Probabilistic λ-calculus, Types and Polymorphism

Alessandra Di Pierro

University of Verona

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$$\{p_i P_i\}_{i=1}^{n}, \qquad \text{with } \sum_{i=1}^{n} p_i = 1$$

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$$\overline{P} = \sum_{i=1}^{n} p_i \cdot \overline{M_i}.$$

Reduction System

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   $\{\frac{1}{4}\bot, \frac{3}{4}42\} \rightarrow_{\frac{1}{4}} \bot$ or $\{\frac{1}{4}\bot, \frac{3}{4}42\} \rightarrow_{\frac{3}{4}} 42$

The complete Example

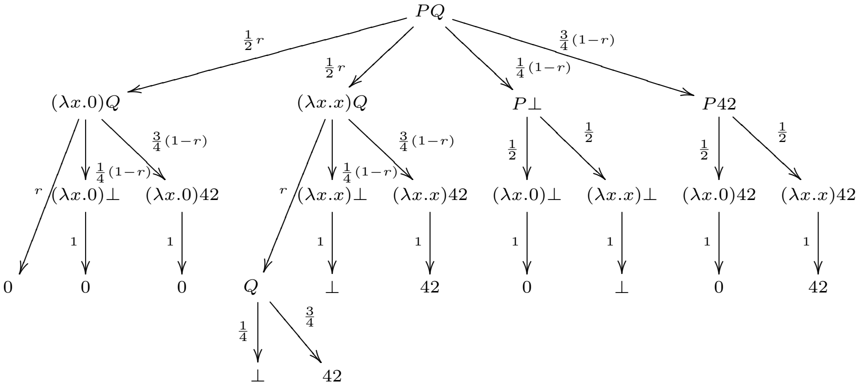

Figure: Reductions with two different strategies

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$$\mathcal{O}(P') = \{\langle 0, 1/2 \rangle, \langle 42, 3/8 \rangle, \langle \bot, 1/8 \rangle\}.$$

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Independence of the scheduling

Theorem

For all parameters r_1 and r_2 and all probabilistic λ-terms $P \in \Lambda_p$ with $P \to^{*p(r_1)} N \not\to$ and $P \to^{*p(r_2)} N \not\to$ we have $p(r_1) = p(r_2)$.

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The term $\{p_1 M_1, \ldots, p_n M_n\} N$, with the M_i's (possibly) differently typed, expresses a form of **ad hoc** polymorphism.

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A type τ is *regular* if for each subexpression $\gamma \oplus \sigma$ in τ, $\gamma \approx \sigma$.

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Theorem

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The set of types is the quotient set

$$\mathcal{T} = \text{Types}_{/\equiv}$$

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$$\frac{\forall i \in [1,n]\ \exists j \in [1,m]\ \tau_i <: \sigma_j \quad \approx \{\tau_i\}_{i=1}^{n} \quad \approx \{\sigma_j\}_{j=1}^{m}}{\bigoplus_{i=1}^{n} \tau_i <: \bigoplus_{j=1}^{m} \sigma_j}$$

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Given a relation $\mathcal{R}$ on $\mathfrak{I}$, the *subtype relation* $<:$ on $\mathcal{T}$ is the smallest relation defined by the transitive and reflexive closure of the rules:

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It suffices to define $<:$ only for regular types.

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When is a probabilistic λ–term is well-typed?

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$$\frac{M : \bigoplus_{i=1}^n(\tau_i \to \sigma_i) \quad N : \rho \qquad \forall i \in [1,n].\rho <: \tau_i}{MN : \bigoplus_{i=1}^n \sigma_i}\ (@)$$

Probabilistic Reduction

For the typed Λ_p we define reduction rules encoding the probabilistic behaviour of a typed Λ_p term.

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$$\frac{P \to_{\alpha}^{p} N}{\lambda x^{\tau}.P \to_{\alpha}^{p} \lambda x^{\tau}.N} \qquad (in\lambda)$$

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Theorem

If $\vdash M : \sigma$ *and* $M \rightarrow^p N$,
then there exists $\tau <: \sigma$ *such that* $\vdash N : \tau$.

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The second type (aka 'overloading') has received less attention regarding its semantics and proof theory.

