter of fact, we normally do not demonstrate the validity of an inference before we use it, and, as has been argued here, a requirement to the effect that we should do so in order that the inference is really to justify us in holding the conclusion true would block the possibility of getting knowledge by inferences.

6. An abstract scheme for how valid inferences justify their conclusions

We may now be inclined to think that the idea that the validity of an inference has to be known in order that the inference is to justify the conclusion was an overreaction to the problem encountered in connection with implication (1). In order that a valid inference is to justify us in holding the conclusion true, it is certainly not enough that there just happens to be such an inference of which we are not at all aware, but it *is* enough, one may suggest, that we actively make the inference, that we infer the conclusion from premisses known to be true. Whether this is a reasonable way out depends on what it is to make an inference.

A simple inference from a sentence A to a sentence B is commonly announced by saying something like "B because of A" or "A, hence B". If

---

[18] The unavoidable circularity in any attempt to justify deductive principles has been discussed by many people. Michael Dummett uses the term *pragmatic circularity* to emphasize that it is not a question of establishing the correctness of a deductive principle by assuming what is to be proved, which would be a gross circularity, but only of using the same deductive principle in the demonstration of its correctness; see for instance Michael Dummett, The Logical Basis of Metaphysics, Duckworth 1991, chapter 9, which also contains a response to my discussion of this kind of circularity in "Remarks on some Approaches to the Concept of Logical Consequence", *Synthese*, 62, 1985, pp 153-171.

Carlo Cellucci has recently discussed this kind of issue in a paper "The Question Hume Didn't Ask", in *Demonstrative and Non-Demonstrative Reasoning in Mathematics and Natural Science*, (eds) C. Cellucci and P. Pecere, Cassino 2006, pp 207-235, where he argues that deductive inference cannot be justified, invoking among other things Carroll's regress. As seen from the above, I place that regress in a different context, but agrees with Cellucci and many others that an attempt to give a suasive argument for the validity of basic deductive principles is doomed to contain a pragmatic circularity in the end. My point here is however the quite different one that we should not demand a justification of an inference before it is legitimately used to justify its conclusion (contrary to what is claimed in (4)). (It follows, as will be argued in the last section 9, that the same form of inference may be used, on the meta-level, to justify a belief in the validity of the inference.)

all that is meant by making an inference is that the conclusion is asserted or believed on the ground of the premisses or because of the premisses, as expressed in such announcements, then it is *not* enough "to make a valid inference" in order to be justified in holding the conclusion true. In the discussion of implication (1) at the beginning of section 3, we have already refuted that idea. A gap in a proof may be located precisely to an inference in that sense, where a sentence *B* is asserted on the ground of an already established premiss *A*, and although that inference from *A* to *B* may be valid, the proof will have to be withdrawn when the gap is discovered, which means that the assertion of *B* has not been given sufficient grounds and is, in other words, not considered to be justified.

Therefore, in order to be justified in holding a sentence true, one must fulfil something more than the condition of having asserted the sentence as the conclusion of a valid inference from premisses known to be true. What could that stronger requirement be? Could there be something like *recognizing* the validity of an inference, understood as less demanding than knowing but as something of sufficient substance to imply that one is justified in holding the conclusion true?

Put more formally, we need to find a relation *R* between a person *P* and an inference *J* in terms of which we can state a condition that satisfies the following demands. On one hand, it is to be substantial enough so that, unlike the antecedent of the implication (1), it implies that the person is justified in holding true the conclusion of the inference, in other words, so that the following implication holds:

(5) *If an inference* J *from a sentence* A *to a sentence* B *is valid, then a person who is justified in holding* A *true and stands in the relation* R *to* J*, is justified in holding* B *true*

But, on the other hand, it is not to be so strong that, like the antecedent of implication (2), it cannot be satisfied when taken as a necessary condition for an inference to justify a belief.

It must be possible to state a condition that is both necessary and sufficient for a person to be justified in holding the conclusion of an inference true. We should therefore be able to describe such a relation *R* and derive the implication (5) from what it means for an inference to be valid.

