Network Science: Diffusion, Percolation, Tipping Points, Contagion and Cascades

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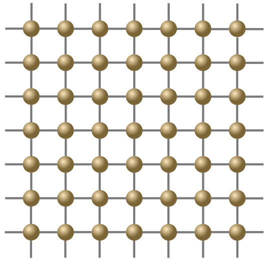
Diffusion

- Viral spread of diseases, information, ideas — *simple* diffusion (contagion)
- Spread of new technologies, behaviors, opinions, fads, fashion — *complex* diffusion (peer-effects)
- Choices, decisions — *games* on networks (cascades)

Spatial networks

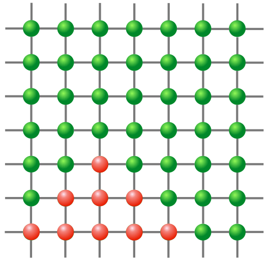
- Implicit networks that arise due to geographic proximity
- Nodes: individuals, edges: physical proximity
- Similar to Kleinberg's small-world model: each node connected to its four compass neighbors



Simple diffusion in spatial networks

- Spread of forest fire in a two-dimensional grid
- Single "density" parameter sets the probability of a grid position being "forest" or "empty"
- Fire starts at a random grid position and spreads to all neighbors
- Observe the "spread" of diffusion (fire) as a function of forestation density

Simple diffusion in spatial networks



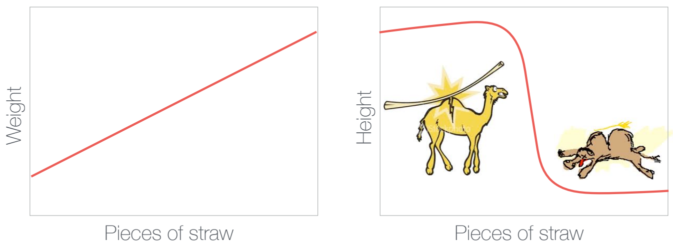
- The density parameter p is the probability that a grid position is forest (empty with probability $(1-p)$)
- Fire will spread to all nodes in the connected component of the network containing the source node
- Run Library/Earth Science/Fire

Tipping phenomenon

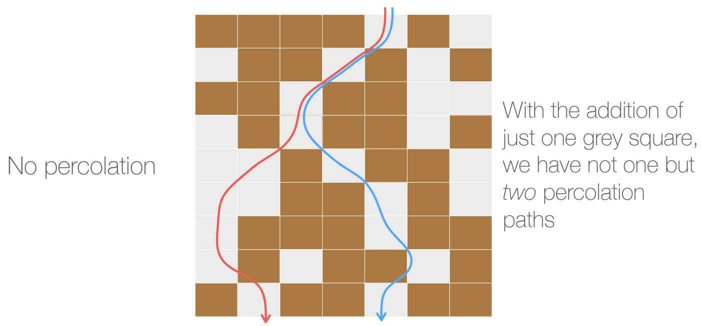
- An abrupt change occurs as the density increases from 57% to 62% — the percentage of burnt forest suddenly increases from small values to almost 100%
- Sudden, massive increase in diffusion is a “tipping” phenomenon (also known as “threshold” or “critical” phenomena)
- Similar to the formation of a giant component in the ER model as the edge probability is increased

Tipping phenomenon

- “The last straw that broke the camel’s back”
- Non-linear relation with a discontinuity
- What property of the camel exhibits a tipping phenomenon?



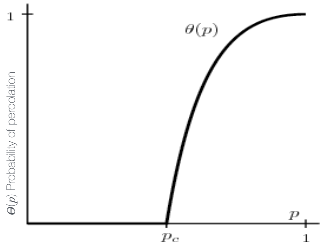
Percolation



- Can we get from *top* to *bottom* touching only grey squares?
- Analogy to water “percolating” through coffee grinds or oil seeping into the ground

Percolation

- Percolation depends on the “density” of the coffee grinds
- Random model where each square is grey (empty) with probability p and brown (coffee) with probability $1-p$