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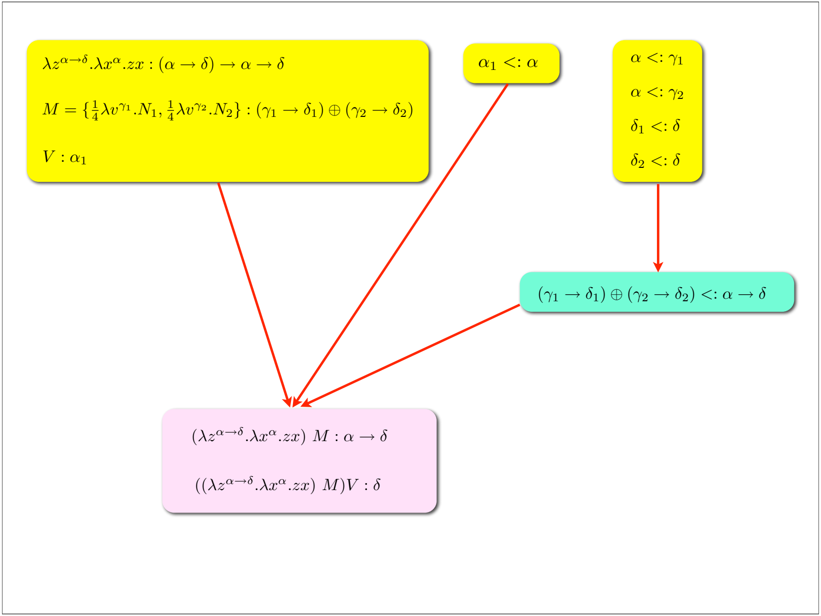
$\rightarrow$ Probabilistic Polymorphism
Note: Subtyping is essential to guarantee the well-typedness of the application term.

An Example


$\lambda z^{\alpha\to\delta}.\lambda x^{\alpha}.zx : (\alpha \to \delta) \to \alpha \to \delta$
$M = \{\frac{1}{4}\lambda v^{\gamma_1}.N_1, \frac{1}{4}\lambda v^{\gamma_2}.N_2\} : (\gamma_1 \to \delta_1) \oplus (\gamma_2 \to \delta_2)$
$V : \alpha_1$
$\alpha_1 <: \alpha$
$\alpha <: \gamma_1$
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$\delta_1 <: \delta$
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$(\gamma_1 \to \delta_1) \oplus (\gamma_2 \to \delta_2) <: \alpha \to \delta$
$(\lambda z^{\alpha\to\delta}.\lambda x^{\alpha}.zx)\ M : \alpha \to \delta$
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An Example

$M : (\gamma_1 \to \delta_1) \oplus (\gamma_2 \to \delta_2)$

$\alpha <: \gamma_1$

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$$\downarrow 1\ \beta$$

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$$N_1[V/v] : \delta_1' <: \delta_1$$

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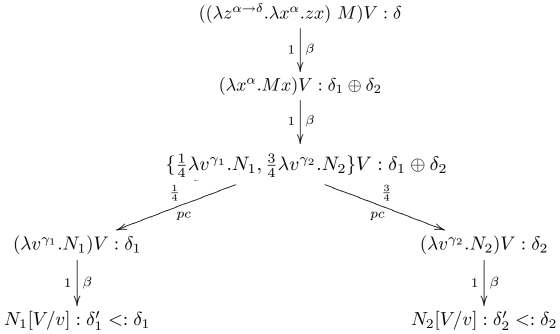
$$\searrow \tfrac{3}{4} \mid pc$$

$$(\lambda v^{\gamma_2}.N_2)V : \delta_2$$

$$\downarrow 1 \mid \beta$$

$$N_2[V/v] : \delta_2' <: \delta_2$$

An Example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The failure of confluence

Let $\mathcal{O}_{CbV}(P)$ and $\mathcal{O}_{CbN}(P)$ be the set of *observables* of P by means of CbV and CbN.
It is possible to find a term M s.t.

$$\mathcal{O}_{CbV}(M) \neq \mathcal{O}_{CbN}(M)$$

Let $M \equiv (\lambda x.(\mathbf{SUM}\ x\ x))\{\frac{1}{2}\mathbf{1}, \frac{1}{2}\mathbf{2}\}$
($\mathbf{1}, \mathbf{2}$ are the usual Church numerals, and **SUM** is the standard term for the sum of Church numerals)

We have that:
$\mathcal{O}_{CbV}(M) = \{\langle \mathbf{2}, \frac{1}{2}\rangle, \langle \mathbf{4}, \frac{1}{2}\rangle\}$
and that
$\mathcal{O}_{CbN}(M) = \{\langle \mathbf{2}, \frac{1}{4}\rangle, \langle \mathbf{3}, \frac{1}{2}\rangle, \langle \mathbf{4}, \frac{1}{4}\rangle\}$