On a discrete Boltzmann–type model of swarming*

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Abstract—A new model for interacting "agents" (organisms, cells, particles etc.) is proposed. We consider the one–dimensional case in which agents are characterized by their position and orientation (+/-) with "majority-based" local (swarming) interaction controlled by a sensitivity parameter (γ). The model possesses equilibrium solutions corresponding to the diffusive (isotropic) and the aligned (swarming) state. In the space-independent case, for $\gamma > 1$ alignment asymptotically occurs while for $0 < \gamma < 1$ alignment is asymptotically destroyed. This behaviour can be interpreted as a phase transition. In the space-dependent case, we provide an existence theory and prove the existence of a Lyapunov functional.

Key words—Nonlinear dynamics, Boltzmann equation, alignment, swarming.

1. Introduction

In this article we focus on biological swarming characterized as coherent motion of groups of agents (organisms or cells) into the same direction. Accordingly, the agents comprising the group are polarized, i.e. oriented into similar directions. Swarming examples on the level of organisms are flocks of sheep or schools of fish. Unicellular organisms can form polarized swarms too, e.g. myxobacteria create aligned streets [DK]. Swarming cell patterns emerge also during wound healing when groups of fibroblasts align and migrate in the direction of the wound. Note that shoals of fish or swarms of mosquitoes describe animal groups that are not necessarily polarized. We are interested in the formation of swarms that do not possess a leader but in which polarization arises as result of local "alignment interactions".

Various mechanisms of swarm formation and maintenance have been discussed. Fish swarms e.g. can be explained as the result of the interplay between hydrodynamic forces and visual perception [PH]. Precondition for street formation of myxobacteria is the existence of extracellular organelles by which the bacteria can sense the orientation of neighbouring cells [DK]. Wound healing cell migration is enforced by extracellular signal molecule gradients whose production is triggerd by the wound [DSMF].

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The principles of swarming viewed as a cooperative phenomenon arising from the interaction of a smaller or larger number of agents can be analyzed by mathematical models. Agents are characterized by their position, orientation and velocity and migrate in space and time. Various microscopic and macroscopic models have already been proposed. The dynamics of alignment without consideration of space has been anlayzed in [GLM], [ME1]. Microscopic models, for example the cellular automaton approach [BGD] and a simulation model by Ben–Jacob *et al.* [BSTCV], allow to distinguish individual agents. By means of a macroscopic model e.g. based on hydrodynamics (see [TT]) it is possible to study the dynamics of agent densities. A theoretical approach to wound-healing models can be found in [SM]. Mogilner *et al.* ([MDC]) compare micro-and macroscopic swarming models while Czirók *et al.* ([CDW]) focus on a comparison of microscopic swarming models. Recently, mesoscopic models have been introduced ([Pf], [AP], [Lu]). These models are of Boltzmann–type, i.e. they are motivated by a statistical description of one or more agents. However, the microscopic basis of most of the meso– and macroscopic models remains unclear. Furthermore, a couple of questions is still unanswered: What is the macroscopic limit for the microscopic swarming models? What is the influence of space in orientation–based models?

Here we introduce a Boltzmann–type model based on a well-defined microscopic "majority-choice" interaction. In the next section the model is defined. The only parameters in the model are agent density ϱ and a sensitivity $\gamma > 0$. In Section 2 we analyze the space–free case and prove existence of a phase-transition behavior. In Sections 3 and 4 an existence theory and the existence of a Lyapunov functional, respectively, are proven. In Section 5 the relevance of our results for the biological applications and the relation of our model to other swarming models are discussed.

2. Mathematical Model

We start with a brief introduction of the discrete (with respect to the orientation) model describing both migration and interaction. We consider the one–dimensional case (for simplicity).

Let $\varrho = \varrho(t, j, x)$ be the probability that an agent is at time t at point x with orientation j, where $t > 0$, $x \in \mathbb{R}$ and $j \in \{-1, 1\}$.