As long as the validity of an inference is equated with the conclusion being a logical consequence of the premisses in the sense of Bolzano and Tarski, the prospects of describing such a relation seem slim. In particular, it seems possible to run again the Bolzano-Carroll regress for a relation *R* of the required kind. We must ask why a person should be justified in holding a sentence *B* true because of being justified in holding a sentence *A* true and standing in the relation *R* to a valid inference *J* from *A* to *B*. Let us go back to the less formal expression "*P* recognizes the validity of *J*" instead of "*P* stands in the relation *R* to *J*", and see what we can infer from the assumption that *P* recognizes the validity of *J*. When "valid" means that the implication "if *A*, then *B*" is true for all variations of the content of the non-logical terms of *A* and *B*, then that is what the person recognizes, and what seems relevant here is just that she recognizes the truth of this implication (without any variation of the content). From this we want to conclude that she is justified to hold *B* true. Trying to infer that by using (5), we may assume that she recognizes the validity of modus ponens, which implies that she recognizes the implication (c) in section 5. But there we are again with this kind of argument that leads to a regress of the Bolzano-Carroll kind.

At this point it may be relevant to remark that we must beware of requiring of a person that *she* demonstrate that something is a justification of a belief in order to be justified in holding the belief. It would be incoherent to require that a person justify that something is a justification before it is counted as a justification. In the discussion above I have made no such requirement; on the contrary I have criticized the idea that one has to know that an inference is valid before it can be used to justify a belief. But in accounting for our deductive practice philosophically, we must be able to explain how valid inferences can yield justifications of beliefs. In this account we must use what it is for an inference to be valid and should be able to derive from that, on the meta-level so to say, that the conclusion of a valid inference is justified when appropriate conditions are satisfied. It has been argued here that it does not seem possible to give such account starting from the Bolzano-Tarski notion of valid inference.

7. Inferences as operations on grounds

To get a fresh approach to the concept of valid inference we should reconsider the concept of inference. As already noted, a typical way of announcing an inference is to make an assertion and state at the same time a ground for the assertion, saying for instance "*B*, because *A*" or "*A*, hence *B*". But there are examples of a more complicated kind. For instance, the premiss *A* need not always be asserted categorically, but may have been asserted only conditionally under some assumptions, in which case the conclusion may be asserted under the same or fewer assumptions. An example: Having arrived at a contradiction under an assumption *A*, we conclude that the assumption is false, saying “hence, by reductio not-*A*”. In that case the conclusion of the inference is asserted categorically or, at least, not any longer under the assumption *A*, which has been discharged.

In the last example, the assertion of the conclusion is accompanied by an indication of some kind of operation that is taken to justify it. This is consonant with an intuitive understanding of an inference as consisting of something more than just a conclusion and some premisses. Although the conclusion and the premisses may be all that we make explicit, there is also some kind of operation involved thanks to which we see that the conclusion is true given that the premisses are. Sometimes we vaguely refer to such an operation as in the example above, but essentially it is left implicit. My suggestion is that in analysing the validity of inferences, we should make these operations explicit, and regard an inference as an act by which we acquire a justification or ground for the conclusion by somehow operating on the already available grounds for the premisses.

Here I can only quite briefly and roughly outline such an alternative approach.[19] Its main idea is thus to take an inference as given not only by its premisses and conclusion (and discharged assumptions if there are such) but also by an operation defined for grounds for the premisses. As before I here use the term *ground for a sentence* to denote what a person needs to be in possession of in order to be justified in holding the sentence

[19] My own first work on such an approach is "Constructive semantics" in *Proceedings of the 1st Scandinavian Logic Symposium Åbo 1968*, pp 96-114, Uppsala 1970. A similar but partly different approach is followed by Martin-Löf in his type-theory, and is also described by Göran Sundholm, for instance in his paper "Proofs as Acts and Proofs as Objects: Some questions for Dag Prawitz", *Theoria* LXIV, 1998, pp187-216.

true; hence, the premisses from which a conclusion is inferred do not constitute grounds for the conclusion in the sense I use the term[20] – rather the premisses have their grounds, and it is by operating on them that we get a ground for the conclusion.

Seen as an act, an *individual inference* is individuated by some premisses and their grounds, a conclusion, an operation performed on the grounds for the premisses, a person who performs the operation, and a time or situation in which it is performed. The act consists in applying the operation to the given grounds for the premisses, claiming that the result obtained is a ground for the conclusion in view of the given grounds for the premisses.

Of course, in logic we usually abstract from the person who performs the operation and the time at which it is performed. We may also abstract from the particular premisses, their grounds, and the conclusion, leaving only the operation and the relation that is to hold between the premisses and the conclusion. We may then speak about an *inference form*. Modus ponens associated with an appropriate operation is an example. Finally, we may also abstract from the operation left in an inference form, and may then speak about an *inference figure* or *schema*.

