Embedding intersection types into MLL

Internship

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Motivations

intersection-based types : used to catch computational behavior

(head, weak, strong normalizability)

Idempotence of $\wedge$:

$$A \wedge A = A$$

"$=$" ?

$(A = B) \equiv (A \leq B)$ and $(B \leq A)$ where

- $A \leq B$ means $(\vdash t : B \to A)$ where $t =_{\beta,\eta} \lambda x.x$
- $A \leq B$ means $(\vdash t : X) \Rightarrow (\vdash t : X[A/B])$

Motivations

Main idea :

- $A \wedge A = A$ (idempotent) $\sim$ intuitionistic conjunction (NJ's $\wedge$)
- $A \wedge A \neq A$ (non idempotent) $\sim$ tensor (LL's $\otimes$)

Logic systems map

	Natural deduction	Sequent calculus
Intuistionistic logic	*NJ* ($D\Omega$)	*LJ* (λLJ)
Multiplicative linear logic	*N-IMLL* ($\lambda N\text{-}IMLL = \mathcal{M}\Omega, R$)	*L-IMLL* ($\lambda L\text{-}IMLL, \lambda L\text{-}IMLL^* = \mathcal{M}\Omega^*$)

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$				
λLJ				
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$				
$\lambda L\text{-}IMLL$				
R				
$\mathcal{M}\Omega^{*}{}_{(\lambda L\text{-}IMLL^{*})}$				

Properties table

	Completeness	Soundness	S. expansion	S. reduction
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Completeness

Every head-normalizable term is non-trivially typable.

Properties table

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R				
$\mathcal{M}\Omega^*{}_{(\lambda L\text{-}IMLL^*)}$				

Soundness

Every non-trivially typable term is head-normalizable.

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R				
$\mathcal{M}\Omega^*{}_{(\lambda L\text{-}IMLL^*)}$				

Subject expansion

If $\Gamma \vdash t' : A$ and $t \rightarrow_\beta t'$ then $\Gamma \vdash t : A$

Used for completeness

Properties table

	Completeness	Soundness	S. expansion	S. reduction
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Subject reduction

If $\Gamma \vdash t : A$ and $t \to_\beta t'$ then $\Gamma \vdash t' : A$

Stronger version for soundness

NJ (fragment $\rightarrow, \wedge, T$)

$$\frac{}{\Gamma, A \vdash A}\ ax \qquad \frac{}{\Gamma \vdash T}\ T$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow_I \qquad \frac{\Gamma \vdash A \rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B} \rightarrow_E$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_I \qquad \frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i} \wedge_{E_i}$$

$D\Omega$: decoration of NJ

$$\frac{}{\Gamma, x : A \vdash x : A}\,ax \qquad \frac{}{\Gamma \vdash t : T}\,T$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\to_I \qquad \frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash tu : B}\to_E$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash t : B}{\Gamma \vdash t : A \wedge B}\wedge_I \qquad \frac{\Gamma \vdash t : A_1 \wedge A_2}{\Gamma \vdash t : A_i}\wedge_{E_i}$$

$D\Omega$: decoration of NJ

$$\frac{}{\Gamma, x : A \vdash x : A}\ ax \qquad \frac{}{\Gamma \vdash t : T}\ T$$

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$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash t : B}{\Gamma \vdash t : A \wedge B} \wedge_I \qquad \frac{\Gamma \vdash t : A_1 \wedge A_2}{\Gamma \vdash t : A_i} \wedge_{E_i}$$

- **Apart from Curry-Howard :** some rules without constructor

$D\Omega$: decoration of NJ

$$\frac{}{\Gamma, x : A \vdash x : A}\ ax \qquad \frac{}{\Gamma \vdash t : T}\ T$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B} \to_I \qquad \frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash tu : B} \to_E$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash t : B}{\Gamma \vdash t : A \wedge B} \wedge_I \qquad \frac{\Gamma \vdash t : A_1 \wedge A_2}{\Gamma \vdash t : A_i} \wedge_{E_i}$$

- **Apart from Curry-Howard :** some rules without constructor
- **Not quite** NJ **:** $A \to B \to A \wedge B$ is provable in NJ but not in $D\Omega$

$D\Omega$: decoration of NJ

$$\frac{}{\Gamma, x : A \vdash x : A}\ ax \qquad \frac{}{\Gamma \vdash t : T}\ T$$

