Jump from parallel to sequential proof: exponentials

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Paolo Di Giamberardino
digiambe@lipn.univ-paris13.fr

LIPN, Paris 13

Introduction

In previous works, by importing ideas from game semantics (notably Faggian-Maurel-Curien's *ludics nets*), we defined a new class of multiplicative/ additive polarized proof nets, *J-proof nets*.

J-proof nets are a generalization of usual proof net syntax, where we can represent nets which are *partially sequentialized*, by using jumps (that is, untyped extra edges) as sequentiality constraints.

In the present work, we extend J-proof nets to the multiplicative/exponential fragment. More precisely, we show how to replace the familiar linear logic notion of exponential box with a less "sequential" one (called *cone*) defined by means of jumps.

As a consequence, we get a syntax for polarized nets where, instead of a structure of boxes nested one into the other, we have one of cones which can be *partially overlapping*.

Plan

- Polarities and proof nets;
- J-nets and cones;
- Correctness;
- Normalization;
- Concluding remarks.

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Polarities in Linear Logic

Polarization: distinction in linear logic between *positive* and *negative* formulas.

Polarized formulas:

$$\begin{array}{lcccccccccc} N & ::= & X^{\perp} & \mid & \perp & \mid & N \parr N & \mid & N \& N & \mid & ?P \\ P & ::= & X & \mid & 1 & \mid & P \otimes P & \mid & P \oplus P & \mid & !N \end{array}$$

Positivity= focalization

Negativity= reversibility

Polarized system: system where all formulas are polarized.

Polarities: what we gain

- *HO* game models for (polarized) linear logic;
- Correspondance with classical logic and $\lambda\mu$ calculus;
- Canonical proof search (*focusing proofs*) and linear logic programming;
- Interactive reconstruction of logical notions from a pre-logical framework (*ludics*);
- Relation with π-calculus.

Polarities: what we lose

- Parallelism: *strict alternance of polarities*;
- We cannot study *uncorrect objects*: all polarized cut-free proof structures are correct.

$MELL_{pol}$

$$N ::= ⅋_{i\in I}(?P_i)$$
$$P ::= \otimes_{i\in I}(!N_i)$$

$$\frac{\vdash \Gamma, ?P_1, \ldots, ?P_n}{\vdash \Gamma, ⅋(?P_1, \ldots, ?P_n)}\ ⅋ \qquad \frac{\vdash \Gamma_1, !N_1 \qquad \vdash \Gamma_n, !N_n}{\vdash \Gamma_1, \ldots, \Gamma_n, \otimes(!N_1, \ldots, !N_n)}\ \otimes$$

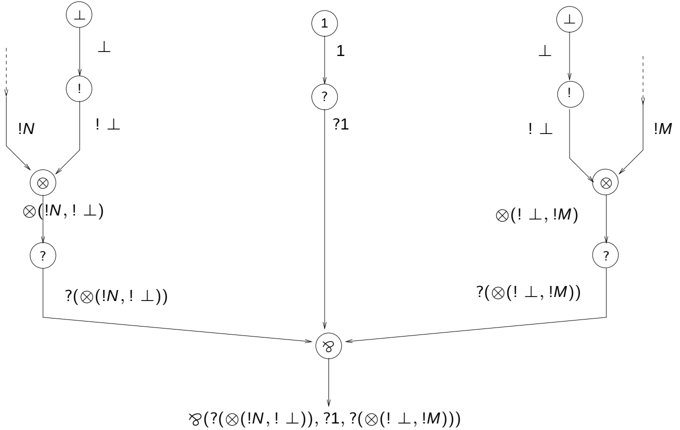
$$\frac{\vdash ?\Gamma, N}{\vdash ?\Gamma, !N}\ ! \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, ?P}\ d$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?P}\ w \qquad \frac{\vdash \Gamma, ?P_1, \ldots, ?P_n}{\vdash \Gamma, ?P}\ c$$

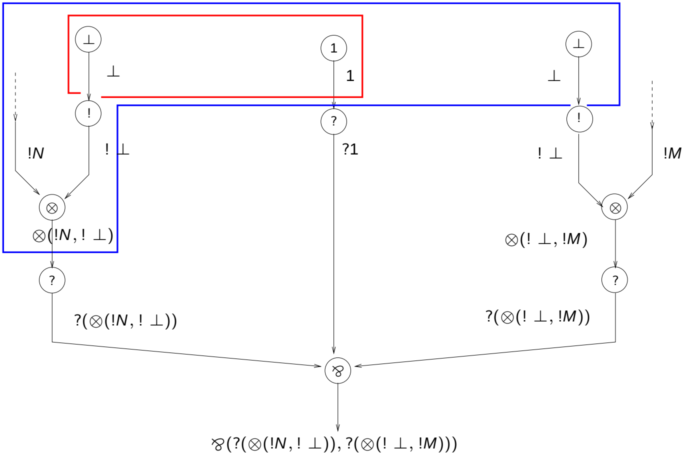
$$\frac{\vdash \Gamma,\ P \qquad \vdash \Delta,\ P^{\perp}}{\vdash \Gamma,\ \Delta}(Cut)$$