- NetLogo Library/Earth Science/Fire and Percolation demos

Percolation

- Related to the formation of giant components in random networks
- “Shape” of the component and not its size
- Model for “breakthrough” developments in different fields
  - All of the pieces have to “fall in the right place”
  - When a missing piece “falls in place”, the breakthrough may be enabled in several different ways

Simple Diffusion (Contagion) in networks

- ***Diffusion*** or ***contagion*** can be formulated on *any* arbitrary network (not just grids)
- Susceptible-Infected (SI) model
- Population divided into two groups
  - Susceptible (S)
  - Infected (I)
- To make the model more realistic, we can add a parameter “infection rate” which is the probability of disease spreading from an infected node to a susceptible node
- NetLogo ERDiffusion (random network)
- NetLogo BADiffusion (preferential attachment network)

Simple diffusion in networks SIS model

- Susceptible-Infected-Susceptible (SIS) model
- After being infected, individuals remain susceptible
- Appropriate for modeling recurring diseases
- NetLogo “SmallWorldDiffusionSIS” (small-world network)

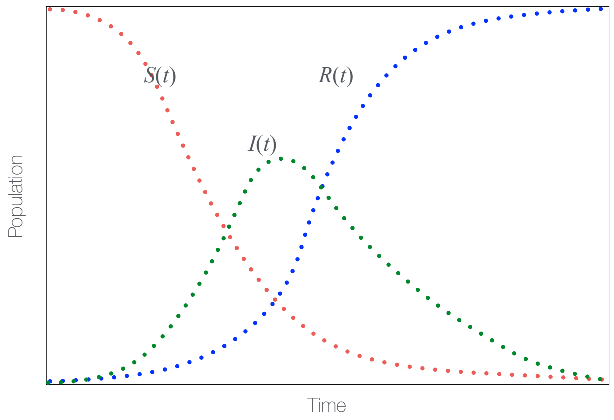
Simple diffusion in networks
SIR model

- Susceptible-Infected-Resistant (SIR) model
- Allows resistance or immunity to be gained after infection
- Add a new "Recovered" state and in addition to the infection rate, add a new parameter "recovery rate"
- We can also add a parameter "gain resistance" if infection does not guarantee resistance with certainty
- NetLogo "Library/Networks/Virus on a Network"

Simple diffusion in networks
SIR model

- Two-parameter SIR model:
  - a: infection rate
  - b: recovery rate
- Population divided into three groups:
  - Let $S(t)$ be the number of susceptible at time t
  - Let $I(t)$ be the number of infected at time t
  - Let $R(t)$ be the number of recovered at time t
- $\frac{dS}{dt} = -aSI$
- $\frac{dI}{dt} = aSI - bI = I(aS - b)$
- $\frac{dR}{dt} = bI$

Simple diffusion in networks
SIR model



Simple diffusion in networks
SIR model

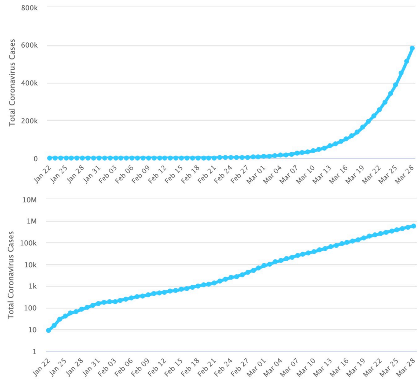
- Assume $S(t)$ is constant around t=0
- We can solve the differential equation $\frac{dI}{dt} = I(aS_0 - b)$
- $I(t) = e^{(aS_0 - b)t}$
- In other words, if $aS_0 - b > 0$ the infected population will start to grow ***exponentially*** and we will have an *epidemic*

Simple diffusion in networks
SIR model

- Rewrite $aS_0 - b > 0$ as $\frac{aS_0}{b} > 1$
- $R = \frac{aS_0}{b}$ is a critical parameter in epidemiology
  - If $R > 1$, the infected population will start to grow exponentially leading to an epidemic
  - If $R < 1$, the infected population will extinguish
- We can decrease R by
  - decreasing the infection rate a (washing hands, social distancing),
  - by decreasing the initial susceptible population S_0 (vaccination),
  - by increasing the recovery rate b (better health care, usually difficult)