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- **Apart from Curry-Howard :** some rules without constructor
- **Not quite** NJ **:** $A \to B \to A \wedge B$ is provable in NJ but not in $D\Omega$
- **No structural rule** for contraction and weakening

LJ (fragment $\to, \wedge, T$)

$$\dfrac{}{\Gamma, A \vdash A}\; ax \qquad \dfrac{\Gamma \vdash A \qquad \Gamma, A \vdash B}{\Gamma \vdash B}\; cut$$

$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\; \to_R \qquad \dfrac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\; \to_L$$

$$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\; \wedge_R \qquad \dfrac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\; \wedge_L$$

λLJ : decoration of LJ

$$\frac{}{\Gamma, x : A \vdash x : A}\,ax \qquad \frac{\Gamma \vdash t : A \qquad \Gamma, x : A \vdash u : B}{\Gamma \vdash u[t/x] : B}\,cut$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\to_R \qquad \frac{\Gamma \vdash t : A \qquad \Gamma, x : B \vdash u : C}{\Gamma, y : A \to B \vdash u[yt/x] : C}\to_L$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash t : B}{\Gamma \vdash t : A \wedge B}\wedge_R \qquad \frac{\Gamma, x : A, y : B \vdash t : C}{\Gamma, z : A \wedge B \vdash t[z/x, y] : C}\wedge_L$$

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R				
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λLJ				
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$				
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R				
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Natural deduction for *IMLL* : *N-IMLL*

$$\frac{}{A \vdash A}\ ax$$

$$\frac{}{\vdash 1}\ 1_I \qquad \frac{\Gamma \vdash 1 \qquad \Delta \vdash C}{\Gamma, \Delta \vdash C}\ 1_E$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \multimap_I \qquad \frac{\Gamma \vdash A \multimap B \qquad \Delta \vdash A}{\Gamma, \Delta \vdash B}\ \multimap_E$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes_I \qquad \frac{\Gamma \vdash A \otimes B \qquad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C}\ \otimes_E$$

$\mathcal{M}\Omega = \lambda N\text{-}IMLL$: decoration of $N\text{-}IMLL$

$$\dfrac{}{x : A \vdash x : A}\, ax$$

$$\dfrac{}{\vdash t : 1}\, 1_I \qquad \dfrac{\Gamma \vdash u : 1 \qquad \Delta \vdash t : C}{\Gamma, \Delta \vdash t : C}\, 1_E$$

$$x\#\Gamma\ \dfrac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}\, \multimap_I \qquad \dfrac{\Gamma \vdash t : A \multimap B \qquad \Delta \vdash u : A}{\Gamma, \Delta \vdash tu : B}\, \multimap_E$$

$$\dfrac{\Gamma \vdash t : A \qquad \Delta \vdash t : B}{\Gamma, \Delta \vdash t : A \otimes B} \qquad x, y\#\Delta\ \dfrac{\Gamma \vdash u : A \otimes B \qquad \Delta, x : A, y : B \vdash t : C}{\Gamma, \Delta \vdash t[u/x, y] : C}\, \otimes_E$$

$\mathcal{M}\Omega = \lambda N$-*IMLL* : decoration of *N-IMLL*

$$\frac{}{x : A \vdash x : A}\ ax$$

$$\frac{}{\vdash t : 1}\ 1_I \qquad \frac{\Gamma \vdash u : 1 \qquad \Delta \vdash t : C}{\Gamma, \Delta \vdash t : C}\ 1_E$$

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- **No structural rules :** no weakening

$\mathcal{M}\Omega = \lambda N\text{-}IMLL$: decoration of *N-IMLL*

$$\frac{}{x : A \vdash x : A}\ ax$$

$$\frac{}{\vdash t : 1}\ 1_I \qquad \frac{\Gamma \vdash u : 1 \qquad \Delta \vdash t : C}{\Gamma, \Delta \vdash t : C}\ 1_E$$

$$x\#\Gamma\ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}\ {\multimap}_I \qquad \frac{\Gamma \vdash t : A \multimap B \qquad \Delta \vdash u : A}{\Gamma, \Delta \vdash tu : B}\ {\multimap}_E$$

$$\frac{\Gamma \vdash t : A \qquad \Delta \vdash t : B}{\Gamma, \Delta \vdash t : A \otimes B} \qquad x, y\#\Delta\ \frac{\Gamma \vdash u : A \otimes B \qquad \Delta, x : A, y : B \vdash t : C}{\Gamma, \Delta \vdash t[u/x, y] : C}\ \otimes_E$$