First we consider only migration. The migration process is a deterministic process described by the equation

$$\varrho(t + \Delta t, j, x + j\Delta x) = \varrho(t, j, x) \tag{2.1}$$

Equation (2.1) defines "free–streaming".

As far as the migration and interactions are concerned, we describe the migration together with the changes of orientation (velocity) variable that are described by the interaction operator. We have the following general expression (valid for every migration —

interaction case):

$$\varrho(t+\Delta t, j, x+\Delta x) = \varrho(t,-j,x)\ \mathrm{Prob}\Big(\text{the change of orientation in } \Delta t \ \Big|\ (t,-j,x)\Big)$$
$$+\ \varrho(t,j,x)\ \mathrm{Prob}\Big(\text{no changes of orientation in } \Delta t \ \Big|\ (t,j,x)\Big)\,. \tag{2.2}$$

Different choices of the probabilities, denoted by "Prob" in Eq. (2.2) give rise to different models. For example, assuming that

$$\mathrm{Prob}\Big(\text{the change of orientation in } \Delta t \ \Big|\ (t,j,x)\Big) \ = \ \varrho(t,j,x)\ \Delta t\,, \tag{2.3}$$

one gets, in the limit $\Delta t = \Delta x \to 0$, the Carleman model of the Boltzmann equation (see [PI])

$$\partial_t \varrho(t,j,x) + j\partial_x \varrho(t,j,x) = \big(\varrho(t,-j,x)\big)^2 - \big(\varrho(t,j,x)\big)^2\,. \tag{2.4}$$

On the other hand the choice

$$\begin{aligned} &\mathrm{Prob}\Big(\text{the change of orientation in } \Delta t \ \Big|\ (t,j,x)\Big) \\ &= \left(g\Big(\sum_{k=\pm 1} \varrho(t,k,x)\Big)\varrho^2(t,-j,x) + \frac{\mu_*}{2}\right)\ \Delta t\,, \end{aligned} \tag{2.5}$$

where $\mu_* \geq 0$ is the *turning rate*, and g is a non–negative function, leads, in the limit $\Delta t = \Delta x \to 0$, to Lutscher's [Lu] alignment model ((7) in [Lu])

$$\begin{aligned} &\partial_t \varrho(t,j,x) + j\partial_x \varrho(t,j,x) \\ &= \left(g\Big(\sum_{k=\pm 1} \varrho(t,k,x)\Big)\varrho(t,j,x)\varrho(t,-j,x) - \frac{\mu_*}{2}\right)\Big(\varrho(t,j,x) - \varrho(t,-j,x)\Big)\,. \end{aligned} \tag{2.6}$$

In this paper we start with

$$\begin{aligned} &\mathrm{Prob}\Big(\text{the change of orientation in } \Delta t \ \Big|\ (t,j,x)\Big) \ = \\ &= \frac{\chi\Big(\sum\limits_{k,l=\pm 1} \varrho(t,k,x+al) > 0\Big)\Big(\sum\limits_{l=\pm 1} \varrho(t,-j,x+al)\Big)^\gamma}{\Big(\sum\limits_{l=\pm 1} \varrho(t,-j,x+al)\Big)^\gamma + \Big(\sum\limits_{l=\pm} \varrho(t,j,x+al)\Big)^\gamma}\ \Delta t\,, \end{aligned} \tag{2.7}$$

where $a > 0$ and $\gamma > 0$ are parameters and $\sum\limits_{l=\pm 1} \varrho(t,j,x+al)$ is the neighbourhood density in direction j, $\chi(\text{true}) = 1$, $\chi(\text{false}) = 0$. The parameter γ describes the sensitivity of interaction. If γ is small (close to 0) then the probability of a change of orientation only weakly depends on the actual orientations. On the other hand for γ large the probability of a change of orientation strongly depends on the actual orientations.

