Linearity: syntax vs. semantics

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- Coherence Spaces has been introduced by Jean-Yves Girard after a fine analysis of stable semantics: stable functions were decomposed in linear functions and exponential. Such a decomposition is patently reflected in linear logic syntax.
- In the context of programming languages, linearity were quickly adopted, at first to eliminate garbage collection and shortly thereafter to handle mutable state.
- Variants, refinements, and improvements on linear type systems have been proposed for many applications, including explicit memory management and control of aliasing, capabilities, tracking state changes for program analysis, typestates for well-behaving API calls, and session types of a channel use agreement.

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Linearity Classification

1. Syntactical linearity claims a linear use of variables in terms.

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4. Operational linearity claims that redexes are not duplicated during the operational evaluation.
5. Denotational linearity claims that programs correspond to linear functions between domains.

Stably-Linear Model: Functions vs. Programs

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- The starting point is the least full subcategory of coherence spaces endowed with linear functions as morphisms and including as object infinite flat domain and closed under linear-function spaces.

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- The starting point is the least full subcategory of coherence spaces endowed with linear functions as morphisms and including as object infinite flat domain and closed under linear-function spaces.

- We avoided the use of exponential domain constructors, thus it should be clear that the considered linear model is not correct (w.r.t. standard interpretation) for a wide number of languages inspired to linear logic.

- We study PCF-like languages able to program only functions of such a purely linear model.

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All stable endofunctions in $\iota \to \iota$ are linear, but the constant-functions (having trace $\{(\emptyset, n) \mid n \in \mathbb{N}\}$).

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All stable endofunctions in $\iota \to \iota$ are linear, but the constant-functions (having trace $\{(\emptyset, n) \mid n \in \mathbb{N}\}$).

As instance, $\{(n, \underbrace{n + \cdots + n}_{n}) | n \in \mathbb{N}\}$,

$$\{\ (n, \underbrace{n * \cdots * n}_{n}) | n \in \mathbb{N}\}$$

and $\{\ (n, \underbrace{n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}}_{n}\ ) | n \in \mathbb{N}\}$ are linear traces.

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In order to represent all functions in $\mathbf{N} \multimap \mathbf{N}$, $\mathcal{S}\ell\mathrm{PCF}_{\mathrm{tk}}$ does not put syntactical-linear constraints on occurrences of ground variables.

High-Order Moral

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Note that this liberality on the management of ground variables entails that we may also use a high-order term many times, provided that we apply it always to the same sequence of arguments.

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Take into account,

$$\lambda \mathtt{F}^{\sigma_0 \to \ldots \to \sigma_k \to \iota}.(\lambda \mathtt{x}^{\iota}.\mathtt{x}\ \mathtt{Op}\ \mathtt{x})(\mathtt{F}\mathtt{M}_1^{\sigma_0} \ldots \mathtt{M}_k^{\sigma_k})^{\iota}\ .$$

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Take into account,

$$\lambda \mathtt{F}^{\sigma_0\to\ldots\to\sigma_\mathtt{k}\to\iota}.(\lambda \mathtt{x}^{\iota}.\mathtt{x}\ \mathtt{Op}\ \mathtt{x})(\mathtt{F}\mathtt{M}_1^{\sigma_0}\ldots\mathtt{M}_\mathtt{k}^{\sigma_\mathtt{k}})^{\iota}\ .$$

The previous term is expected to be equivalent to

$$\lambda \mathtt{F}^{\sigma_0\to\ldots\to\sigma_\mathtt{k}\to\iota}.(\mathtt{F}\mathtt{M}_1^{\sigma_0}\ldots\mathtt{M}_\mathtt{k}^{\sigma_\mathtt{k}})\ \mathtt{Op}\ (\mathtt{F}\mathtt{M}_1^{\sigma_0}\ldots\mathtt{M}_\mathtt{k}^{\sigma_\mathtt{k}})$$

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Take into account,

$$\lambda \mathtt{F}^{\sigma_0 \to \ldots \to \sigma_k \to \iota}.(\lambda \mathtt{x}^{\iota}.\mathtt{x}\ \mathtt{Op}\ \mathtt{x})(\mathtt{F}\mathtt{M}_1^{\sigma_0} \ldots \mathtt{M}_k^{\sigma_k})^{\iota}\ .$$

The previous term is expected to be equivalent to

$$\lambda \mathtt{F}^{\sigma_0 \to \ldots \to \sigma_k \to \iota}.(\mathtt{F}\mathtt{M}_1^{\sigma_0} \ldots \mathtt{M}_k^{\sigma_k})\ \mathtt{Op}\ (\mathtt{F}\mathtt{M}_1^{\sigma_0} \ldots \mathtt{M}_k^{\sigma_k})$$

Now, if $\mathtt{N}_i^{\sigma_i}$ is the evaluation of $\mathtt{M}_i^{\sigma_i}$ then the previous term is expected to be equivalent to

$$\lambda \mathtt{F}^{\sigma_0 \to \ldots \to \sigma_k \to \iota}.(\mathtt{F}\mathtt{N}_1^{\sigma_0} \ldots \mathtt{N}_k^{\sigma_k})\ \mathtt{Op}\ (\mathtt{F}\mathtt{M}_1^{\sigma_0} \ldots \mathtt{M}_k^{\sigma_k})$$

Strictness & Fixpoints

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All linear functions are strict, in case:

- Ground Variables: call-by-value parameter passing.
- High-Order Variables: syntactical linearity.

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All linear functions are strict, in case:

- Ground Variables: call-by-value parameter passing.
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In order to add all first-order strict stable-functions to our linear-language, we need fixpoints. Unfortunately, the least fixpoint of a linear function is always the bottom of the considered domain, because strictness.

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$\mathcal{S}\ell\mathrm{PCF}_{\mathrm{tk}}$ will contain a special kind of variables: stable variables. We don't permit to λ-abstract those variables, they will be used only in order to obtain fixpoints.

Typing the Core of $\mathcal{S}\ell\mathrm{PCF}_{\mathrm{tk}}$

$$\frac{}{\mathtt{x}^{\iota} \vdash \mathtt{x} : \iota}\ \text{(gv)} \qquad \frac{}{\mathtt{f}^{\sigma \multimap \tau} \vdash \mathtt{f} : \sigma \multimap \tau}\ \text{(hv)} \qquad \frac{}{F^{\sigma} \vdash F : \sigma}\ \text{(sv)}$$

