An introduction to Linear Logic

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1 october 2010

Consider, instead of true formulas of arithmetic, valid formulas of predicate calculus.

We shall introduce Gentzen's Sequent Calculus.

$$\Gamma \vdash \Delta$$

Identity

$$A \vdash A$$

Structural Rules:

$$\frac{\Gamma \vdash \Sigma, A, B, \Sigma_2}{\Gamma \vdash \Sigma, B, A, \Sigma_2}\,(\vdash E) \qquad \frac{\Gamma, A, B, \Gamma_2 \vdash \Sigma}{\Gamma, B, A, \Gamma_2 \vdash \Sigma}\,(E \vdash)$$

$$\frac{\Gamma \vdash \Sigma}{\Gamma, A \vdash \Sigma}\,(W \vdash) \qquad \frac{\Gamma \vdash \Sigma}{\Gamma \vdash A, \Sigma}\,(\vdash W)$$

$$\frac{\Gamma, A, A \vdash \Sigma}{\Gamma, A \vdash \Sigma}\,(C \vdash) \qquad \frac{\Gamma \vdash A, A, \Sigma}{\Gamma \vdash A, \Sigma}\,(\vdash C)$$

Logical Rules

$$\frac{\Gamma, A \vdash B, \Sigma}{\Gamma \vdash A \rightarrow B, \Sigma}\ (\vdash\rightarrow) \qquad \frac{\Gamma, B \vdash \Sigma \qquad \Lambda \vdash A, \Delta}{\Gamma, \Lambda, A \rightarrow B \vdash \Sigma, \Delta}\ (\rightarrow\vdash)$$

$$\frac{\Gamma \vdash A, \Sigma}{\Gamma \vdash \forall x.A, \Sigma}\ (\vdash \forall)^1 \qquad \frac{\Gamma, A(t) \vdash \Sigma}{\Gamma, \forall x.A(x) \vdash \Sigma}\ (\forall \vdash)^2$$

$$\frac{\Gamma \vdash A(t), \Sigma}{\Gamma \vdash \exists x.A(x), \Sigma}\ (\vdash \exists)^2 \qquad \frac{\Gamma, A \vdash \Sigma}{\Gamma, \exists x.A(x) \vdash \Sigma}\ (\exists \vdash)^1$$

[1] $x \notin FV(\Gamma, \Sigma)$.

[2] t arbitrary term.

(Logical Rules)

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\vdash \wedge) \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge \vdash) \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge \vdash)$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee \vdash) \quad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vdash \vee) \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vdash \vee)$$

$$\frac{\Gamma, A \vdash \Sigma}{\Gamma \vdash \neg A, \Sigma}\ (\vdash \neg) \qquad \frac{\Gamma \vdash A, \Sigma}{\Gamma, \neg A \vdash \Sigma}\ (\neg \vdash)$$

Cut rule (redundant):

$$\frac{\Gamma \vdash A, \Delta \qquad \Lambda, A \vdash \Sigma}{\Gamma, \Lambda \vdash \Delta, \Sigma}\ (cut)$$

We could economize and use less rules:

$$\frac{\Gamma \vdash A, \Sigma \qquad \Lambda \vdash B, \Delta}{\Gamma, \Lambda, \vdash A \wedge B, \Sigma, \Delta} (\vdash \wedge)' \qquad \frac{\Gamma, A, B \vdash \Sigma}{\Gamma, A \wedge B \vdash \Sigma} (\wedge \vdash)'$$

$$\frac{\Gamma, A \vdash \Sigma \qquad \Lambda, B \vdash \Delta}{\Gamma, \Lambda, A \vee B \vdash \Sigma, \Delta} (\vee \vdash)' \qquad \frac{\Gamma \vdash A, B, \Sigma}{\Gamma \vdash A \vee B, \Sigma} (\vdash \vee)'$$

and internalize the meaning of "," (commas) on the left and on the right.

Are the two formulations equal? YES!

The first way to introduce roule is called additive and the last one is called multiplicative.

Classical $\wedge$ and $\vee$ are now splitted in two.