Considering the migration and interaction model, assuming that $a = \Delta x$ together with (2.7), in the limit $\Delta t = \Delta x \to 0$, we obtain

$$\begin{aligned} \partial_t \varrho(t,j,x) + j\partial_x \varrho(t,j,x) &= \frac{\chi\big(\sum\limits_{k=\pm 1} \varrho(t,k,x) > 0\big)}{\big(\varrho(t,-j,x)\big)^\gamma + \big(\varrho(t,j,x)\big)^\gamma} \\ &\times \Big(\varrho(t,-j,x)\big(\varrho(t,j,x)\big)^\gamma - \varrho(t,j,x)\varrho(t,-j,x)\big)^\gamma\Big) . \end{aligned} \tag{2.8}$$

Note some similarities of Eq. (2.8) for $\gamma = 2$ and Lutscher's equation (2.6) for $\mu_* = 0$.

On the other hand, considering the migration and interaction model, assuming that a remains fixed, in the limit $\Delta t = \Delta x \to 0$, we obtain

$$\begin{aligned} \partial_t \varrho(t,j,x) + j\partial_x \varrho(t,j,x) = &\frac{\chi\big(\sum\limits_{k,l=\pm 1} \varrho(t,k,x+al) > 0\big)}{\Big(\sum\limits_{l=\pm 1} \varrho(t,-j,x+al)\Big)^\gamma + \Big(\sum\limits_{l=\pm} \varrho(t,j,x+al)\Big)^\gamma} \\ \times \Bigg(\varrho(t,-j,x)\Big(\sum_{l=\pm 1} \varrho(t,j,x+al)\Big)^\gamma &- \varrho(t,j,x)\Big(\sum_{l=\pm 1} \varrho(t,-j,x+al)\Big)^\gamma\Bigg) . \end{aligned} \tag{2.9}$$

Finally, we may consider the following "*averaged in the neighbourhood*" model

$$\begin{aligned} \partial_t \varrho(t,j,x) + j\partial_x \varrho(t,j,x) = &\frac{\chi\big(\sum\limits_{k=\pm 1} \int\limits_{x-a}^{x+a} \varrho(t,k,y)\,\mathrm{d}y > 0\big)}{\Big(\int\limits_{x-a}^{x+a} \varrho(t,-j,y)\,\mathrm{d}y\Big)^\gamma + \Big(\int\limits_{x-a}^{x+a} \varrho(t,j,y)\,\mathrm{d}y\Big)^\gamma} \\ \times \Bigg(\varrho(t,-j,x)\Big(\int_{x-a}^{x+a} \varrho(t,j,y)\,\mathrm{d}y\Big)^\gamma &- \varrho(t,j,x)\Big(\int_{x-a}^{x+a} \varrho(t,-j,y)\,\mathrm{d}y\Big)^\gamma\Bigg) . \end{aligned} \tag{2.10}$$

We are going to study Eq. (2.8). Note that for $\gamma = 1$ the model reduces to the free–streaming only model. Thus we consider $\gamma \neq 1$. We are interested in possible *phase transitions* in the model.

Consider now the space–homogeneous (i.e. x–independent) version of Eq. (2.8). We write $\varrho_j = \varrho(t,j)$.

The (non–negative) equilibrium solution $\bar{\varrho}$ corresponding to the space homogeneous version of Eq. (2.8) is defined by

$$\bar{\varrho}_{-1}\Big(\bar{\varrho}_1\Big)^\gamma - \bar{\varrho}_1\Big(\bar{\varrho}_{-1}\Big)^\gamma = 0 . \tag{2.11}$$

It is easy to see that Eq. (2.11) is satisfied if and only if either

$$\bar{\varrho}_1 = \bar{\varrho}_{-1} , \tag{2.12}$$

or

$$\bar{\varrho}_j = 0\,, \quad \text{and } \bar{\varrho}_{-j} > 0\,, \qquad \text{for some } j:\; j = 1 \text{ or } j = -1 \tag{2.13}$$

holds.