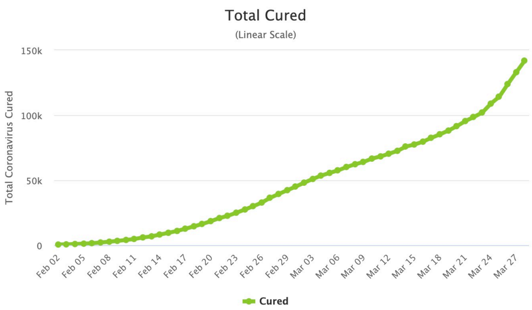
Simple diffusion in networks
SIR model

- Covid-19 total infections as of 28 March 2020



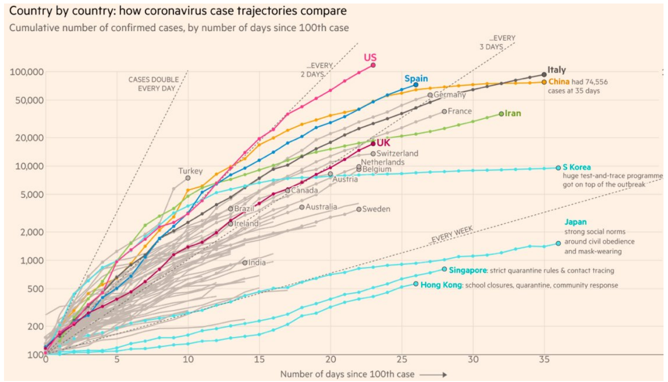
Simple diffusion in networks
SIR model

- Covid-19 total recovered as of 28 March 2020



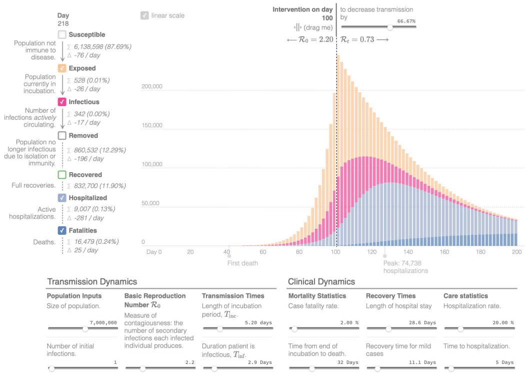
Simple diffusion in networks
SIR model

- Covid-19 confirmed cases as of 26 March 2020



Simple diffusion in networks
SEIR model

- **S**usceptible → **E**xposed → **I**nfected → **R**emoved



Simple diffusion in networks
SEIR model

$$\frac{dS}{dt} = -\frac{\mathcal{R}_t}{T_{\text{inf}}} \cdot IS, \quad \frac{dE}{dt} = \frac{\mathcal{R}_t}{T_{\text{inf}}} \cdot IS - T_{\text{inc}}^{-1}E, \quad \frac{dI}{dt} = T_{\text{inc}}^{-1}E - T_{\text{inf}}^{-1}I, \quad \frac{dR}{dt} = T_{\text{inf}}^{-1}I$$

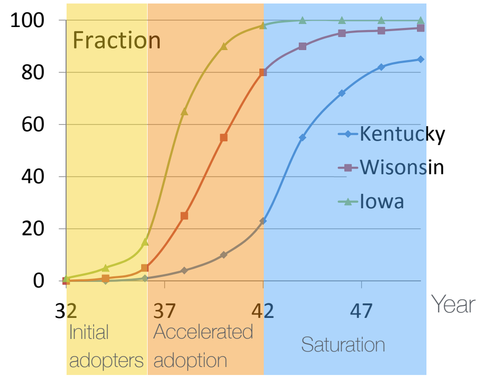
- Epidemic Calculator
- http://gabgoh.github.io/COVID/index.html

Diffusion in networks
Peer-effects

- The "infection rate" is often not a constant but depends on how many nodes are already infected
- Captures the notion of "peer-effects" (or "network-effects")— we are more likely to adopt the behavior or choices of our peers
- The greater the number of my peers who dress, talk, walk, eat or vote one way, the higher the probability that I will dress, talk, walk, eat or vote that way too

Peer-effects
S-shape curves

Adoption rate of hybrid corn in three states



Peer-effects
Bass model

- No explicit network
- Two states/behaviors: 0 and 1
- States are irreversible — cannot switch back and forth
- Let $F(t)$ denote the fraction of population who have adopted state 1 at time t
- Let p denote the rate of *spontaneous* adoption
- Let q denote the rate of *peer-effect* adoption
- How does the fraction of adoption vary with time?