Typing the Core of $\mathcal{S}\ell\mathrm{PCF}_{\mathrm{tk}}$

$$\frac{}{\mathtt{x}^{\iota} \vdash \mathtt{x} : \iota}\ \text{(gv)} \qquad \frac{}{\mathtt{f}^{\sigma \multimap \tau} \vdash \mathtt{f} : \sigma \multimap \tau}\ \text{(hv)} \qquad \frac{}{\mathit{F}^{\sigma} \vdash \mathit{F} : \sigma}\ \text{(sv)}$$

$$\frac{\Gamma \vdash \mathtt{M} : \tau}{\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M} : \tau}\ \text{(gw)} \qquad \frac{\Gamma, \mathtt{x_1}^{\iota}, \mathtt{x_2}^{\iota} \vdash \mathtt{M} : \tau}{\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M}[\mathtt{x}/\mathtt{x_1}, \mathtt{x_2}] : \tau}\ \text{(gc)} \qquad \frac{\Gamma, \varkappa_2^{\sigma_2}, \varkappa_1^{\sigma_1}, \Delta \vdash \mathtt{M} : \tau}{\Gamma, \varkappa_1^{\sigma_1}, \varkappa_2^{\sigma_2}, \Delta \vdash \mathtt{M} : \tau}\ \text{(ex)}$$

$$\frac{\Gamma, \mathit{F}_1^{\sigma}, \mathit{F}_2^{\sigma} \vdash \mathtt{M} : \tau}{\Gamma, \mathit{F}^{\sigma} \vdash \mathtt{M}[\mathit{F}/\mathit{F}_1, \mathit{F}_2] : \tau}\ \text{(sc)} \qquad \frac{\Gamma \vdash \mathtt{M} : \tau}{\Gamma, \mathit{F}^{\sigma} \vdash \mathtt{M} : \tau}\ \text{(sw)}$$

Typing the Core of $\mathcal{S}\ell\mathrm{PCF}_{\mathrm{tk}}$

$$\frac{}{\mathtt{x}^{\iota} \vdash \mathtt{x} : \iota}\ (\mathsf{gv}) \qquad \frac{}{\mathtt{f}^{\sigma \multimap \tau} \vdash \mathtt{f} : \sigma \multimap \tau}\ (\mathsf{hv}) \qquad \frac{}{F^{\sigma} \vdash F : \sigma}\ (\mathsf{sv})$$

$$\frac{\Gamma \vdash \mathtt{M} : \tau}{\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M} : \tau}\ (\mathsf{gw}) \qquad \frac{\Gamma, \mathtt{x_1}^{\iota}, \mathtt{x_2}^{\iota} \vdash \mathtt{M} : \tau}{\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M}[\mathtt{x}/\mathtt{x_1}, \mathtt{x_2}] : \tau}\ (\mathsf{gc}) \qquad \frac{\Gamma, \varkappa_2^{\sigma_2}, \varkappa_1^{\sigma_1}, \Delta \vdash \mathtt{M} : \tau}{\Gamma, \varkappa_1^{\sigma_1}, \varkappa_2^{\sigma_2}, \Delta \vdash \mathtt{M} : \tau}\ (\mathsf{ex})$$

$$\frac{\Gamma, F_1^{\sigma}, F_2^{\sigma} \vdash \mathtt{M} : \tau}{\Gamma, F^{\sigma} \vdash \mathtt{M}[F/F_1, F_2] : \tau}\ (\mathsf{sc}) \qquad \frac{\Gamma \vdash \mathtt{M} : \tau}{\Gamma, F^{\sigma} \vdash \mathtt{M} : \tau}\ (\mathsf{sw})$$

$$\frac{\Gamma, \mathtt{x}^{\sigma} \vdash \mathtt{M} : \tau}{\Gamma \vdash \lambda \mathtt{x}^{\sigma}.\mathtt{M} : \sigma \multimap \tau}\ (\lambda) \qquad \frac{\Gamma \cap \Delta = \emptyset \quad \Gamma \vdash \mathtt{M} : \sigma \multimap \tau \quad \Delta \vdash \mathtt{N} : \sigma}{\Gamma, \Delta \vdash \mathtt{MN} : \tau}\ (\mathsf{ap})$$

$$\frac{}{\vdash \underline{\mathtt{0}} : \iota}\ (\mathsf{z}) \qquad \frac{}{\vdash \mathtt{succ} : \iota \multimap \iota}\ (\mathsf{s}) \qquad \frac{}{\vdash \mathtt{pred} : \iota \multimap \iota}\ (\mathsf{p})$$

$$\frac{\Gamma \cap \Delta = \emptyset \quad \Gamma \vdash \mathtt{M} : \iota \quad \Delta \vdash \mathtt{L} : \iota \quad \Delta \vdash \mathtt{R} : \iota}{\Gamma, \Delta \vdash \ell\mathtt{if}\ \mathtt{M}\ \mathtt{L}\ \mathtt{R} : \iota}\ (\ell\mathtt{if}) \qquad \frac{\Gamma^{\iota}, \Delta^{*}, F^{\sigma} \vdash \mathtt{M} : \sigma}{\Gamma^{\iota}, \Delta^{*} \vdash \mu F.\mathtt{M} : \sigma}\ (\mu)$$

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$$\frac{}{\underline{\mathtt{0}} \Downarrow \underline{\mathtt{0}}}\ (0) \qquad \frac{\mathtt{M} \Downarrow \underline{\mathtt{n}}}{\mathtt{succ}\,\mathtt{M} \Downarrow \mathtt{succ}\,\underline{\mathtt{n}}}\ (\mathsf{s}) \qquad \frac{\mathtt{M} \Downarrow \mathtt{succ}\,\underline{\mathtt{n}}}{\mathtt{pred}\,\mathtt{M} \Downarrow \underline{\mathtt{n}}}\ (\mathsf{p})$$

$$\frac{\mathtt{M} \Downarrow \underline{\mathtt{0}} \quad \mathtt{L} \Downarrow \underline{\mathtt{m}}}{\ell\mathtt{if}\ \mathtt{M}\ \mathtt{L}\ \mathtt{R} \Downarrow \underline{\mathtt{m}}}\ (\mathsf{ifl}) \qquad \frac{\mathtt{M} \Downarrow \mathtt{succ}(\underline{\mathtt{n}}) \quad \mathtt{R} \Downarrow \underline{\mathtt{m}}}{\ell\mathtt{if}\ \mathtt{M}\ \mathtt{L}\ \mathtt{R} \Downarrow \underline{\mathtt{m}}}\ (\mathsf{ifr})$$

$$\frac{\mathtt{N} \Downarrow \underline{\mathtt{m}} \quad \mathtt{M}[\underline{\mathtt{m}}/\mathtt{x}]\mathtt{P}_1 \cdots \mathtt{P}_i \Downarrow \underline{\mathtt{v}}}{(\lambda \mathtt{x}^{\iota}.\mathtt{M})\mathtt{N}\mathtt{P}_1 \cdots \mathtt{P}_i \Downarrow \underline{\mathtt{v}}}\ (\lambda^{\iota}) \qquad \frac{\mathtt{M}[\mathtt{N}/\mathtt{f}]\mathtt{P}_1 \cdots \mathtt{P}_i \Downarrow \underline{\mathtt{v}}}{(\lambda \mathtt{f}^{\sigma \multimap \tau}.\mathtt{M})\mathtt{N}\mathtt{P}_1 \cdots \mathtt{P}_i \Downarrow \underline{\mathtt{v}}}\ (\lambda^{\multimap})$$

$$\frac{\mathtt{M}[\mu F.\mathtt{M}/F]\mathtt{P}_1 \cdots \mathtt{P}_i \Downarrow \underline{\mathtt{v}}}{(\mu F.\mathtt{M})\mathtt{P}_1 \cdots \mathtt{P}_i \Downarrow \underline{\mathtt{v}}}\ (\mu)$$