Equation (2.12) corresponds to equal probabilities of both orientations (*a diffusive picture*), whereas Eq. (2.13) does to the *aligned picture*.

In the space–homogeneous case it is easy to see that the trajectories of the corresponding ODE's are contained in the straight lines defined by

$$\varrho_{-1} + \varrho_1 = c\,, \qquad \text{where } c = \varrho_{-1}(0) + \varrho_1(0) > 0\,, \tag{2.14}$$

and have different types of behavior according to the value of γ.

For $\gamma > 1$

$$\lim_{t\to\infty} \varrho_{-1} = 0\,, \quad \lim_{t\to\infty} \varrho_1 = \varrho_1(0) + \varrho_{-1}(0)\,, \quad \text{for } \varrho_1(0) > \varrho_{-1}(0)\,, \tag{2.15}$$

$$\lim_{t\to\infty} \varrho_{-1} = \varrho_1(0) + \varrho_{-1}(0)\,, \quad \lim_{t\to\infty} \varrho_1 = 0\,, \quad \text{for } \varrho_1(0) < \varrho_{-1}(0)\,, \tag{2.16}$$

whereas for $0 < \gamma < 1$ we have

$$\lim_{t\to\infty} \varrho_1 = \lim_{t\to\infty} \varrho_{-1} = \frac{\varrho_1(0) + \varrho_{-1}(0)}{2}\,. \tag{2.17}$$

In the spatially–nonhomogeneous case simple solutions of Eq. (2.8), when both the left and right hand side terms are equal to zero, are given by

$$\varrho(t, j, x) = \varrho(t, -j, x) = \text{const} \geq 0\,, \tag{2.18}$$

or

$$\varrho(t, j, x) = 0\,, \qquad \varrho(t, -j, x) = \phi(x + j\,t)\,, \tag{2.19}$$

for some j: $j = 1$ or $j = -1$ and a given non–negative function ϕ.

3. Existence

In this section we consider the problem of the global existence and uniqueness results in an appropriate Banach space for $\gamma > 1$. Here we are dealing with the case $x \in \mathbb{R}$, but similar results are possible for $x \in \mathbb{R}^d$, or $x \in \mathbb{T}^d$, where $d \geq 1$ and $\mathbb{T}^d$ is a d–dimensional torus (corresponding to the periodic boundary conditions).

Let X_1 be the space of real–valued functions equipped with the norm

$$\|\, f \,\| = \sum_{j=\pm 1} \int_{\mathbb{R}} |f(j, x)|\, \mathrm{d}x$$

The cone of non–negative functions in X_1 is denoted by X_1^+. We define the following operators

$$Q[f] = Q^+[f] - Q^-[f]\,, \tag{3.1}$$

where

$$Q^+[f](j,x) = \chi\Big(\sum_{k=\pm 1} f(k,x) > 0\Big)\frac{f^\gamma(j,x)f(-j,x)}{f^\gamma(j,x)+f^\gamma(-j,x)}\,,$$

and

$$Q^-[f](j,x) = \chi\Big(\sum_{k=\pm 1} f(k,x) > 0\Big)\frac{f(j,x)f^\gamma(-j,x)}{f^\gamma(j,x)+f^\gamma(-j,x)}\,.$$

Note that, for $f > 0$,

$$Q[f] = R[f] - f\,, \tag{3.2}$$

where

$$R[f](j,x) = f^\gamma(j,x)\frac{f(1,x)+f(-1,x)}{f^\gamma(1,x)+f^\gamma(-1,x)}\,.$$

Consider the following function

$$F(x,y) = \begin{cases} \dfrac{x^\gamma y}{x^\gamma + y^\gamma} & \text{for } x > 0\,,\ y > 0 \\ 0 & \text{otherwise}\,, \end{cases} \tag{3.3}$$

defined for $\gamma > 1$. It easy to see that

$$0 \le F(x,y) \le |y|\,, \tag{3.4}$$

and that the function F is differentiable in $\mathfrak{A} = \mathbb{R}^2 \setminus [0,\infty[\,\times\{0\}$. Its first order derivatives are continuous and uniformly bounded in $\mathfrak{A}$. These properties lead to the conclusion that the operator Q is Lipschitz continuous in the cone X_1^+.