- **No structural rules :** no weakening, multiplicative contexts

$\mathcal{M}\Omega = \lambda N\text{-}IMLL$: decoration of *N-IMLL*

$$\frac{}{x : A \vdash x : A}\ ax$$

$$\frac{}{\vdash t : 1}\ 1_I \qquad \frac{\Gamma \vdash u : 1 \qquad \Delta \vdash t : C}{\Gamma, \Delta \vdash t : C}\ 1_E$$

$$x\#\Gamma\ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}\ \multimap_I \qquad \frac{\Gamma \vdash t : A \multimap B \qquad \Delta \vdash u : A}{\Gamma, \Delta \vdash tu : B}\ \multimap_E$$

$$\frac{\Gamma \vdash t : A \qquad \Delta \vdash t : B}{\Gamma, \Delta \vdash t : A \otimes B} \qquad x, y\#\Delta\ \frac{\Gamma \vdash u : A \otimes B \qquad \Delta, x : A, y : B \vdash t : C}{\Gamma, \Delta \vdash t[u/x, y] : C}\ \otimes_E$$

- **No structural rules :** no weakening, multiplicative contexts
- n occurences of the same variable :
  - $n = 0$ (erasing) : 1_E

$\mathcal{M}\Omega = \lambda N\text{-}IMLL$: decoration of *N-IMLL*

$$\frac{}{x : A \vdash x : A}\ ax$$

$$\frac{}{\vdash t : 1}\ 1_I \qquad \frac{\Gamma \vdash u : 1 \qquad \Delta \vdash t : C}{\Gamma, \Delta \vdash t : C}\ 1_E$$

$$x\#\Gamma\ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}\ \multimap_I \qquad \frac{\Gamma \vdash t : A \multimap B \qquad \Delta \vdash u : A}{\Gamma, \Delta \vdash tu : B}\ \multimap_E$$

$$\frac{\Gamma \vdash t : A \qquad \Delta \vdash t : B}{\Gamma, \Delta \vdash t : A \otimes B} \qquad x, y\#\Delta\ \frac{\Gamma \vdash u : A \otimes B \qquad \Delta, x : A, y : B \vdash t : C}{\Gamma, \Delta \vdash t[u/x, y] : C}\ \otimes_E$$

- **No structural rules :** no weakening, multiplicative contexts
- n occurences of the same variable :
  - $n = 0$ (erasing) : 1_E
  - $n \geq 2$ (duplication) : $\otimes_E$

$\mathcal{M}\Omega = \lambda N\text{-}IMLL$: decoration of $N\text{-}IMLL$

$$\frac{}{x : A \vdash x : A}\ ax$$

$$\frac{}{\vdash t : 1}\ 1_I \qquad \frac{\Gamma \vdash u : 1 \qquad \Delta \vdash t : C}{\Gamma, \Delta \vdash t : C}\ 1_E$$

$$x\#\Gamma\ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}\ \multimap_I \qquad \frac{\Gamma \vdash t : A \multimap B \qquad \Delta \vdash u : A}{\Gamma, \Delta \vdash tu : B}\ \multimap_E$$

$$\frac{\Gamma \vdash t : A \qquad \Delta \vdash t : B}{\Gamma, \Delta \vdash t : A \otimes B} \qquad x, y\#\Delta\ \frac{\Gamma \vdash u : A \otimes B \qquad \Delta, x : A, y : B \vdash t : C}{\Gamma, \Delta \vdash t[u/x, y] : C}\ \otimes_E$$

- **No structural rules :** no weakening, multiplicative contexts
- n occurences of the same variable :
  - $n = 0$ (erasing) : 1_E
  - $n \geq 2$ (duplication) : $\otimes_E$
- Still not quite *IMLL* : $\nvdash_{\lambda N\text{-}IMLL}$: $A \multimap B \multimap A \otimes B$

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
λLJ	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$				
$\lambda L\text{-}IMLL$				
R				
$\mathcal{M}\Omega^*_{(\lambda L\text{-}IMLL^*)}$				