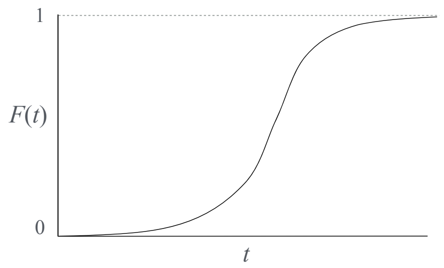
$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = (p + qF(t))(1 - F(t))$$

Peer-effects
Bass model

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = (p + qF(t))(1 - F(t))$$

- When $F(t)$ nears 1, $\mathrm{d}F(t)/\mathrm{d}t$ nears 0
- When $F(t)=0$, $\mathrm{d}F(t)/\mathrm{d}t=p$
- When $F(t)=\varepsilon$ for some ε, $\mathrm{d}F(t)/\mathrm{d}t=(p+q\varepsilon)(1-\varepsilon)$
- To get initial convexity, $(p+q\varepsilon)(1-\varepsilon)$ must be greater than p
- Or $q(1-\varepsilon)>p$ and for small ε, this is equivalent to $q > p$
- Thus, we get the "S-shape" if $q > p$

Peer-effects
Bass model



where

$$F(t) = \frac{(1 - e^{-(p+q)t})}{(1 + \frac{qe^{-(p+q)t}}{p})}$$

Complex diffusion in networks

- *Simple diffusion*: just one node sufficient to "infect" another node (with some fixed probability)
- *Peer-effects diffusion*: probability of infection depends on number of nodes already infected
- *Complex diffusion*: a node being "infected" depends on choices or decisions made by the node
- Appropriate for modeling adoption of new technologies, behaviors, opinions, fashion trends, etc.

Complex diffusion in networks

- *Example*: Suppose a node will adopt a behavior (get a tattoo) only if two of its neighbors have adopted the behavior
- NetLogo "SmallWorldDiffusionComplex" (adopt if two neighbors share opinion)
- In general, the decision to adopt or not could be much more complex
- Can be modeled as a "network coordination game"

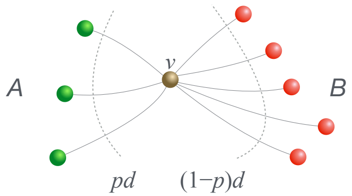
Network coordination game

- Two possible behaviors: *A* and *B*
- Two nodes that are neighbors have an incentive to adopt the same behavior
  - Units of measure (Metric vs Imperial)
  - Sports (Basketball vs Soccer)
  - Social networking (Instagram vs TikTok)
- Payoff matrix

	A	**B**
A	a, a	$0, 0$
B	$0, 0$	b, b

Network coordination game

- Each node plays a copy of the game with each of its neighbors
- The *utility* of a node is the sum of the payoffs obtained from the individual games
- Consider a node v with d neighbors of which a fraction p have chosen *A*



Network coordination game

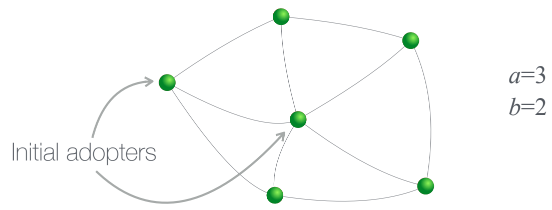
- If node v chooses *A*, its utility will be pda
- If it chooses *B*, its utility will be $(1-p)db$
- Thus, *A* is a better choice if $pda \geq (1-p)db$ or $p \geq b/(a+b)$
- In other words, if at least $b/(a+b)$ fraction of v's neighbors chose *A*, then v should chose *A* as well
  - if $b/(a+b)$ is small, *A* is the more attractive choice
  - if $b/(a+b)$ is large, *B* is the more attractive choice