- $\otimes$ (multiplicative) and & (additive) are the two conjunctions.

$$\frac{\Gamma \vdash A, \Sigma \qquad \Lambda \vdash B, \Delta}{\Gamma, \Lambda, \vdash A \otimes B, \Sigma, \Delta} (\vdash \otimes) \qquad \frac{\Gamma, A, B \vdash \Sigma}{\Gamma, A \otimes B \vdash \Sigma} (\otimes \vdash)$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta} (\vdash \&) \; \frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta} (\& \vdash) \; \frac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash \Delta} (\& \vdash)$$

- $⅋$ (multiplicative) and $\oplus$ (additive) are the two disjuctions.

$$\frac{\Gamma, A \vdash \Sigma \qquad \Lambda, B \vdash \Delta}{\Gamma, \Lambda, A ⅋ B \vdash \Sigma, \Delta} (⅋ \vdash) \qquad \frac{\Gamma \vdash A, B, \Sigma}{\Gamma \vdash A ⅋ B, \Sigma} (\vdash ⅋)$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} (\oplus \vdash) \; \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} (\vdash \oplus) \; \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta} (\vdash \oplus)$$

Here all the rules.

$$\frac{}{\vdash A^{\perp}, A}\ (\text{Axiom}) \qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (\text{cut})$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ (\text{right plus}) \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ (\text{left plus})$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ (\text{with}) \qquad \frac{}{\vdash \top, \Gamma}\ (\top) \quad \text{No rule for } 0$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ (\text{times}) \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ (\text{par})$$

$$\frac{}{\vdash 1}\ (1) \qquad \frac{\vdash \Gamma}{\vdash \perp, \Gamma}\ (\perp)$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \forall \alpha.A}\ (\text{for all}) \qquad \frac{\vdash \Gamma, A[t/x]}{\vdash \Gamma, \exists x.A}\ (\text{there is})$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ (\text{weakening}) \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash ?\Gamma, A}\ (\text{contraction})$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ (\text{deleriction}) \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ (\text{of course})$$

- **Additive Conjunction**: the symbol for this connective is & and it's called with. It rappresents the idea of simultaneity of resources, that cannot be occour in the same time. Suppose to have money and have to choose one of three object of the same price. It is not possible to choose more then one: $Object1 \,\&\, (Oobject2 \,\&\, Oobject3)$.
- **Additive Disjunction**: the symbol of this connective is $\oplus$ and it's called plus. It rappresents the idea of simultaneity of resources, but with a difference. You cannot know which of the option will be choose. It's like in the previous example, where the choice is made by someone else: $Object1 \oplus (Oobject2 \oplus Oobject3)$.

- **Multiplicative Conjunction**: the symbol is $\otimes$ and it's called tensor. It rappresents the "so called" computational parallelism. Suppose to have money needful to buy two object, and suppose to buy all of them: $Object1 \otimes Object2$.
- **Multiplicative Disjuction**: the symbol is $⅋$ and it's called par (parallelization). We read $A ⅋ B$ as "if not A, then B". It can be understood better if we consider the linear implication. $A \multimap B$ ($A^{\perp} ⅋ B$) it means B can be derived by using only and just one time A.

- **Exponential**: In Linear Logic we have two modal operators. The first one is "!", called bang, and rappresents the possibility to re-use or duplicate a resource. The second is "?", called why not, and rappresents the dual of bang. It hold, indeed, the equality $(?A)^{\perp} = !(A^{\perp})$.

As the $\wedge$ and $\vee$ splits, also the neutral elements (n.e.) splits.

- 1 is the n.e. for $\otimes$
- $\top$ is the n.e. for $\&$
- $\perp$ is the n.e. for $\parr$
- 0 is the n.e. for $\oplus$

Consider A,B,C,D,.. as the set of atomic formulas. Are formulas of LL:

- neutral elemements: $1, 0, \bot, \top$
- atomic formulas: $A, B, C, D, ...$
- negation of atomic formulas: $A^\bot, B^\bot, C^\bot, D^\bot, ...$
- the result of applied connectives. If X,Y are formulas of LL then also $X \& Y, X ⅋ Y, X \otimes Y, \oplus Y, !X, ?X, \forall\alpha X, \exists\alpha X, \exists\alpha Y$ are formulas of LL.