Subject reduction in λN-$IMLL$

$$x(Iz)(x(Iz)) \to_\beta xz(x(Iz))$$

$$\dfrac{\dfrac{\dfrac{}{x : D \vdash x : D}\,ax \qquad \dfrac{\cdots}{z : C \vdash Iz : C}}{z : C, x : D \vdash x(Iz) : A \otimes (A \multimap B)}\,\multimap_E \qquad \dfrac{\cdots}{a : A, b : A \multimap B \vdash ba : B}}{z : C, x : D = C \multimap (A \otimes (A \multimap B)) \vdash x(Iz)(x(Iz)) : B}\,\otimes_E$$

$$\dfrac{\dfrac{z_1 : C, z_2 : C, x_1 : C \multimap A, x_2 : C \multimap A \multimap B \vdash x_1 z_1(x_2(Iz_2)) : B}{z_1 : C, z_2 : C, x : (C \multimap A) \otimes (C \multimap A \multimap B) \vdash xz_1(x(Iz_2)) : B}\,ax, \otimes_E}{z : C \otimes C, x : (C \multimap A) \otimes (C \multimap A \multimap B) \vdash xz(x(Iz)) : B}\,ax, \otimes_E$$

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
λLJ	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$				$\times$
$\lambda L\text{-}IMLL$				
R				
$\mathcal{M}\Omega^*_{(\lambda L\text{-}IMLL^*)}$				

Soundness

Theorem (Soundness)

If $\Gamma \vdash_{\lambda N\text{-}IMLL} t : A$ and A is non trivial then t is head-normalizable.

Proof : Krivine's realizability

- $\mathcal{N}$ = head-normalizable terms
- $\mathcal{N}_0 = \{y u_1 \dots u_n\}$

Lemma (Adequation)

If $x_1 : A_1, \dots, x_n : A_n \vdash_{\lambda N\text{-}IMLL} t : B$ and $\forall i\ u_i \in [\![A_i]\!]$ then $t[u_1/x_1, \dots, u_n/x_n] \in [\![A_i]\!]$

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	✓	✓$\mathcal{I}$	✓	✓
λLJ	✓	✓$\mathcal{I}$	✓	✓
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$		✓$\mathcal{I}$		×
$\lambda L\text{-}IMLL$				
R				
$\mathcal{M}\Omega^*{}_{(\lambda L\text{-}IMLL^*)}$				

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
λLJ	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$		$\checkmark\mathcal{I}$	?	$\times$
$\lambda L\text{-}IMLL$				
R				
$\mathcal{M}\Omega^{*}{}_{(\lambda L\text{-}IMLL^{*})}$				

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	✓	✓$\mathcal{I}$	✓	✓
λLJ	✓	✓$\mathcal{I}$	✓	✓
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$		✓$\mathcal{I}$	?	×
$\lambda L\text{-}IMLL$				
R				
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The system R

from : Daniel de Carvalho
Origin : relational semantics of the linear logic

$$\frac{}{x : A \vdash x : A}\ \text{ax}$$

$$x \# \Gamma\ \frac{\Gamma, x : A_1, \ldots x : A_n \vdash t : B}{\Gamma \vdash \lambda x.t : A_1 \otimes \cdots \otimes A_n \multimap B}\ \lambda$$

$$\frac{\Gamma \vdash t : A_1 \otimes \cdots \otimes A_n \multimap B \qquad \Delta_i \vdash u : A_i \qquad i \in [1, n]}{\Gamma, \Delta_1, \ldots, \Delta_n \vdash tu : B}\ @_n$$

Properties of the system R

Theorem (Subject expansion)

If $\Gamma \vdash_R t' : A$ and $t \to_\beta t'$ then $\Gamma \vdash_R t : A$.

Proof sketch :

- induction on $\pi' :: \Gamma \vdash t' : A$, following the structure of t.
- substitution lemma for the expansion :
  If $\Sigma \vdash t[u/x] : A$ then there exists $n, (B_i), (\Delta_i), \Gamma$ s.t. :
  - $\Gamma, x : B_1, \ldots, x : B_n \vdash t : A$
  - $\Delta_i \vdash u : B_i$ for all $i \in [1, n]$
  - $\Sigma = \Gamma, \Delta_1, \ldots, \Delta_n$

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	✓	$\checkmark\mathcal{I}$	✓	✓
λLJ	✓	$\checkmark\mathcal{I}$	✓	✓
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$		$\checkmark\mathcal{I}$	?	×
$\lambda L\text{-}IMLL$				
R			✓	
$\mathcal{M}\Omega^*_{(\lambda L\text{-}IMLL^*)}$				

Properties of the system R

Lemma (Typing head normal forms)