Cascading behavior

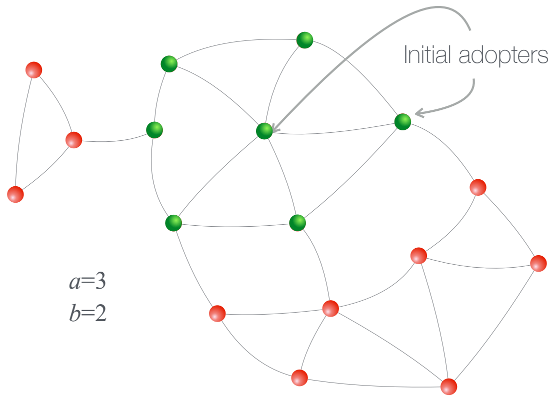
- There are two obvious equilibria: everyone chooses *A*, or everyone chooses *B*
- Suppose that the network is in the second equilibrium: everyone has chosen *B*
- Can the network be "tipped over" to the other equilibrium by flipping the choices of a small number of "initial adopter" nodes?
- Answer depends on the network structure, the ratio $b/(a+b)$ and the choice of initial adopters
- Initial adopters switch for reasons external to the game
- The other nodes continue to play the coordination game

Cascading behavior

- Chain reaction of switches from decision *B* to decision *A* is called a *cascade*
- Cascades can be either *complete* — the entire network eventually switches to the other decision — or they can be *partial*



Cascading behavior



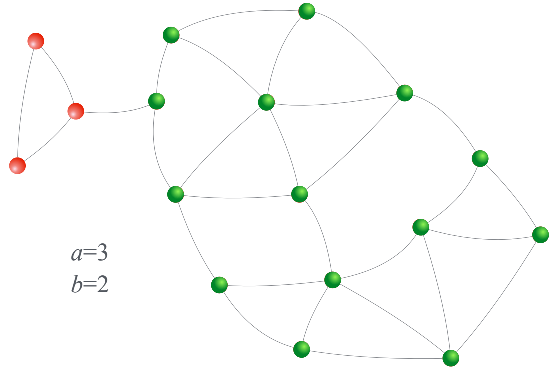
Cascading behavior
Application to viral marketing

- Partial cascades result in situations in which two (or more) decisions coexist
- Suppose *A* and *B* are two competing products or technologies and the manufacturer of *A* wants to dominate the market (obtain a complete cascade)
- Manufacturer of *A* has two possible strategies:
  - Make its product more "competitive" (increase a)
  - Pick very carefully the set of its initial adopters
  - Usually, modifying the network structure is *not* an option

Cascading behavior
Application to viral marketing

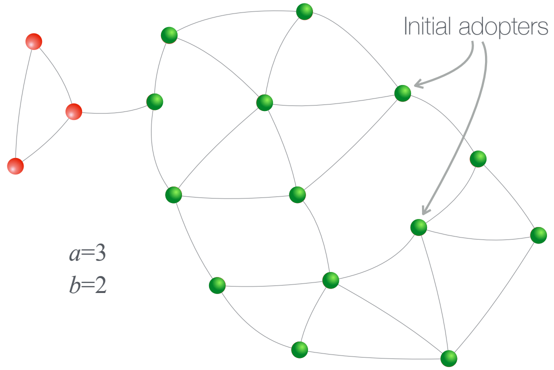
- Pursuing the first strategy, suppose the manufacturer of *A* is able to increase a from 3 to 4 (while b remains at 2)
- The threshold for adopting *A* reduces from 2/5 to 1/3
- In the example, the adoption of *A* becomes a complete cascade
- Pursuing the second strategy, suppose the manufacturer of *A* is able to convince two additional nodes

Cascading behavior
Application to viral marketing



- Convince two additional nodes

Cascading behavior
Application to viral marketing



- Or, pick a different set of initial adopters