Therefore, the classical results from the non–linear semigroups theory (see [I1] and [I2]) can be applied. In fact, the operator

$$Af(j,x) = -j\partial_x f(j,x) \tag{3.5}$$

is an infinitesimal generator of a linear contraction semigroup $\{\exp(tA)\}_{t\ge 0}$ on X_1 (see [Pa]) that is invariant on X_1^+, and the nonlinear operator Q is locally Lipschitz continuous such that

$$Q[0] = 0\,, \tag{3.6a}$$

$$\lambda f + Q[f] \in X_1^+\,, \qquad \forall\, f \in X_1^+\,, \quad \forall\, \lambda > 1\,, \tag{3.6b}$$

$$\lambda\|f\| \le \|\lambda f - Q[f]\|\,, \qquad \forall\, f \in X_1^+\,, \quad \forall\, \lambda > 0\,. \tag{3.6c}$$

The statement (3.6b) follows by (3.2), whereas (3.6c) by

$$\lambda\|f\| = \lambda\sum_{j=\pm 1}\int_{\mathbb{R}} f(j,x)\,\mathrm{d}x = \sum_{j=\pm 1}\int_{\mathbb{R}}\Big(\lambda f(j,x) - Q[f](j,x)\Big)\,\mathrm{d}x\,, \qquad f \in X_1^+\,. \tag{3.7}$$

Hence, $A+Q\,:\,D(A)\cap X_1^+ \to X_1$ is the infinitesimal generator of a non–linear semigroup of type B on X_1^+ (see [I1] and [Db]).

Consider now the following Cauchy problem in X_1

$$\begin{aligned} \frac{\mathrm{d}\,f}{\mathrm{d}\,t} &= Af + Q[f]\,, \qquad t > 0\,, \\ f\Big|_{t=0} &= f_0\,; \end{aligned} \tag{3.8}$$

and its integral ("*mild*") version

$$f(t) = \exp\big(tA\big) f_0 + \int\limits_0^t \exp\Big((t-s)A\Big) Q[f](s)\,\mathrm{d}s\,, \quad t > 0\,. \tag{3.9}$$

With the notation

$$f^\sharp(t,j,x) = f(t,j,x+jt) \tag{3.10}$$

the other integral version of (3.8) is

$$f^\sharp(t) = f_0 + \int\limits_0^t \Big(Q[f]\Big)^\sharp(s)\,\mathrm{d}s\,, \quad t > 0\,. \tag{3.11}$$

The following global theorem holds (cf. [I1] and [Db])

Theorem 3.1. *For each $f_0 \in X_1^+$ and each $T > 0$ there exists a unique solution f in $C^0\big([0,T];X_1\big)$ of Eq. (3.9). For each $t > 0$ the solution satisfies*

$$f^\sharp \in C^1\big(]0,t[\,;X_1\big)\,, \tag{3.12}$$

and

$$f(t) \in X_1^+\,, \qquad \|f(t)\| = \|f_0\|\,. \tag{3.13}$$

■

In the case of $0 < \gamma < 1$ one finds some difficulties related to possible behavior of solutions close to the value 0. In this case an existence (without uniqueness) result is possible. However we do not deal with this problem here.

4. Entropy

We consider the entropy functional defined by

$$\mathcal{E}[f](t) = -\sum_{j=\pm 1} \int\limits_{\mathbb{R}} f(t,j,x)\log f(t,j,x)\,\mathrm{d}x\,. \tag{4.1}$$

We will show that if $\gamma > 1$ and f is a solution of Eq. (2.8) then the entropy $\mathcal{E}[f]$ is a decreasing function of time. Note that in the case $0 < \gamma < 1$ the corresponding entropy is formally increasing function. Here we consider only the case $\gamma > 1$.