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If $\mathtt{M}^{(\iota\multimap\iota)\multimap\iota}$ program a linear function, then $\mathtt{M}(\lambda \mathtt{x}^{\iota}.\mathtt{x}) \Downarrow \underline{\mathtt{n}}$ implies that there is $\underline{\mathtt{k}}$ such that

$$\mathtt{M}\Big(\lambda \mathtt{x}^{\iota}.\mathtt{if}\,(\mathtt{x} \doteq \mathtt{k})\;\underline{\mathtt{k}}\;\Omega^{\iota}\Big) \Downarrow \underline{\mathtt{n}}$$

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$$\mathtt{M}\Big(\lambda \mathtt{x}^{\iota}.\mathtt{if}\,(\mathtt{x} \doteq \mathtt{k})\ \underline{\mathtt{k}}\ \Omega^{\iota}\Big) \Downarrow \underline{\mathtt{n}}$$

A further operator is $\mathtt{which?} : ((\iota \multimap \iota) \multimap \iota) \multimap \iota$

$$\frac{}{\mathtt{which?}\,\mathtt{M}^{(\iota\multimap\iota)\multimap\iota} \Downarrow \langle \underline{\mathtt{n}}, \underline{\mathtt{k}}\rangle}\ (\mathtt{w}^{\underline{\mathtt{k}}})$$

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A further operator is $\mathtt{which?} : ((\iota \multimap \iota) \multimap \iota) \multimap \iota$

$$\frac{\mathtt{M}\Big(\lambda \mathtt{x}^{\iota}.\mathtt{if}\,(\underline{\mathtt{k}} \doteq \mathtt{x})\;\underline{\mathtt{k}}\;\Omega^{\iota}\Big) \Downarrow \underline{\mathtt{n}}}{\mathtt{which?}\,\mathtt{M}^{(\iota\multimap\iota)\multimap\iota} \Downarrow \langle \underline{\mathtt{n}}, \underline{\mathtt{k}} \rangle}\;(\mathtt{w}^{\underline{\mathtt{k}}})$$

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$\mathcal{S}\ell\mathrm{PCF}_{\mathrm{tk}}$ was introduced in [Gaboardi and Paolini 2007,Paolini and Piccolo 2008] as the syntactical counterpart of Linear Stable Functions among Coherence Spaces.

- Formalization of a Turing-complete linear language

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- Language assures token-definability

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- Formalization of a Turing-complete linear language
- Language assures token-definability
- Restricted Full Abstraction for Linear Stable Functions

$\mathcal{S}\ell\mathrm{PCF}_{\mathrm{tk}}$ in a Nutshell

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- Formalization of a Turing-complete linear language
- Language assures token-definability
- Restricted Full Abstraction for Linear Stable Functions
- Introduction of novel linear operators

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- Formalization of a Turing-complete linear language
- Language assures token-definability
- Restricted Full Abstraction for Linear Stable Functions
- Introduction of novel linear operators
- New knowledge on higher-type computation

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$$\left\{\begin{array}{c} \big((0,0)\,,\,(1,0)\,,\,(0,1)\,,\,0\big) \\ \big((0,1)\,,\,(0,0)\,,\,(1,0)\,,\,1\big) \\ \big((1,0)\,,\,(0,1)\,,\,(0,0)\,,\,2\big) \\ \big((1,1)\,,\,(1,1)\,,\,(1,1)\,,\,3\big) \end{array}\right\}$$

in $(\iota \multimap \iota) \multimap (\iota \multimap \iota) \multimap (\iota \multimap \iota) \multimap \iota$ is linear,

but it cannot be defined in $\mathcal{S}\ell\mathrm{PCF}_{\mathrm{tk}}$.

$$\left\{\begin{array}{l} \big((0,0)\,,\,(1,0)\,,\,(0,1)\,,\,0\big) \\ \big((0,1)\,,\,(0,0)\,,\,(1,0)\,,\,1\big) \\ \big((1,0)\,,\,(0,1)\,,\,(0,0)\,,\,2\big) \\ \big((1,1)\,,\,(1,1)\,,\,(1,1)\,,\,3\big) \end{array}\right\}$$

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We can add $\overset{2}{\mathtt{Gor}} : (\iota \multimap \iota) \multimap (\iota \multimap \iota) \multimap (\iota \multimap \iota) \multimap \iota$ to ${\mathcal{S}\ell\mathrm{PCF}_{\mathrm{tk}}}^{+}$ with the following operational semantics

$$\left\{\begin{array}{l}((0,0)\,,\,(1,0)\,,\,(0,1)\,,\,0)\\((0,1)\,,\,(0,0)\,,\,(1,0)\,,\,1)\\((1,0)\,,\,(0,1)\,,\,(0,0)\,,\,2)\\((1,1)\,,\,(1,1)\,,\,(1,1)\,,\,3)\end{array}\right\}$$

in $(\iota \multimap \iota) \multimap (\iota \multimap \iota) \multimap (\iota \multimap \iota) \multimap \iota$ is linear,

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$$\frac{\mathtt{P}_1\underline{0} \Downarrow \underline{0} \quad \mathtt{P}_1\underline{1} \Downarrow \underline{0} \quad \mathtt{P}_2\underline{0} \Downarrow \underline{1}}{\overset{2}{\mathtt{Gor}}\ \mathtt{P}_0\mathtt{P}_1\mathtt{P}_2 \Downarrow \underline{0}}\ (\overset{2}{\mathtt{Gor}}_0) \qquad \frac{\mathtt{P}_1\underline{0} \Downarrow \underline{1} \quad \mathtt{P}_1\underline{0} \Downarrow \underline{0} \quad \mathtt{P}_2\underline{1} \Downarrow \underline{0}}{\overset{2}{\mathtt{Gor}}\ \mathtt{P}_0\mathtt{P}_1\mathtt{P}_2 \Downarrow \underline{1}}\ (\overset{2}{\mathtt{Gor}}_1)$$

$$\frac{\mathtt{P}_1\underline{1} \Downarrow \underline{0} \quad \mathtt{P}_1\underline{0} \Downarrow \underline{1} \quad \mathtt{P}_2\underline{0} \Downarrow \underline{0}}{\overset{2}{\mathtt{Gor}}\ \mathtt{P}_0\mathtt{P}_1\mathtt{P}_2 \Downarrow \underline{2}}\ (\overset{2}{\mathtt{Gor}}_2) \qquad \frac{\mathtt{P}_1\underline{1} \Downarrow \underline{1} \quad \mathtt{P}_1\underline{1} \Downarrow \underline{1} \quad \mathtt{P}_2\underline{1} \Downarrow \underline{1}}{\overset{2}{\mathtt{Gor}}\ \mathtt{P}_0\mathtt{P}_1\mathtt{P}_2 \Downarrow \underline{3}}\ (\overset{2}{\mathtt{Gor}}_3)$$

Tokens are Trees of Integers

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Tokens are Trees of Integers respecting the type-trees!