If t is in head normal form then $\Gamma \vdash_R t : A$

Proof : If $\forall i, y \neq x_i$

$$\pi = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{y : 1 \multimap \cdots \multimap 1 \multimap A \vdash y : 1 \multimap \cdots \multimap 1 \multimap A}\ ax}{\vdots}\ @_0}{y : 1 \multimap \cdots \multimap 1 \multimap A \vdash y\ u_1 \ldots u_{m-1} : 1 \multimap A}\ @_0}{y : 1 \multimap \cdots \multimap 1 \multimap A \vdash y\ u_1 \ldots u_{m-1} u_m : A}\ \lambda}{y : 1 \multimap \cdots \multimap 1 \multimap A \vdash \lambda x_n.\ y\ u_1 \ldots u_m : 1 \multimap A}\ \lambda}{\vdots}}{y : 1 \multimap \cdots \multimap 1 \multimap A \vdash \lambda x_2 \ldots \lambda x_n.\ y\ u_1 \ldots u_m : 1 \multimap \cdots \multimap 1 \multimap A}}{y : 1 \multimap \cdots \multimap 1 \multimap A \vdash \lambda x_1 \lambda x_2 \ldots \lambda x_n.\ y\ u_1 \ldots u_m : 1 \multimap \cdots \multimap 1 \multimap A}\ \lambda$$

Properties of the system R

Lemma (Typing head normal forms)

If t is in head normal form then $\Gamma \vdash_R t : A$

Proof : If $y = x_i$

$$
\pi = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x_i : 1 \multimap \cdots \multimap 1 \multimap A \vdash x_i : 1 \multimap \cdots \multimap 1 \multimap A}\,ax}{\vdots}\,@_0}{x_i : 1 \multimap \cdots \multimap 1 \multimap A \vdash x_i\ u_1 \ldots u_{m-1} : 1 \multimap A}}{x_i : 1 \multimap \cdots \multimap 1 \multimap A \vdash x_i\ u_1 \ldots u_{m-1} u_m : A}\,@_0}{x_i : 1 \multimap \cdots \multimap 1 \multimap A \vdash \lambda x_n.\ x_i\ u_1 \ldots u_m : 1 \multimap A}\,\lambda}{\dfrac{\vdots}{\vdash \lambda x_2 \ldots \lambda x_n.\ x_i\ u_1 \ldots u_m : 1 \multimap \cdots \multimap 1 \multimap A}}\,\lambda
$$

$$
\dfrac{\vdash \lambda x_2 \ldots \lambda x_n.\ x_i\ u_1 \ldots u_m : 1 \multimap \cdots \multimap 1 \multimap A}{\vdash \lambda x_1 \lambda x_2 \ldots \lambda x_n . x_i\ u_1 \ldots u_m : 1 \multimap \cdots \multimap (1 \multimap \cdots \multimap 1 \multimap A) \multimap \cdots \multimap 1 \multimap A}\,\lambda
$$

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	✓	✓$\mathcal{I}$	✓	✓
λLJ	✓	✓$\mathcal{I}$	✓	✓
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$		✓$\mathcal{I}$	?	×
$\lambda L\text{-}IMLL$				
R	✓		✓	
$\mathcal{M}\Omega^*_{(\lambda L\text{-}IMLL^*)}$				

$R \subseteq \lambda N\text{-}IMLL$

Theorem ($R \subseteq \lambda N\text{-}IMLL$)

If $\Gamma \vdash_R t : A$ *then* $\Gamma \vdash_{N\cdot IMLL} t : A$

(R's rules not directly provable in N-IMLL) Proof :

- simple induction on $size(\pi)$
- $+$ lemma : If
  $\pi :: \Gamma, x : B_1, \ldots, x : B_n \vdash_R t : A$ then
  $\pi' :: \Gamma, x_1 : B_1, \ldots, x_n : B_n \vdash_R t' : A$
  where $t = t'[x/x_1 \ldots x_n]$
  and $size(\pi') = size(\pi)$

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	✓	$\checkmark\mathcal{I}$	✓	✓
λLJ	✓	$\checkmark\mathcal{I}$	✓	✓
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$	✓	$\checkmark\mathcal{I}$	?	×
$\lambda L\text{-}IMLL$				
R	✓		✓	
$\mathcal{M}\Omega^{*}{}_{(\lambda L\text{-}IMLL^{*})}$				

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
λLJ	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$	$\checkmark$	$\checkmark\mathcal{I}$	?	$\times$
$\lambda L\text{-}IMLL$				
R	$\checkmark$		$\checkmark$	
$\mathcal{M}\Omega^*_{(\lambda L\text{-}IMLL^*)}$				

Properties of the system R

Theorem (Subject reduction)

If $\Gamma \vdash_R t : A$ *and* $t \rightarrow_\beta t'$ *then* $\Gamma \vdash t' : A$.