How the validity of an inference is to be seen now suggests itself:

An individual inference is valid if and only if the given grounds for the premisses are grounds for them and the result of applying the given operation to these grounds is in fact a ground for the conclusion.

An inference form is valid if and only if all its instances are valid, which is also to say that *when the operation in question is applied to grounds for sentences that may occur as premisses of the inference, it yields as value a ground for the corresponding conclusion*.

An inference scheme is valid if and only if it can be associated with an operation so as to get a valid inference form.

---

[20] Their truth (or a justified belief in their truth) alone does not justify a belief in the conclusion. But those who nevertheless prefer to use the term ground for the truth of the premisses may substitute "justification" when I speak of ground. As for Bolzano's use of the term ground compare footnote 17.

Since the premisses and the conclusion of an inference may be held true only under some assumptions, we also need to be able to speak of what we may call unsaturated grounds that become grounds when supplied with grounds for these assumptions.

Although there is no room here for a precise development of these ideas, the general idea can be illustrated by some examples. Let us consider the inference form of mathematical induction, in which it is concluded that a sentence $A(n)$ holds for an arbitrary natural number n, having established the induction base that $A(0)$ holds and the induction step that A holds for the successor n' of any natural number n given that A holds for n. The ground for the induction step may be thought of as a chain of operations that results in a ground for $A(n')$ when applied to a ground for $A(n)$. The operation that is involved in this inference form may roughly be described as the operation which, for any given n, takes the given ground for $A(0)$ and then successively applies the chain of operations given as ground for the induction step n times. Obviously, given that the arguments to which this operation is applied are grounds for the premisses, i.e. the induction base and the induction step, the result of applying this operation is a ground for $A(n)$, for any n, and hence the inference form is valid in the sense defined. This way of looking at the validity of mathematical induction accords quite well with the way in which this inference form is often explained to a beginning student, saying for instance that $A(n)$, clearly holds for any n, since having established the induction base, we know that it holds for 0, and then we can apply the proof of the induction step as many times as needed.

That the operation associated with the inference form of mathematical induction yields a ground for the conclusion is immediately clear, given that the grounds to which the operation is applied are of the stated kinds. But in some cases the validity of an inference must depend on what we take as a ground for the conclusion. Consider the simple example of conjunction introduction – the premisses are here two arbitrary sentences A and B, and the corresponding conclusion has the form A & B. The operation, which we may call &I, brings together given grounds for A and B, say g and h. To carry out this inference is to apply &I to g and h and to claim that the result &I(g,h) is a ground for A & B.

How do we see that conjunction introduction so described is valid? Somewhere the claim that something is a ground for a sentence must rest on what the sentence means. That we have a ground for a conjunction A & B when we bring together a ground for A and a ground for B is an example of this: we have to take it for granted because of what conjunction means. To give an account of valid inferences for a specific language, one must therefore also specify for sentences of various forms that certain things count as grounds in virtue of what the sentence in question means.

When inferences are understood in the way suggested, it is immediate that a person who makes a valid inference obtains a justification for holding the conclusion true, given that she has grounds for holding the premisses true. To make an inference is to apply a certain operation to the given grounds for the premisses, and that the inference is valid is now defined just to mean that the result of applying this operation to grounds for the premisses is a ground for the conclusion, and hence it justifies the person in question in holding the conclusion true.

We have thus found a relation *R* between a person and an inference of the kind sought for in section 5: it consists simply in the person making the inference. But to make an inference has now been given a meaning more substantial than just claiming that the conclusion is true because of the truth of the premisses. It now also means that a specific operation is applied to given grounds for the premisses. The carrying out of this operation may not result in a ground for the conclusion, the inference then being invalid, of course. But if the inference is valid, then by the definition of what that means, the result is a ground for the conclusion. Therefore, as already said, when a person makes a valid inference, she gets in possession of a ground for the conclusion, and we must thus grant that she is justified in holding the conclusion true.

What it is to apply an operation to some arguments is something fairly well known from other fields, for instance arithmetic, but should anyway be discussed more thoroughly than I can do here; one had better do that in parallel with a more systematic development of the ideas concerning grounds for sentences of various forms. But let us consider one further example, beyond the ones already given. The inference form conjunction elimination exists in two forms: the premiss is always a conjunction, say *A* & *B*, and the conclusion is then either *A* or *B*. Corresponding to these two forms of conjunction elimination, we have two operations, call them $\&E_1$ and $\&E_2$. They are applicable to all grounds for conjunctions, and the value of applying the operations to such a ground is given by the equations

$$\&E_1(\&I(p,q)) = p \text{ and } \&E_2(\&I(p,q)) = q,$$

respectively. That a person applies one of these operations implies that she is able to handle the equation in question. By the meaning of conjunction, g is ground for A & B, if and only if g = &I(p,q) for some p and q such that p is a ground for A and q is a ground for B. Applying for instance &E_2 to a ground &I(p,q) for A & B thus yields q, which by assumption is a ground for B, as required for this inference form to be valid.