Tokens are Trees of Integers

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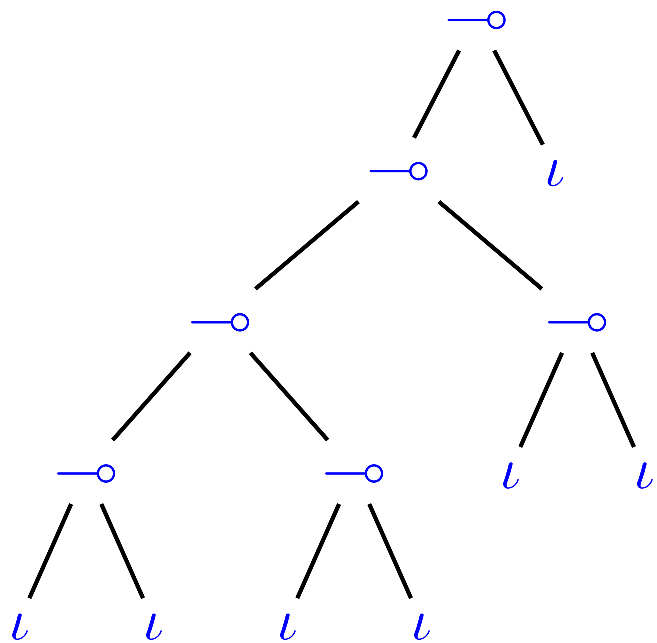
Tokens are Trees of Integers respecting the type-trees!

As instance, the first token of $\overset{2}{\mathtt{Gor}}$ is $((0,0)\,,\,(1,0)\,,\,(0,1)\,,\,0)$



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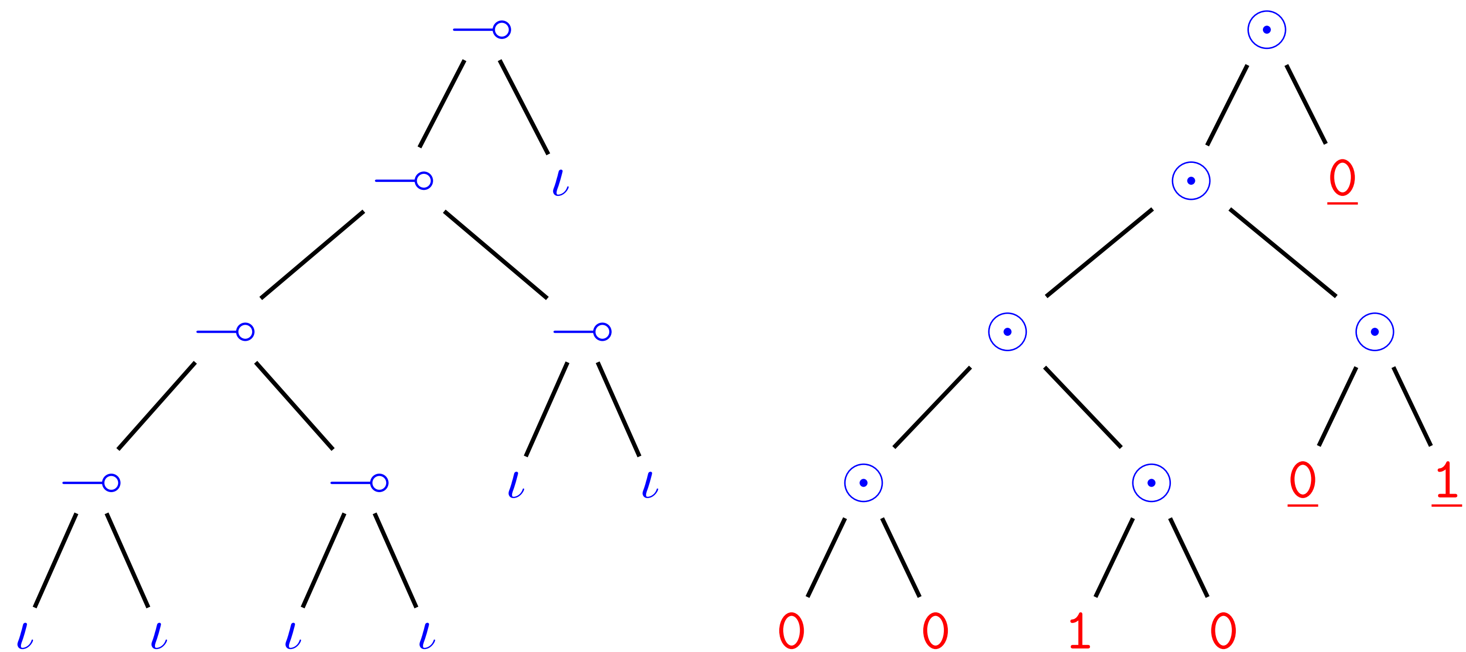

Token Embedding

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$\mathtt{Tkn}^{\sigma}_{\mathtt{n}} : \sigma$ and $\mathtt{Chk}^{(\sigma)}_{\mathtt{n}} : \sigma \multimap \iota$ are defined by mutual induction on σ.

- case $\sigma = \iota$.

$$\mathtt{Tkn}^{\iota}_{\mathtt{n}} = \underline{\mathtt{n}} \qquad \mathtt{Chk}^{(\iota)}_{\mathtt{n}} = \lambda \mathtt{y}^{\iota}.\ell\mathtt{if}\ \underline{\mathtt{n}} \doteq \mathtt{y}\ \underline{\mathtt{0}}\ \Omega^{\iota}$$

- case $\sigma = \sigma_1 \multimap \sigma_2$ (where $\sigma_2 = \tau_1 \multimap \ldots \multimap \tau_k \multimap \iota$).

$$\begin{aligned}\mathtt{Tkn}^{\sigma}_{\mathtt{n}} &= \lambda \mathtt{f}^{\sigma_1}.\lambda \mathtt{g}_1^{\tau_1} \ldots \lambda \mathtt{g}_{\mathtt{k}}^{\tau_{\mathtt{k}}}.\\ &\quad \ell\mathtt{if}\ (\mathtt{Chk}^{(\sigma_1)}_{\pi_1(\mathtt{n})}\ \mathtt{f})\ (\mathtt{Tkn}^{\sigma_2}_{\pi_2(\mathtt{n})}\mathtt{g}_1 \ldots \mathtt{g}_{\mathtt{k}})\ (\Omega^{\sigma_2}\mathtt{g}_1 \ldots \mathtt{g}_{\mathtt{k}})\end{aligned}$$

$$\mathtt{Chk}^{(\sigma)}_{\mathtt{n}} = \lambda \mathtt{f}^{\sigma}.\ell\mathtt{if}\ (\mathtt{Chk}^{(\sigma_2)}_{\pi_2(\mathtt{n})}(\mathtt{f}\ \mathtt{Tkn}^{(\sigma_1)}_{\pi_1(\mathtt{n})}))\ \underline{\mathtt{0}}\ \Omega^{\iota}$$

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Let $a \in |[\![\sigma_1 \multimap \ldots \sigma_m \multimap \iota]\!]|$ and let $n = (\!|(a_1, \ldots, a_m, k)|\!)$.