Linear negation: We have used, previously, the linear negation. Here's the definition:

- $1^\perp = \perp$ e $\perp^\perp = 1$
- $0^\perp = \top$ e $\top^\perp = 0$
- $(A \otimes B)^\perp = A^\perp ⅋ B^\perp$
- $(A ⅋ B)^\perp = A^\perp \otimes B^\perp$
- $(A \oplus B)^\perp = A^\perp \& B^\perp$
- $(A \& B)^\perp = A^\perp \oplus B^\perp$
- $(\forall\alpha.A)^\perp = \exists\alpha.A^\perp$
- $(\exists\alpha.A)^\perp = \forall\alpha.A^\perp$

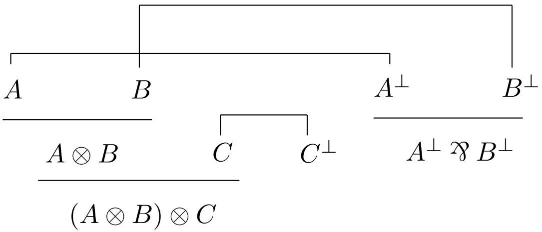
N.B.: The negation of a multiplicative connective remains multiplicative and the same happens with the additive ones.

As we have seen, we can divide the rules in sets: multiplicative, additive and exponential.
We can define some sub-linear logic as:

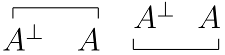
- MLL: Multiplicative Linear Logic.
- MELL: Multiplicative Exponential Linear Logic.
- MALL: Multiplicative Additive Linear Logic.

Is it possible to rappresents the Linear Logic in a different way? maybe in a graphical manner?

YES: Proof Nets !



In this kind of net we have two basic links: Axiom link and cut-link

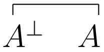


There are also other kind of links: Times link and Par link.

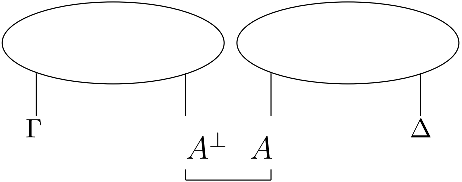
$$\frac{A \qquad\qquad B}{A \otimes B} \qquad \frac{A \qquad\qquad B}{A \parr B}$$

We can associate every proof $\vdash \Gamma$ in MLL with a proof-net, whose conclusions are Γ.

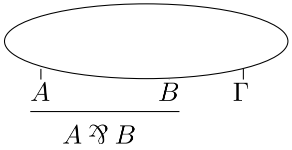
$\vdash A^\perp, A \quad \rightarrow$



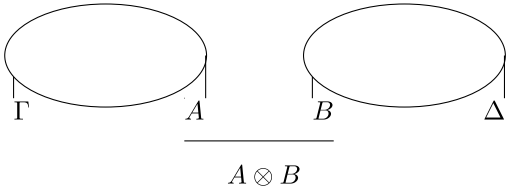
$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta} \quad \rightarrow$$



$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \quad \rightarrow$$



$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, A \otimes B} \quad \rightarrow$$

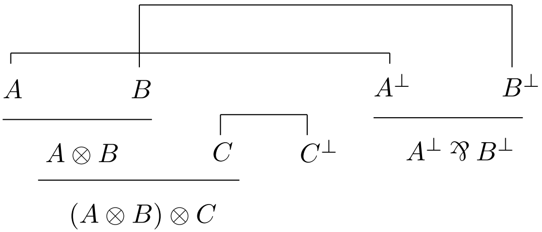


Exchange rule?

Example

$$\dfrac{\dfrac{\dfrac{\vdash A, A^{\perp} \qquad \vdash B, B^{\perp}}{\vdash A^{\perp}, B^{\perp}, A \otimes B} \qquad \vdash C, C^{\perp}}{\vdash C^{\perp}, A^{\perp}, B^{\perp}, (A \otimes B) \otimes C}}{\vdash C^{\perp}, (A^{\perp} \parr B^{\perp}), (A \otimes B) \otimes C}$$

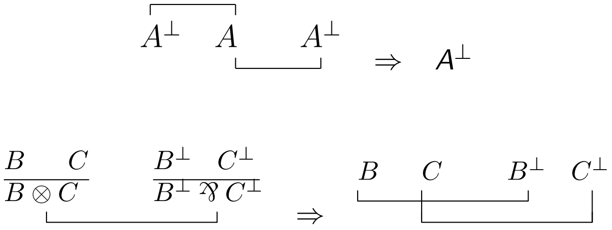
By using the rules just seen we obtain:



Theorem

Given a proof $\vdash A_1, .., A_n$ *in LL we can always build a proof net, whose concluson are* $A_1, .., A_n$.

Cut-elimination on proof net G = semplify the structure of G, so that the number of cut-links go to zero.



Theorem

Be Π a proof-net with a d nodes. The execution time of cut-elimination is $O(d)$