Theorem 4.1. *Let $f_0 \in X_1^+$ and $f_0 \log f_0 \in X_1$. Then the unique solution f in $C^0\big([0,T];X_1\big)$ of Eq. (2.8) is such that $\mathcal{E}[f]$ is a decreasing function of $t > 0$.*

Proof. Let

$$g(t,j,x) = e^t f(t,j,x)\,, \tag{4.2}$$

where f is the solution of Eq. (3.9) given by Theorem 3.1 (corresponding to the intial datum $f_0 \in X_1^+$). Clearly we have

$$\frac{\mathrm{d}}{\mathrm{d}\,t} g^\sharp = e^t \Big(\frac{\mathrm{d}}{\mathrm{d}\,t} f^\sharp + f^\sharp \Big) = e^t \big(R[f]\big)^\sharp = \big(R[g]\big)^\sharp\,. \tag{4.3}$$

Now we consider the function

$$G(x,y) = \begin{cases} \dfrac{x^\gamma (x+y)}{x^\gamma + y^\gamma} & \text{for } x>0\,,\ y>0 \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.4}$$

For $x > 0$, $y > 0$ we obtain

$$\frac{\partial G}{\partial y}(x,y) = \frac{x^{2\gamma} + (1-\gamma)x^\gamma y^\gamma - \gamma x^{\gamma+1} y^{\gamma-1}}{(x^\gamma + y^\gamma)^2}\,,$$

and putting $y = \xi x$ yields

$$\frac{\partial G}{\partial y}(x,\xi x) = \frac{1 + (1-\gamma)\xi^\gamma - \gamma \xi^{\gamma-1}}{(1+\xi^\gamma)^2}\,.$$

It is easy to see that there exists $0 < \xi_0 < 1$ such that $\frac{\partial G}{\partial y}(x,\xi x)$ is positive for $0 < \xi < \xi_0$ and negative for $\xi > \xi_0$. In other words for fixed $x > 0$ and for all y one has $0 \le G(x,y) \le G(x,\xi_0 x)$. Hence Eq. (4.3) implies

$$0 \le \frac{\mathrm{d}}{\mathrm{d}\,t} g^\sharp \le M\, g^\sharp\,, \qquad \forall\, t > 0\,, \tag{4.5}$$

where

$$1 < M = \frac{1+\xi_0}{1+\xi_0^\gamma} < 2\,.$$

Therefore

$$f_0 \le g^\sharp(t) \le f_0 e^{M\,t}\,, \qquad \forall\, t > 0\,. \tag{4.6}$$

From inequality (4.6) we obtain

$$|\log g^\sharp(t)| \le |\log f_0| + M\,t\,, \qquad t > 0\,. \tag{4.7}$$

Inequalities (4.6) and (4.7) show that, if $f_0 \in X_1^+$ and $f_0 \log f_0 \in X_1$ then for all $t > 0$, $j = \pm 1$ function $g^\sharp(t, j, \cdot\,) \log g^\sharp(t, j, \cdot\,)$ is Lebesgue integrable on $\mathbb{R}$. Therefore for all $t > 0$, $j = \pm 1$ the function $f^\sharp(t, j, \cdot\,) \log f^\sharp(t, j, \cdot\,)$ is Lebesgue integrable on $\mathbb{R}$. Note that

$$\mathcal{E}[f](t) = - \sum_{j=\pm 1} \int_{\mathbb{R}} f^\sharp(t, j, x) \log f^\sharp(t, j, x)\, \mathrm{d}x \,. \tag{4.8}$$