1. $[\![\mathtt{Tkn}^{(\sigma)}(\underline{\mathtt{n}})]\!]\rho = \{(a_1, \ldots, a_m, k)\}$

2. If $\mathtt{N}^\sigma \in \mathcal{S}\ell\mathrm{PCF}_{\mathrm{tk}}$ and $[\![\mathtt{Chk}^{(\sigma)}_{\underline{\mathtt{n}}}\ \mathtt{N}]\!]\rho = [\![\underline{\mathtt{0}}]\!]\rho$

   then $(a_1, \ldots, a_m, k) \in [\![\mathtt{N}]\!]\rho$

Linear Non-Determinism: $\ell\mathtt{et}\text{-}\ell\mathtt{or}$

$$\frac{\Delta\restriction_{\ell} = \mathtt{f}_1^{\sigma_1},\ldots,\mathtt{f}_k^{\sigma_k} \quad \Gamma_1 \vdash \mathtt{N}_1 : \sigma_1 \;\ldots\; \Gamma_k \vdash \mathtt{N}_k : \sigma_k \quad \Delta \vdash \mathtt{M}_i : \iota \; \left(i\in\{1,2,3\}\right)}{\Gamma_1,\ldots,\Gamma_k, \Delta\restriction_{\mathbb{S}}, \Delta\restriction_{\iota} \vdash \underline{\ell\mathtt{et}}\,\mathtt{f}_1 = \mathtt{N}_1,\ldots,\mathtt{f}_k = \mathtt{N}_k\,\underline{\mathtt{in}\ell\mathtt{or}}\;\mathtt{M}_1\;\mathtt{M}_2\;\mathtt{M}_3 : \iota}\;(\ell\mathtt{et}\text{-}\ell\mathtt{or})$$

Linear Non-Determinism: $\ell\texttt{et-}\ell\texttt{or}$

$$\frac{\Delta{\restriction_{\ell}} = \mathtt{f}_1^{\sigma_1}, \ldots, \mathtt{f}_\mathtt{k}^{\sigma_\mathtt{k}} \quad \Gamma_1 \vdash \mathtt{N}_1 : \sigma_1 \ \ \ldots \ \ \Gamma_k \vdash \mathtt{N}_\mathtt{k} : \sigma_k \quad \Delta \vdash \mathtt{M}_\mathtt{i} : \iota \ \big(i \in \{1,2,3\}\big)}{\Gamma_1, \ldots, \Gamma_k, \Delta{\restriction_{\mathbb{S}}}, \Delta{\restriction_{\iota}} \vdash \underline{\ell\mathtt{et}}\,\mathtt{f}_1 = \mathtt{N}_1, \ldots, \mathtt{f}_\mathtt{k} = \mathtt{N}_\mathtt{k}\, \underline{\mathtt{in}\ell\mathtt{or}}\ \mathtt{M}_1\ \mathtt{M}_2\ \mathtt{M}_3 : \iota} \ (\ell\mathtt{et}\text{-}\ell\mathtt{or})$$

$$\frac{\begin{array}{lll} \exists \mathtt{n}_1 \ldots \mathtt{n}_\mathtt{k} & \mathtt{M}_1[\mathtt{Tkn}_{\mathtt{n}_1}^{\sigma_1}/\mathtt{f}_1, \ldots, \mathtt{Tkn}_{\mathtt{n}_\mathtt{k}}^{\sigma_\mathtt{k}}/\mathtt{f}_\mathtt{k}] \Downarrow \underline{\mathtt{0}} & \big(j \in \{1,\ldots,k\}\big) \\ & \mathtt{M}_2[\mathtt{Tkn}_{\mathtt{n}_1}^{\sigma_1}/\mathtt{f}_1, \ldots, \mathtt{Tkn}_{\mathtt{n}_\mathtt{k}}^{\sigma_\mathtt{k}}/\mathtt{f}_\mathtt{k}] \Downarrow \mathtt{succ}\,\underline{\mathtt{m}} & \mathtt{Chk}_{\mathtt{n}_\mathtt{j}}^{(\sigma_\mathtt{k})}\mathtt{N}_\mathtt{j} \Downarrow \underline{\mathtt{0}} \end{array}}{\underline{\ell\mathtt{et}}\,\mathtt{f}_1^{\sigma_1} = \mathtt{N}_1, \ldots, \mathtt{f}_\mathtt{k}^{\sigma_\mathtt{k}} = \mathtt{N}_\mathtt{k}\, \underline{\mathtt{in}\ell\mathtt{or}}\ \mathtt{M}_1\ \mathtt{M}_2\ \mathtt{M}_3 \Downarrow \underline{\mathtt{m}}} \ (\mathtt{1lgor})$$

Linear Non-Determinism: $\ell\mathtt{et}$-$\ell\mathtt{or}$

$$\frac{\Delta{\restriction_\ell} = \mathtt{f}_1^{\sigma_1},\ldots,\mathtt{f}_k^{\sigma_k} \quad \Gamma_1 \vdash \mathtt{N}_1 : \sigma_1 \;\ldots\; \Gamma_k \vdash \mathtt{N}_k : \sigma_k \quad \Delta \vdash \mathtt{M}_i : \iota \;(i{\in}\{1,2,3\})}{\Gamma_1,\ldots,\Gamma_k,\Delta{\restriction_{\mathbb{S}}},\Delta{\restriction_\iota} \vdash \underline{\ell\mathtt{et}}\,\mathtt{f}_1 = \mathtt{N}_1,\ldots,\mathtt{f}_k = \mathtt{N}_k\,\underline{\mathtt{in}\ell\mathtt{or}}\;\mathtt{M}_1\;\mathtt{M}_2\;\mathtt{M}_3 : \iota}\;(\ell\mathtt{et}\text{-}\ell\mathtt{or})$$

$$\frac{\begin{array}{lll}\exists \mathtt{n}_1\ldots\mathtt{n}_k & \mathtt{M}_1[\mathtt{Tkn}_{\mathtt{n}_1}^{\sigma_1}/\mathtt{f}_1,\ldots,\mathtt{Tkn}_{\mathtt{n}_k}^{\sigma_k}/\mathtt{f}_k] \Downarrow \underline{0} & (j{\in}\{1,\ldots,k\}) \\ & \mathtt{M}_2[\mathtt{Tkn}_{\mathtt{n}_1}^{\sigma_1}/\mathtt{f}_1,\ldots,\mathtt{Tkn}_{\mathtt{n}_k}^{\sigma_k}/\mathtt{f}_k] \Downarrow \mathtt{succ}\,\underline{\mathtt{m}} & \mathtt{Chk}_{\mathtt{n}_j}^{(\sigma_k)}\mathtt{N}_j \Downarrow \underline{0}\end{array}}{\underline{\ell\mathtt{et}}\,\mathtt{f}_1^{\sigma_1} = \mathtt{N}_1,\ldots,\mathtt{f}_k^{\sigma_k} = \mathtt{N}_k\,\underline{\mathtt{in}\ell\mathtt{or}}\;\mathtt{M}_1\;\mathtt{M}_2\;\mathtt{M}_3 \Downarrow \underline{\mathtt{m}}}\;(1\mathtt{lgor})$$

