Unique expansions with digits in ternary alphabets

Anna Chiara Lai

Istituto per le Applicazioni del Calcolo "Mauro Picone", CNR

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- Existence of unique expansions and critical base;
- Minimal unique expansions.

Expansions and positional number systems

Fix a *base* $q > 1$ and a finite *alphabet* $A \subset \mathbb{R}$.

An *expansion* for the value x is a sequence (x_i) with digits in A s.t.

$$x = \sum_{i=1}^{\infty} \frac{x_i}{q^i}$$

The value x is *representable* if there exists an expansion of x/q^N for some $N \in \mathbb{N}$, namely

$$x = x_1 q^{N-1} + x_2 q^{N-2} + \cdots + x_N + \frac{x_{N+1}}{q} + \frac{x_{N+2}}{q^2} + \cdots$$

If any number in the set Λ is representable, then the couple (q, A) is a *positional number system for* Λ.

Examples

Set $\Lambda = \mathbb{R}^+ \cup \{0\}$:

- decimal number system $(10, \{0, \ldots, 9\})$;
- binary number system $(2, \{0, 1\})$;
- usual number system in base b: $b \in \mathbb{N}, b > 1,\ (b, \{0, \ldots, b-1\})$;

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- binary number system $(2, \{0, 1\})$;
- usual number system in base b: $b \in \mathbb{N}, b > 1$, $(b, \{0, \ldots, b-1\})$;

If $b \in \mathbb{N}, b > 1$ then $(-b, \{0, \ldots, b-1\})$ is a positional numeration system for $\mathbb{R}$.

A non-integer based number system

Theorem (A. Rényi, 1957)

Every non-negative real number can be represented in base $q > 1$ and with alphabet $\{0, \dots, \lfloor q \rfloor\}$.

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Example: Golden Mean numeration system.

If $G^2 = G + 1$, namely $G = (1 + \sqrt{5})/2$ and $\lfloor G \rfloor = 1$, every non-negative real number x satisfies:

$$x = x_1 G^{N-1} + \cdots + x_N + \frac{x_{N+1}}{G} + \frac{x_{N+2}}{G^2} + \cdots$$

for some $N \in \mathbb{N}$ and some $(x_i) \in \{0, 1\}^\omega$.

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Example: expansions of 1 in base G

$$1 = \frac{1}{G} + \frac{1}{G^2} = \frac{1}{G} + \frac{1}{G^3} + \cdots + \frac{1}{G^{2n+1}} + \cdots$$

Greedy expansions

Greedy expansions

The *greedy expansion* of x is the lexicographically greatest expansion of x.

Example

The sequence $11(0)^\infty$ is the greedy expansion of 1 in base G and with alphabet $\{0, 1\}$

$$1 = \frac{1}{G} + \frac{1}{G^2}.$$

Digit distribution of greedy expansions

Set $q > 1$ and consider the alphabet $A_q = \{0, \ldots, \lfloor q \rfloor\}$:

- the greedy expansion (x_i) of x is generated by the iteration of the map $T_q(x) = qx - \lfloor qx \rfloor$, in particular $x_i = \lfloor qT_q^{i-1}(x) \rfloor$ [Rényi, 1957];

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The measure μ_q induces an invariant measure on the closure of the greedy expansions endowed with the shift operation.

Redundancy

An expansion is *finite* if it is definitively equal to the lowest digit of the alphabet.

If $b \in \mathbb{N},\ b > 1$ and $A = \{0, \dots, b-1\}$

- every infinite expansion is unique;
- for every finite expansion there exists exactly one different expansion representing the same number:

$$\frac{x_1}{b} + \cdots + \frac{x_n}{b^n} = \frac{x_1}{b} + \cdots + \frac{x_n - 1}{b^n} + \frac{b-1}{b^{n+1}} + \frac{b-1}{b^{n+2}} + \cdots$$

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If $q \in \mathbb{R} \setminus \mathbb{N},\ q > 1$ and $A = \{0, \ldots, \lfloor q \rfloor\}$

- almost every number in $[0, \lfloor q \rfloor/(q-1)]$ has a continuum of different expansions [Sidorov, 2001].

Expansions in non-integer base with general alphabets

Let $q > 1$ and $A = \{a_1, \cdots, a_J\}$ such that

$$\max_{j=1,\ldots,J-1} a_{j+1} - a_j \leq \frac{a_J - a_1}{q-1};$$

define $I := [a_1/(q-1), a_J/(q-1)]$.
Then:

- every number in I has at a least an expansion [Pedicini, 2005];

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Then:

- every number in I has at at least an expansion [Pedicini, 2005];
- if

$$\min_{j=1,\dots,J-1} a_{j+1} - a_j < \frac{a_J - a_1}{q-1},$$

  then almost every number in I has a continuum of different expansions [L. and Pedicini, 2010].

Redundancy with general alphabets: the critical base

Theorem (Komornik, L. and Pedicini, 2009)

For every alphabet A there exists a critical base G_A such that

- *if $1 < q < G_A$ then every number in the interior of I has at least two different expansions;*
- *if $q > G_A$ then there exists some value in I with a unique expansion.*

Example. If $A = \{0, 1\}$ then G_A equals to the Golden Mean. [Daròczy and Katai, 1993].