Properties of the system R

Theorem (Subject reduction)

If $\Gamma \vdash_R t : A$ and $t \to_\beta t'$ then $\Gamma \vdash t' : A$.

Theorem (Subject head reduction)

If $\pi :: \Gamma \vdash_R t : A$ and $t \to_h t'$ then there exists π' such that :

- $\pi' :: \Gamma \vdash_R t' : A$.
- $m(\pi') < m(\pi)$.

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
λLJ	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$	$\checkmark$	$\checkmark\mathcal{I}$	?	$\times$
$\lambda L\text{-}IMLL$				
R	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$(m)
$\mathcal{M}\Omega^*{}_{(\lambda L\text{-}IMLL^*)}$				

L-IMLL : *IMLL* sequent calculus

$$\frac{}{A \vdash A}\ ax \qquad \frac{\Gamma \vdash A \qquad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}\ cut$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \multimap_R \qquad \frac{\Gamma \vdash A \multimap B \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ \multimap_L$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes_R \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes_L$$

$$\frac{}{\vdash 1}\ 1_R \qquad \frac{\Gamma \vdash C}{\Gamma, 1 \vdash C}\ 1_L$$

λL-$IMLL$: decoration of L-$IMLL$

$$\frac{}{x : A \vdash x : A}\,ax \qquad x\#\Gamma\ \frac{\Gamma \vdash t : A \qquad \Delta, x : A \vdash u : C}{\Gamma, \Delta \vdash u[t/x] : C}\,cut$$

$$x\#\Gamma\ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}\multimap_R \qquad x\#\Delta\ \frac{\Gamma \vdash t : A \multimap B \qquad \Delta, x : B \vdash u : C}{\Gamma, \Delta, y : A \multimap B \vdash u[yt/x] : C}\multimap_L$$

$$\frac{\Gamma \vdash t : A \qquad \Delta \vdash t : B}{\Gamma, \Delta \vdash t : A \otimes B}\otimes_R \qquad x, y\#\Gamma\ \frac{\Gamma, x : A, y : B \vdash t : C}{\Gamma, z : A \otimes B \vdash t[z/x, y] : C}\otimes_L$$

$$\frac{}{\vdash t : 1}\,1_R \qquad \frac{\Gamma \vdash u : C}{\Gamma, x : 1 \vdash u : C}\,1_L$$

λ*L-IMLL* : decoration of *L-IMLL*

$$\frac{}{x : A \vdash x : A}\ ax \qquad x\#\Gamma\ \frac{\Gamma \vdash t : A \qquad \Delta, x : A \vdash u : C}{\Gamma, \Delta \vdash u[t/x] : C}\ cut$$

$$x\#\Gamma\ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}\ \multimap_R \qquad x\#\Delta\ \frac{\Gamma \vdash t : A \multimap B \qquad \Delta, x : B \vdash u : C}{\Gamma, \Delta, y : A \multimap B \vdash u[yt/x] : C}\ \multimap_L$$

$$\frac{\Gamma \vdash t : A \qquad \Delta \vdash t : B}{\Gamma, \Delta \vdash t : A \otimes B}\ \otimes_R \qquad x, y\#\Gamma\ \frac{\Gamma, x : A, y : B \vdash t : C}{\Gamma, z : A \otimes B \vdash t[z/x, y] : C}\ \otimes_L$$

$$\frac{}{\vdash t : 1}\ 1_R \qquad \frac{\Gamma \vdash u : C}{\Gamma, x : 1 \vdash u : C}\ 1_L$$