$$\frac{\begin{array}{lll}\exists \mathtt{n}_1\ldots\mathtt{n}_k & \mathtt{M}_2[\mathtt{Tkn}_{\mathtt{n}_1}^{\sigma_1}/\mathtt{f}_1,\ldots,\mathtt{Tkn}_{\mathtt{n}_k}^{\sigma_k}/\mathtt{f}_k] \Downarrow \underline{0} & (j{\in}\{1,\ldots,k\}) \\ & \mathtt{M}_3[\mathtt{Tkn}_{\mathtt{n}_1}^{\sigma_1}/\mathtt{f}_1,\ldots,\mathtt{Tkn}_{\mathtt{n}_k}^{\sigma_k}/\mathtt{f}_k] \Downarrow \mathtt{succ}\,\underline{\mathtt{m}} & \mathtt{Chk}_{\mathtt{n}_j}^{(\sigma_k)}\mathtt{N}_j \Downarrow \underline{0}\end{array}}{\underline{\ell\mathtt{et}}\,\mathtt{f}_1^{\sigma_1} = \mathtt{N}_1,\ldots,\mathtt{f}_k^{\sigma_k} = \mathtt{N}_k\,\underline{\mathtt{in}\ell\mathtt{or}}\;\mathtt{M}_1\;\mathtt{M}_2\;\mathtt{M}_3 \Downarrow \underline{\mathtt{m}}}\;(2\mathtt{lgor})$$

$$\frac{\begin{array}{lll}\exists \mathtt{n}_1\ldots\mathtt{n}_k & \mathtt{M}_3[\mathtt{Tkn}_{\mathtt{n}_1}^{\sigma_1}/\mathtt{f}_1,\ldots,\mathtt{Tkn}_{\mathtt{n}_k}^{\sigma_k}/\mathtt{f}_k] \Downarrow \underline{0} & (j{\in}\{1,\ldots,k\}) \\ & \mathtt{M}_1[\mathtt{Tkn}_{\mathtt{n}_1}^{\sigma_1}/\mathtt{f}_1,\ldots,\mathtt{Tkn}_{\mathtt{n}_k}^{\sigma_k}/\mathtt{f}_k] \Downarrow \mathtt{succ}\,\underline{\mathtt{m}} & \mathtt{Chk}_{\mathtt{n}_j}^{(\sigma_k)}\mathtt{N}_j \Downarrow \underline{0}\end{array}}{\underline{\ell\mathtt{et}}\,\mathtt{f}_1^{\sigma_1} = \mathtt{N}_1,\ldots,\mathtt{f}_k^{\sigma_k} = \mathtt{N}_k\,\underline{\mathtt{in}\ell\mathtt{or}}\;\mathtt{M}_1\;\mathtt{M}_2\;\mathtt{M}_3 \Downarrow \underline{\mathtt{m}}}\;(3\mathtt{lgor})$$

Pre-orders Coincidence

Let $\mathtt{x_1}^{\iota}, \ldots, \mathtt{x_n}^{\iota}, \mathtt{f_1}^{\tau_1}, \ldots, \mathtt{f_m}^{\tau_m}, F_1^{\mu_1}, \ldots, F_l^{\mu_l} \vdash \mathtt{M}, \mathtt{N} : \sigma_1 \multimap \ldots \sigma_k \multimap \iota$.

1. $\mathtt{M} \lessapprox_{\sigma} \mathtt{N}$ whenever, for all $C[\cdot^{\sigma}]$ s.t. $C[\mathtt{M}], C[\mathtt{N}] \in \mathcal{P}$,

   if $\mathtt{C}[\mathtt{M}] \Downarrow \underline{\mathtt{n}}$ then $\mathtt{C}[\mathtt{N}] \Downarrow \underline{\mathtt{n}}$.

Pre-orders Coincidence

Let $\mathtt{x_1}^{\iota}, \ldots, \mathtt{x_n}^{\iota}, \mathtt{f_1}^{\tau_1}, \ldots, \mathtt{f_m}^{\tau_m}, \digamma_1^{\mu_1}, \ldots, \digamma_l^{\mu_l} \vdash \mathtt{M}, \mathtt{N} : \sigma_1 \multimap \ldots \sigma_k \multimap \iota$.

1. $\mathtt{M} \lessapprox_\sigma \mathtt{N}$ whenever, for all $C[\cdot^\sigma]$ s.t. $C[\mathtt{M}], C[\mathtt{N}] \in \mathcal{P}$,

   if $\mathtt{C}[\mathtt{M}] \Downarrow \underline{\mathtt{n}}$ then $\mathtt{C}[\mathtt{N}] \Downarrow \underline{\mathtt{n}}$.

2. $\mathtt{M} \lesssim_\sigma \mathtt{N}$ whenever, for all closed term $\mathtt{P}_1^{\sigma_1}, \ldots \mathtt{P}_\mathtt{n}^{\sigma_\mathtt{n}}$, for all $C[\cdot^\sigma]$ s.t. $C[\mathtt{M}[\mathtt{P}_1/\digamma_1, \ldots, \mathtt{P}_\mathtt{n}/\digamma_\mathtt{n}]], C[\mathtt{N}[\mathtt{P}_1/\digamma_1, \ldots, \mathtt{P}_\mathtt{n}/\digamma_\mathtt{n}]] \in \mathcal{P}$,

   if $\mathtt{C}[\mathtt{M}[\mathtt{P}_1/\digamma_1, \ldots, \mathtt{P}_\mathtt{n}/\digamma_\mathtt{n}]] \Downarrow \underline{\mathtt{n}}$ then $\mathtt{C}[\mathtt{N}[\mathtt{P}_1/\digamma_1, \ldots, \mathtt{P}_\mathtt{n}/\digamma_\mathtt{n}]] \Downarrow \underline{\mathtt{n}}$.

Let $\mathtt{x_1}^{\iota}, \ldots, \mathtt{x_n}^{\iota}, \mathtt{f_1}^{\tau_1}, \ldots, \mathtt{f_m}^{\tau_m}, F_1^{\mu_1}, \ldots, F_l^{\mu_l} \vdash \mathtt{M}, \mathtt{N} : \sigma_1 \multimap \ldots \sigma_k \multimap \iota$.

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2. $\mathtt{M} \lesssim_{\sigma} \mathtt{N}$ whenever, for all closed term $\mathtt{P}_1^{\sigma_1}, \ldots \mathtt{P}_n^{\sigma_n}$, for all $C[\cdot^{\sigma}]$ s.t. $C[\mathtt{M}[\mathtt{P}_1/F_1, \ldots, \mathtt{P}_n/F_n]], C[\mathtt{N}[\mathtt{P}_1/F_1, \ldots, \mathtt{P}_n/F_n]] \in \mathcal{P}$,

   if $\mathtt{C}[\mathtt{M}[\mathtt{P}_1/F_1, \ldots, \mathtt{P}_n/F_n]] \Downarrow \underline{\mathtt{n}}$ then $\mathtt{C}[\mathtt{N}[\mathtt{P}_1/F_1, \ldots, \mathtt{P}_n/F_n]] \Downarrow \underline{\mathtt{n}}$.