By definition (4.2) we have

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\big(f^\sharp(t) \log f^\sharp(t)\big) = e^{-t}\Big(1 - t + \log g^\sharp(t)\Big)\Big(- g^\sharp(t) + \frac{\mathrm{d}}{\mathrm{d}t} g^\sharp(t)\Big) \,. \tag{4.9}$$

Now combining Eq. (4.3), inequalities (4.5), (4.6) and (4.7) yields

$$\Big|\frac{\mathrm{d}}{\mathrm{d}\,t}\Big(f^\sharp(t) \log f^\sharp(t)\Big)\Big| \leq e^{Mt}(M+1)\Big((1 + t + Mt) f_0 + f_0 |\log f_0|\Big) \,. \tag{4.10}$$

This inequality shows that the derivative $\frac{\mathrm{d}}{\mathrm{d}\,t}\mathcal{E}[f](t)$ exists and can be computed interchanging the derivative with the integral.

We have

$$\sum_{j=\pm 1} \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}\,t}\Big(f \log f\Big)^\sharp (t, j, x)\, \mathrm{d}x = \sum_{j=\pm 1} \int_{\mathbb{R}} \Big(\big(1 + \log f\big)\, Q[f]\Big)^\sharp (t, j, x)\, \mathrm{d}x \,. \tag{4.11}$$

Therefore denoting

$$\hat{f}(t, x) = \frac{f(t, 1, x)\, f(t, -1, x)}{f^\gamma(t, 1, x) + f^\gamma(t, -1, x)} \tag{4.12}$$

we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\,t}\mathcal{E}[f](t) \\ &= \int_{\mathbb{R}} \hat{f}(t, x) \sum_{j=\pm 1} \Big(f^{\gamma-1}(t, -j, x) - f^{\gamma-1}(t, j, x)\Big) \log f(t, j, x)\, \mathrm{d}x \\ &= \int_{\mathbb{R}} \hat{f}(t, x)\Big(f^{\gamma-1}(t, -1, x) - f^{\gamma-1}(t, 1, x)\Big) \log \frac{f(t, 1, x)}{f(t, -1, x)}\, \mathrm{d}x \\ &= \frac{1}{\gamma - 1} \int_{\mathbb{R}} \hat{f}(t, x)\Big(f^{\gamma-1}(t, -1, x) - f^{\gamma-1}(t, 1, x)\Big) \log \frac{f^{\gamma-1}(t, 1, x)}{f^{\gamma-1}(t, -1, x)}\, \mathrm{d}x \,. \end{aligned} \tag{4.13}$$

Thus applying the elementary inequality

$$\big(y - x\big) \log \frac{x}{y} \leq 0 \,, \qquad x, y \geq 0 \,, \tag{4.14}$$

we conclude that $\mathcal{E}[f]$ is a decreasing function of time $t > 0$.

■

5. Discussion

In the present paper we provided mathematical results for the model defined by Eq. (2.8). In the space independent case the full characterization of the asymptotic behaviour was presented. For $\gamma > 1$ the highest initial distribution of orientations decides on the asymptotic behaviour (*asymptotic alignment*), whereas for $0 < \gamma < 1$ the *asymptotic diffusive picture* is verified. The case of space–dependent distributions is more difficult. The question of asymptotic behavior remains open. However, for $\gamma > 1$, we have provided the global existence result (taking advantage of the Lipschitz continuity and conservativity properties in the X_1 space setting). Moreover we have proved the existence of a Lyapunov functional which can be considered as a first step towards the complete asymptotic theory in the space dependent case. The case $0 < \gamma < 1$ is much more complex even as the existence theory concerns. In fact for distributions approaching 0 the Lipschitz continuity is not verified any more. In this case however a weak existence result is possible or alternatively the existence result (like in Theorem 3.1) for $x \in \mathbb{T}^d$ and for the initial data separate from 0:

$$f_0 \geq c_0 > 0\,, \tag{5.1}$$

where c_0 is a constant. The details are left to the reader.

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