As this should illustrate, to make a valid inference is to apply a specific operation to given grounds and to evaluate the result, and is not to show that the inference is valid. But by simple reflection on what one has done, one realizes that one is in possession of a ground for the conclusion, and by some further reflection that the inference is valid. Knowledge of the validity of an inference has not been made a prerequisite for using the inference to justify a belief, but, as seen, such knowledge is easily obtained by reflecting upon the inference act. When making such a reflection explicit, it will involve a number of inferences, and typically among them will be an inference of the very kind that we are reflecting on. But because of the fact that a valid inference justifies a belief in the conclusion without it being known that the inference is valid, there is no vicious circle involved here.

8. The compelling force of an inference

Let me finish by discussing briefly from the perspective of inferences sketched here what kind of necessity is involved in connection with a valid inference. My suggestion is that this necessity has to do with the compelling force of a valid inference, which, as remarked in the introduction, is a feature of a valid inference. Can we account for this force of an inference from the present perspective?

To connect again with Carroll's story about Achilles and Tortoise, but now in its original formulation, where Achilles is given the task of making the Tortoise compelled (or with Carroll's words, "to force [the Tortoise], logically") to accept as true a sentence B that obviously follows from a sentence A that the Tortoise has accepted, we recall that Achilles' strategy was quite unsuccessful: all that Achilles succeeds in doing is to get the Tortoise to accept a series of additional premisses in the form of implications, starting with "if A is true, then B must be true". It would be equally pointless to invoke the validity of the inference from A to B, even in the sense that we have now given to it; accepting the validity, the Tortoise would thereby again be sitting with a number of additional premisses, which in itself leads nowhere. The right strategy must be instead to ask the Tortoise to act, for instance: "Please, infer *B* from *A*; you know how to do it!" And then: "Look at what you have got! You see that it is a ground for *B*, don't you!"

I think that the strategy suggested is a good illustration of how one may get compelled to form a belief. The first step is to act and make an inference. If the inference is valid and one has grounds for the premisses, then the inference act does in fact result in a ground for the conclusion. One may say that this fact does not yet have any compelling force. But we just saw in the case of conjunction elimination how, knowing the meaning of the sentences involved and reflecting upon the inference that one has made, one easily realizes that one has got in the possession of a ground for the conclusion. At least at this point, when we have become aware of having a ground for the conclusion, it would be irrational not to hold that sentence true; by pain of being irrational, we have to hold the sentence true, and are in that sense compelled. As I have argued elsewhere,[21] this is a kind of necessity that can suitably be called *necessity of thought* – it refers to how one should or must think.

It is clearly not an ontic necessity – there is no reference to all possible worlds – and the necessity has not to do with variations of the content of non-logical terms. The necessity is not particularly connected with logically valid inferences; simply valid inferences are equally compelling.

This kind of necessity makes it quite appropriate to place the word "must" in front of the conclusion.[22] However, when placed in that position,

---

[21] Dag Prawitz, "Logical Consequence from a Constructivist Point of View", in *The Oxford Handbook of Philosophy of Mathematics and Logic*, Oxford 2005, pp 671-695.

[22] I think that even non-deductive inference may have a compelling force. If overwhelming evidence points to the truth of a sentence, it would be irrational no to hold the sentence true, no matter whether the evidence is conclusive or not. As linguists have observed, in several indo-european languages "must" or its synonym is often inserted when an assertion is not based on observations but is reached by inferences. Admittedly, the compelling force may be quite weak in such cases, and inserting "must" then comes to indicate, not so much that the grounds are compelling, but rather that they are indirect; see for instance Dag Prawitz, "Meaning and Experience", *Synthese* 98, pp 131-141, 1994 or "Logisk intuitionism, sanning og mening", *Norsk Filosofisk Tidskrift*, pp 139-172, 1977.

saying “hence *B* must be true” should be understood to say, not that it must be the case that the *B* is true, but rather that *we must* hold *B* true.

Dag Prawitz

Stockholm, July 2007