3. $\mathtt{M} \lesssim_{\sigma}^{\mathtt{A}} \mathtt{N}$ whenever,
   for all closed terms $\mathtt{P}_1^{\iota}, \ldots, \mathtt{P}_n^{\iota}, \mathtt{Q}_1^{\tau_1}, \ldots, \mathtt{Q}_m^{\tau_m}, \mathtt{R}_1^{\mu_1}, \ldots, \mathtt{R}_1^{\mu_l}, \mathtt{L}_1^{\sigma_1} \ldots, \mathtt{L}_k^{\sigma_k}$

   if $(\lambda\vec{\mathtt{x}}.\lambda\vec{\mathtt{f}}.\mathtt{M}[\vec{\mathtt{R}}/\vec{F}])\vec{\mathtt{P}}\vec{\mathtt{Q}}\vec{\mathtt{L}} \Downarrow \underline{\mathtt{p}}$ then $(\lambda\vec{\mathtt{x}}.\lambda\vec{\mathtt{f}}.\mathtt{N}[\vec{\mathtt{R}}/\vec{F}])\vec{\mathtt{P}}\vec{\mathtt{Q}}\vec{\mathtt{L}} \Downarrow \underline{\mathtt{p}}$.

$$\frac{}{\langle \underline{\mathtt{0}} | e \rangle \Mapsto \langle \underline{\mathtt{0}} | e \rangle}\ (0) \qquad \frac{\langle \mathtt{M} | e_0 \rangle \Mapsto \langle \underline{\mathtt{n}} | e_1 \rangle}{\langle \mathtt{succ}\,\mathtt{M} | e_0 \rangle \Mapsto \langle \mathtt{succ}\,\underline{\mathtt{n}} | e_1 \rangle}\ (\mathsf{s}) \qquad \frac{\langle \mathtt{M} | e_0 \rangle \Mapsto \langle \mathtt{succ}\,\underline{\mathtt{n}} | e_1 \rangle}{\langle \mathtt{pred}\,\mathtt{M} | e_0 \rangle \Mapsto \langle \underline{\mathtt{n}} | e_1 \rangle}\ (\mathsf{p})$$

$$\frac{\langle \mathtt{M} | e_0 \rangle \Mapsto \langle \underline{\mathtt{0}} | e_1 \rangle \quad \langle \mathtt{L} | e_1 \rangle \Mapsto \langle \underline{\mathtt{m}} | e_2 \rangle}{\langle \ell\mathtt{if}\ \mathtt{M}\ \mathtt{L}\ \mathtt{R} | e_0 \rangle \Mapsto \langle \underline{\mathtt{m}} | e_2 \rangle}\ (\mathsf{ifl}) \qquad \frac{\langle \mathtt{M} | e_0 \rangle \Mapsto \langle \mathtt{succ}(\underline{\mathtt{n}}) | e_1 \rangle \quad \langle \mathtt{R} | e_1 \rangle \Mapsto \langle \underline{\mathtt{m}} | e_2 \rangle}{\langle \ell\mathtt{if}\ \mathtt{M}\ \mathtt{L}\ \mathtt{R} | e_0 \rangle \Mapsto \langle \underline{\mathtt{m}} | e_2 \rangle}\ (\mathsf{ifr})$$

$$\frac{\langle \mathtt{M}[\mathtt{h}/\mathtt{f}]\mathtt{P}_1 \ldots \mathtt{P}_\mathtt{k} | e_0[\mathtt{h} := \mathtt{N}] \rangle \Mapsto \langle \underline{\mathtt{n}} | e_1 \rangle}{\langle (\lambda \mathtt{f}^{\sigma \multimap \tau}.\mathtt{M})\mathtt{N}\mathtt{P}_1 \ldots \mathtt{P}_\mathtt{k} | e_0 \rangle \Mapsto \langle \underline{\mathtt{n}} | e_1 \restriction_{\{\mathtt{h}\}} \rangle}\ (\lambda^{\multimap})$$

$$\frac{\langle \mathtt{N} | e_0 \rangle \Mapsto \langle \underline{\mathtt{m}} | e_1 \rangle \quad \langle \mathtt{M}[\mathtt{z}/\mathtt{x}]\mathtt{P}_1 \ldots \mathtt{P}_\mathtt{k} | e_1[\mathtt{z} := \underline{\mathtt{m}}] \rangle \Mapsto \langle \underline{\mathtt{n}} | e_2 \rangle}{\langle (\lambda \mathtt{x}^{\iota}.\mathtt{M})\mathtt{N}\mathtt{P}_1 \ldots \mathtt{P}_\mathtt{k} | e_0 \rangle \Mapsto \langle \underline{\mathtt{n}} | e_2 \restriction_{\{\mathtt{z}\}} \rangle}\ (\lambda^{\iota})$$

Constructive Semantics

$$\frac{\langle \mathtt{g}\mathtt{P}_2 \ldots \mathtt{P}_\mathtt{k} | e_1[\mathtt{h} := \mathtt{P}_1, \mathtt{g} := \mathtt{M}[\mathtt{h}/\mathtt{x}]] - \mathtt{f}\rangle \mapsto \langle \underline{\mathtt{n}} | e_2 \rangle}{\langle \mathtt{f}\mathtt{P}_1 \ldots \mathtt{P}_\mathtt{k} | e_0[\mathtt{f} := \lambda \mathtt{x}^{\sigma \multimap \tau}.\mathtt{M}]\rangle \mapsto \langle \underline{\mathtt{n}} | e_2[\mathtt{f} = e_2(\mathtt{h}) \odot e_2(\mathtt{g})] \upharpoonright_{\{\mathtt{h},\mathtt{g}\}} \rangle} \ (\mathsf{abs}^{\multimap})$$

$$\frac{\langle \mathtt{hNP}_1 \ldots \mathtt{P}_\mathtt{k} | e_0[\mathtt{h} := \mathtt{M}] - \mathtt{f}\rangle \mapsto \langle \underline{\mathtt{n}} | e_1 \rangle}{\langle \mathtt{f}\mathtt{P}_1 \ldots \mathtt{P}_\mathtt{k} | e_0[\mathtt{f} := \mathtt{MN}]\rangle \mapsto \langle \underline{\mathtt{n}} | e_1[\mathtt{f} := \pi_2(e_1(\mathtt{h}))] \upharpoonright_{\{\mathtt{h}\}} \rangle} \ (\mathsf{app})$$

$$\frac{\langle \mathtt{hP}_1 \ldots \mathtt{P}_\mathtt{k} | e_0 - \mathtt{f}\rangle \mapsto \langle \underline{\mathtt{n}} | e_1 \rangle}{\langle \mathtt{f}\mathtt{P}_1 \ldots \mathtt{P}_\mathtt{k} | e_0[\mathtt{f} := \mathtt{h}]\rangle \mapsto \langle \underline{\mathtt{n}} | e_1[\mathtt{f} := \mathtt{h}] \rangle} \ (\mathsf{var})$$

$$\frac{\langle \mathtt{hI} | e_0[\mathtt{h} := \mathtt{M}]\rangle \mapsto \langle \underline{\mathtt{n}} | e_1 \rangle}{\langle \mathtt{which?M}^{(\iota \multimap \iota) \multimap \iota} | e_0 \rangle \mapsto \langle \underline{\mathtt{n}} \odot \pi_1(e_1(\mathtt{h})) | e_1 \rangle} \ (\mathtt{w})$$