The ternary case

Due to a normalization we may consider only alphabets of the form

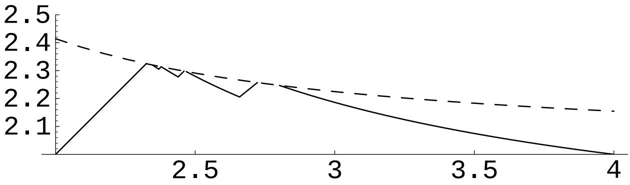
$$A_m = \{0, 1, m\}$$

with $m \geq 2$.

Theorem

Let G_{A_m} be the critical base of the alphabet $A_m = \{0, 1, m\}$ with $m \geq 2$. Then the greedy expansion of either of $m - 1$ or of $\frac{m}{G_{A_m} - 1} - 1$ in base G_{A_m} is a sturmian sequence.

Further properties of the critical base



- $G_{A_m} \in [2, 1 + \sqrt{\frac{m}{m-1}}]$;
- $G_{A_m} = 2$ if and only if $m = 2^k$ for some $k \in \mathbb{N}$;
- $G_{A_m} = 1 + \sqrt{\frac{m}{m-1}}$ if and only if m belongs to a Cantor set C;
- G_{A_m} is continuous w.r.t. m in $[2, \infty)$;
- in every connected component of $[2, \infty) \setminus C$ the critical base G_{A_m} is a convex function of m.

Minimal unique expansions

Let $U_{q,A}$ be the set of unique expansions in base q and alphabet A.

- if $1 < q < G_A$ then $U_{q,A} = \{(a_1)^\omega, (a_J)^\omega\}$:
- if $q < q'$ then $U_{q,A} \subseteq U_{q',A}$

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- if $q < q'$ then $U_{q,A} \subseteq U_{q',A}$

An expansion is *minimal* if it belongs to $U_{q,A}$ for every $q > G_A$.

We denote U_A the set of minimal expansions.

Characterization of minimal expansions

If $G_{A_m} < 1 + \sqrt{m/(m-1)}$ then the greedy expansion in base G_{A_m} either of $m-1$ or of $G_{A_m}\left(\frac{m}{G_{A_m}-1} - 1\right) - 1$ belongs to $\{1, m\}^\omega$ and it is a periodic characteristic sturmian sequence, which we denote (s_n). [Komornik, L. and Pedicini, 2009]

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Theorem

If $q_{A_m} < 1 + \sqrt{m/(m-1)}$ *then*

$$U_A = U_{q,A}$$

for every $q \in (q_{A_m}, 1 + \sqrt{m/(m-1)}]$. *Moreover*

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$$\begin{aligned} U_A = &\{(0)^\infty, (m)^\infty\} \cup \{m^* s_{n+1} s_{n+2} \cdots ; n \in \mathbb{N}\} \\ &\cup \{0^* s_{n+1} s_{n+2} \cdots ; s_n = 1, \sum_{k \geq 1} s_{n+k}/q^k < 1; n \in \mathbb{N}\}. \end{aligned}$$

Example: critical base for $A_3 = \{0, 1, 3\}$

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The critical base is the solution of

$$\sum_{i=1}^{\infty} \frac{s_i^{(3)}}{q^i} = 2$$

namely

$$\sum_{i=1}^{\infty} \frac{3}{q^{2i-1}} + \sum_{i=1}^{\infty} \frac{1}{q^{2i}} = 2$$

and

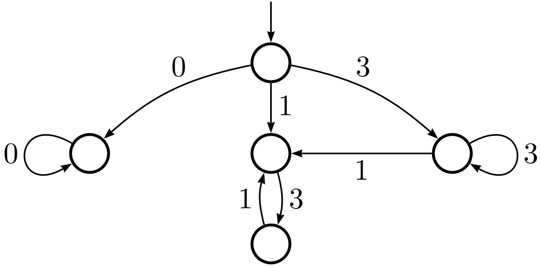
$$G_{A_3} \sim 2.18614$$

Example: minimal unique expansions for $A_3 = \{0, 1, 3\}$

- If $q \in (1, G_{A_3}]$ then $U_{q,A_3} = \{(0)^\omega, (3)^\omega\}$.
- If $q \in (G_{A_3}, 1 + \sqrt{\frac{3}{2}}]$ then

$$U_{q,A_3} = \{(0)^\omega, (3)^\omega, 3^t(13)^\omega \mid t = 0, 1, \dots\}$$

and it is recognized by



Conclusions

If $q \in [1, 1+\sqrt{\frac{m}{m-1}})$ the set U_{A_m} is explicitely known, in fact

- if $q_{A_m} < 1 + \sqrt{m/(m-1)}$ the previous theorem applies;
- if $q_{A_m} = 1 + \sqrt{m/(m-1)}$ then $U_{q,A_m} = \{(0)^\omega, (m)^\omega\}$.

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Since uniqueness is preserved by some digit-set operations, the restriction to the normal alphabets does not imply a loss of generality.

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... thank you for your attention...