- contraction

λL-IMLL : decoration of L-IMLL

$$\frac{}{x : A \vdash x : A}\, ax \qquad x\#\Gamma\ \frac{\Gamma \vdash t : A \qquad \Delta, x : A \vdash u : C}{\Gamma, \Delta \vdash u[t/x] : C}\, cut$$

$$x\#\Gamma\ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}\, \multimap_R \qquad x\#\Delta\ \frac{\Gamma \vdash t : A \multimap B \qquad \Delta, x : B \vdash u : C}{\Gamma, \Delta, y : A \multimap B \vdash u[yt/x] : C}\, \multimap_L$$

$$\frac{\Gamma \vdash t : A \qquad \Delta \vdash t : B}{\Gamma, \Delta \vdash t : A \otimes B}\, \otimes_R \qquad x, y\#\Gamma\ \frac{\Gamma, x : A, y : B \vdash t : C}{\Gamma, z : A \otimes B \vdash t[z/x, y] : C}\, \otimes_L$$

$$\frac{}{\vdash t : 1}\, 1_R \qquad \frac{\Gamma \vdash u : C}{\Gamma, x : 1 \vdash u : C}\, 1_L$$

- contraction
- weakening

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
λLJ	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$	$\checkmark$	$\checkmark\mathcal{I}$	?	$\times$
$\lambda L\text{-}IMLL$				
R	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$(m)
$\mathcal{M}\Omega^*_{(\lambda L\text{-}IMLL^*)}$				

N-IMLL=L-IMLL

Theorem (NL equivalence)

$\Gamma \vdash_{N\cdot IMLL} t : A \Leftrightarrow \Gamma \vdash_{L\cdot IMLL} t : A$

Proof.

rules provable in each other system preserving decoration

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
λLJ	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$	$\checkmark$	$\checkmark\mathcal{I}$	?	$\times$
$\lambda L\text{-}IMLL$	$\checkmark$	$\checkmark\mathcal{I}$	?	$\times$
R	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$(m)
$\mathcal{M}\Omega^*_{(\lambda L\text{-}IMLL^*)}$				

$\lambda L\text{-}IMLL^*$

$$\frac{}{x : A \vdash x : A}\ ax \qquad x\#\Gamma\ \frac{\Gamma \vdash t : A \qquad \Delta, x : A \vdash u : C}{\Gamma, \Delta \vdash u[t/x] : C}\ cut$$

$$x\#\Gamma\ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}\ \multimap_R \qquad x\#\Delta\ \frac{\Gamma \vdash t : A \multimap B \qquad \Delta, x : B \vdash u : C}{\Gamma, \Delta, y : A \multimap B \vdash u[yt/x] : C}\ \multimap_L$$

$$(B \neq 1, \_ \otimes \_)$$

$$\frac{\Gamma \vdash t : A \qquad \Delta \vdash t : B}{\Gamma, \Delta \vdash t : A \otimes B}\ \otimes_R \qquad x, y\#\Gamma\ \frac{\Gamma, x : A, y : B \vdash t : C}{\Gamma, z : A \otimes B \vdash t[z/x, y] : C}\ \otimes_L$$

$$\frac{}{\Gamma \vdash t : 1}\ 1_R \qquad \frac{\Gamma \vdash u : C}{\Gamma, x : 1 \vdash u : C}\ 1_L$$

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
λLJ	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$	$\checkmark$	$\checkmark\mathcal{I}$	?	$\times$
$\lambda L\text{-}IMLL$	$\checkmark$	$\checkmark\mathcal{I}$	?	$\times$
R	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$(m)
$\mathcal{M}\Omega^*{}_{(\lambda L\text{-}IMLL^*)}$				

Subject expansion

Theorem (Subject expansion)

If $\Gamma \vdash t' : A$ *and* $t \rightarrow_\beta t'$ *then* $\Gamma \vdash t : A$

Proof by induction on t, using proof reversing and $\otimes$-elimination

Completeness

Lemma (Typing HNF)

If t is in head normal form then $\exists \Gamma, A \quad \Gamma \vdash t : A$

Theorem (Completeness of N-IMLL*)

If t is head-normalizable then $\exists \Gamma, A \quad \Gamma \vdash t : A$

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	✓	✓$\mathcal{I}$	✓	✓
λLJ	✓	✓$\mathcal{I}$	✓	✓
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$	✓	✓$\mathcal{I}$	?	×
$\lambda L\text{-}IMLL$	✓	✓$\mathcal{I}$	?	×
R	✓	✓	✓	✓(m)
$\mathcal{M}\Omega^*{}_{(\lambda L\text{-}IMLL^*)}$	✓		✓	

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
λLJ	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$	$\checkmark$	$\checkmark\mathcal{I}$	?	$\times$
$\lambda L\text{-}IMLL$	$\checkmark$	$\checkmark\mathcal{I}$	?	$\times$
R	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$(m)
$\mathcal{M}\Omega^*{}_{(\lambda L\text{-}IMLL^*)}$	$\checkmark$		$\checkmark$	