Constructive Semantics

$$\frac{\langle \mathtt{P}_1 | e_0 - \mathtt{f}\rangle \Mapsto \langle \underline{\mathtt{m}} | e_1\rangle \quad \langle \mathtt{f}\mathtt{P}_2 \ldots \mathtt{P}_\mathtt{k} | e_1[\mathtt{z} := \underline{\mathtt{m}}, \mathbf{g} := \mathtt{M}[\mathtt{z}/\mathtt{x}]]\rangle \Mapsto \langle \underline{\mathtt{n}} | e_2\rangle}{\langle \mathtt{f}\mathtt{P}_1 \ldots \mathtt{P}_\mathtt{k} | e_0[\mathtt{f} := \lambda \mathtt{x}^\iota.\mathtt{M}]\rangle \Mapsto \langle \underline{\mathtt{n}} | e_2[\mathtt{f} = e_2(\mathbf{h}) \odot e_2(\mathbf{g})]\restriction_{\{\mathtt{z},\mathbf{g}\}}\rangle} \; (\mathsf{abs}^\iota)$$

$$\frac{\langle \mathtt{M}[\mu F.\mathtt{M}/F]\mathtt{P}_1 \ldots \mathtt{P}_\mathtt{k} | e_0\rangle \Mapsto \langle \underline{\mathtt{n}} | e_1\rangle}{\langle (\mu F.\mathtt{M})\mathtt{P}_1 \ldots \mathtt{P}_\mathtt{k} | e_0\rangle \Mapsto \langle \underline{\mathtt{n}} | e_1\rangle} \; (\mu)$$

$$\frac{}{\langle \mathtt{x}^\iota | e[\mathtt{x} := \underline{\mathtt{n}}]\rangle \Mapsto \langle \underline{\mathtt{n}} | e[\mathtt{x} := \underline{\mathtt{n}}]\rangle} \; (\mathsf{gvar})$$

$$\frac{\langle \mathtt{M}_1 | e_0\rangle \Mapsto \langle \underline{\mathtt{n}}_1 | e_1\rangle \quad \langle \mathtt{M}_2 | e_1\rangle \Mapsto \langle \underline{\mathtt{n}}_2 | e_2\rangle}{\langle \mathtt{M}_1 \odot \mathtt{M}_2 | e_0\rangle \Mapsto \langle \underline{\mathtt{n}}_1 \odot \underline{\mathtt{n}}_2 | e_2\rangle} \; (\odot)$$

$$\frac{\langle \mathtt{M} | e_0\rangle \Mapsto \langle \underline{\mathtt{n}}_1 \odot \underline{\mathtt{n}}_2 | e_1\rangle \quad \langle \mathtt{M}[\mathtt{z}_1/\mathtt{x}_1, \mathtt{z}_2/\mathtt{x}_2]\mathtt{P}_1 \ldots \mathtt{P}_\mathtt{k} | e_1[\mathtt{z}_1 := \underline{\mathtt{n}}_1, \mathtt{z}_2 := \underline{\mathtt{n}}_2]\rangle \Mapsto \langle \underline{\mathtt{n}} | e_2\rangle}{\langle (\underline{\mathtt{let}}\ \mathtt{x}_1, \mathtt{x}_2 = \mathtt{M}\ \underline{\mathtt{in}}\ \mathtt{N})\mathtt{P}_1 \ldots \mathtt{P}_\mathtt{n} | e_0\rangle \Mapsto \langle \underline{\mathtt{n}} | e_2\rangle} \; (\underline{\mathtt{let}})$$

Constructive Semantics

$$\frac{\begin{array}{l}\langle \mathtt{M}_1 | e_0[\mathtt{f}_1 := \mathtt{N}_1, \ldots, \mathtt{f}_k := \mathtt{N}_k] \rangle \Mapsto \langle \underline{0} | e_1 \rangle \\ \langle \mathtt{M}_2 | e_0[\mathtt{f}_1 := \mathtt{N}_1, \ldots, \mathtt{f}_k := \mathtt{N}_k] \rangle \Mapsto \langle \underline{\mathtt{m}+1} | e_2 \rangle\end{array} \quad \forall \mathtt{i} \leq \mathtt{k},\ e_1(\mathtt{f}_\mathtt{i}) \doteq e_2(\mathtt{f}_\mathtt{i})}{\langle \underline{\ell\mathtt{et}}\, \mathtt{f}_1^{\sigma_1} = \mathtt{N}_1, \ldots, \mathtt{f}_k^{\sigma_k} = \mathtt{N}_k\, \underline{\mathtt{in}\ell\mathtt{or}}\ \mathtt{M}_1\ \mathtt{M}_2\ \mathtt{M}_3 | e_0 \rangle \Mapsto \langle \underline{\mathtt{m}} | e_1 \restriction_{\{\mathtt{f}_1, \ldots, \mathtt{f}_k\}} \rangle} \ (1\mathtt{lgor})$$

$$\frac{\begin{array}{l}\langle \mathtt{M}_2 | e_0[\mathtt{f}_1 := \mathtt{N}_1, \ldots, \mathtt{f}_k := \mathtt{N}_k] \rangle \Mapsto \langle \underline{0} | e_2 \rangle \\ \langle \mathtt{M}_3 | e_0[\mathtt{f}_1 := \mathtt{N}_1, \ldots, \mathtt{f}_k := \mathtt{N}_k] \rangle \Mapsto \langle \underline{\mathtt{m}+1} | e_3 \rangle\end{array} \quad \forall \mathtt{i} \leq \mathtt{k},\ e_2(\mathtt{f}_\mathtt{i}) \doteq e_3(\mathtt{f}_\mathtt{i})}{\langle \underline{\ell\mathtt{et}}\, \mathtt{f}_1^{\sigma_1} = \mathtt{N}_1, \ldots, \mathtt{f}_k^{\sigma_k} = \mathtt{N}_k\, \underline{\mathtt{in}\ell\mathtt{or}}\ \mathtt{M}_1\ \mathtt{M}_2\ \mathtt{M}_3 | e_0 \rangle \Mapsto \langle \underline{\mathtt{m}} | e_2 \restriction_{\{\mathtt{f}_1, \ldots, \mathtt{f}_k\}} \rangle} \ (2\mathtt{lgor})$$

$$\frac{\begin{array}{l}\langle \mathtt{M}_3 | e_0[\mathtt{f}_1 := \mathtt{N}_1, \ldots, \mathtt{f}_k := \mathtt{N}_k] \rangle \Mapsto \langle \underline{0} | e_3 \rangle \\ \langle \mathtt{M}_1 | e_0[\mathtt{f}_1 := \mathtt{N}_1, \ldots, \mathtt{f}_k := \mathtt{N}_k] \rangle \Mapsto \langle \underline{\mathtt{m}+1} | e_1 \rangle\end{array} \quad \forall \mathtt{i} \leq \mathtt{k},\ e_3(\mathtt{f}_\mathtt{i}) \doteq e_1(\mathtt{f}_\mathtt{i})}{\langle \underline{\ell\mathtt{et}}\, \mathtt{f}_1^{\sigma_1} = \mathtt{N}_1, \ldots, \mathtt{f}_k^{\sigma_k} = \mathtt{N}_k\, \underline{\mathtt{in}\ell\mathtt{or}}\ \mathtt{M}_1\ \mathtt{M}_2\ \mathtt{M}_3 | e_0 \rangle \Mapsto \langle \underline{\mathtt{m}} | e_3 \restriction_{\{\mathtt{f}_1, \ldots, \mathtt{f}_k\}} \rangle} \ (3\mathtt{lgor})$$

Outline of Results

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- Clique-Definability
- Full-Abstraction and Equivalences coincidence
- Constructive Semantics

Future works

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- Applications