Subject reduction

with this restriction on the $\multimap_L$ rule :

Theorem (Subject reduction)

If $\Gamma \vdash t : A$ *and* $t \to_\beta t'$ *then* $\Gamma \vdash t' : A$

Proof :

- induction on t about proof tree reversing and $\otimes$ elimination

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
λLJ	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$	$\checkmark$	$\checkmark\mathcal{I}$	?	$\times$
$\lambda L\text{-}IMLL$	$\checkmark$	$\checkmark\mathcal{I}$	?	$\times$
R	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$(m)
$\mathcal{M}\Omega^*{}_{(\lambda L\text{-}IMLL^*)}$	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
λLJ	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$	$\checkmark$	$\checkmark\mathcal{I}$	?	$\times$
$\lambda L\text{-}IMLL$	$\checkmark$	$\checkmark\mathcal{I}$	?	$\times$
R	$\checkmark$	$\checkmark$(m)	$\checkmark$	$\checkmark$(m)
$\mathcal{M}\Omega^*{}_{(\lambda L\text{-}IMLL^*)}$	$\checkmark$	$\checkmark\mathcal{I}$	$\checkmark$	$\checkmark$

Properties table

	Completeness	Soundness	S. expansion	S. reduction
$D\Omega_{(\lambda NJ)}$	✓	✓$\mathcal{I}$	✓	✓
λLJ	✓	✓$\mathcal{I}$	✓	✓
$\mathcal{M}\Omega_{(\lambda N\text{-}IMLL)}$	✓	✓$\mathcal{I}$	?	weaker ?
$\lambda L\text{-}IMLL$	✓	✓$\mathcal{I}$	?	weaker ?
R	✓	✓(m)	✓	✓(m)
$\mathcal{M}\Omega^*{}_{(\lambda L\text{-}IMLL^*)}$	✓	$\mathcal{I}$(m ?)	✓	✓

Weaker subject reduction

If $\Gamma \vdash t : A$ and $t \rightarrow_\beta t'$ there exists t'' such that :

- $t' \rightarrow_\beta^* t''$
- $\Gamma \vdash t'' : A$

Conclusion

- Soundness of $\mathcal{M}\Omega^*$ (bound ?)
- Subject expansion for $\mathcal{M}\Omega$
- Weaker subject reduction for $\mathcal{M}\Omega$
  If $\Gamma \vdash t : A$ and $t \to_\beta t'$ there exists t'' such that :
  - $t' \to_\beta^* t''$
  - $\Gamma \vdash t'' : A$
- MLL's cuts for subject reduction
  $\pi :: \Gamma \vdash t : A$, π cut-free, A non trivial $\Rightarrow$ t in head normal form
  $+$ MLL cut elimination
  $+$ link reduction to cut elimination
- Approximation lemma : $\Gamma \vdash_{IMELL} A \Rightarrow \Gamma_{\otimes/!} \vdash_{IMLL} A_{\otimes/!}$
  $D\Omega \to \mathcal{M}\Omega \overset{?}{\sim} IMELL \to IMLL$

Any question ?

?

Realisability

$(\mathcal{N}_0, \mathcal{N})$ adapted

- $\mathcal{N}_0 \subseteq \mathcal{N}$
- $\mathcal{N}_0 \subseteq (\mathcal{N} \to \mathcal{N}_0)$
- $(\mathcal{N}_0 \to \mathcal{N}) \subseteq \mathcal{N}$

- $[\![A \to B]\!] = \{t | \forall a \in [\![A]\!] \; ta \in [\![B]\!]$
- $[\![\alpha]\!] = \mathcal{N}_0$
- $[\![A \otimes B]\!] = [\![A]\!] \cap [\![B]\!]$

Definition

Definition

Trivial types are :

- 1 (Ω)
- $A \multimap T$ $(A \to T)$ where T is trivial
- $T \otimes T'$ $(T \wedge T')$ where both T and T' are trivial

Properties of the system R and pals

Properties (head normalization) :

- Subject reduction (with decreasing mesure)
- Soundness
- Subject expansion
- Completeness

Alternative systems :

- $R\backslash 1$ (terms without 1 in the typing relation)
  $\rightarrow$ characterise the weakly normalizable terms
- R^* (R without the $@_0$ rule)
  $\rightarrow$ characterise the strongly normalizable terms