Every sequent has at most *one positive formula*

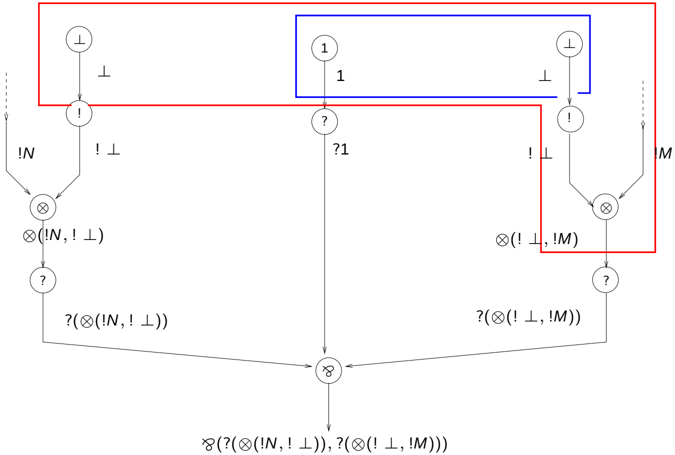
Polarized proof net and alternance (1)



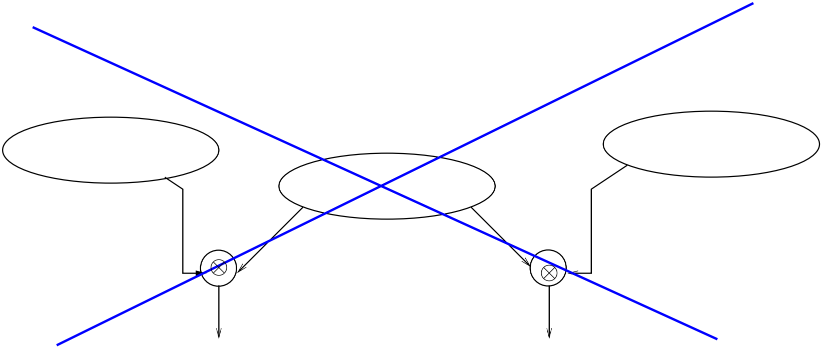
Polarized proof net and alternance (2)



Polarized proof net and alternance (3)



Polarized proof net and alternance (4)



Polarization = sequentiality?

Many contributions have been given recently in the direction of freeing polarities from a strong sequential framework (Mellies, Faggian-Maurel-Curien, Mimran et many others)

Our aim: to provide a more parallel notion of polarized proof net, in the setting of multiplicative exponential polarized linear logic.

Our tool: jumps, that is untyped extra edges which express sequentiality constraints.

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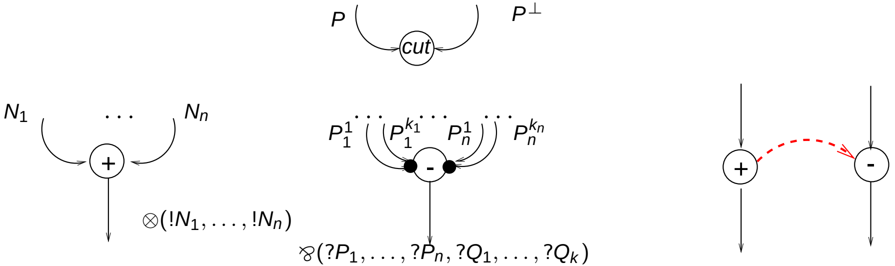
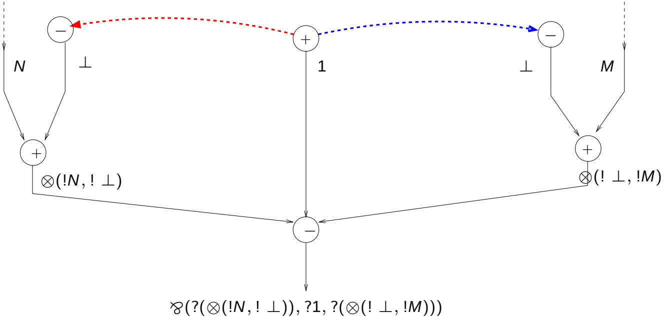
J-nets



P
$P^{\perp}$
cut
N_1
...
N_n
+
$\otimes(!N_1, \ldots, !N_n)$
P_1^1
$P_1^{k_1}$
P_n^1
$P_n^{k_n}$
-
$⅋(?P_1, \ldots, ?P_n, ?Q_1, \ldots, ?Q_k)$
+
-

An example of J-net



Cones and jumps

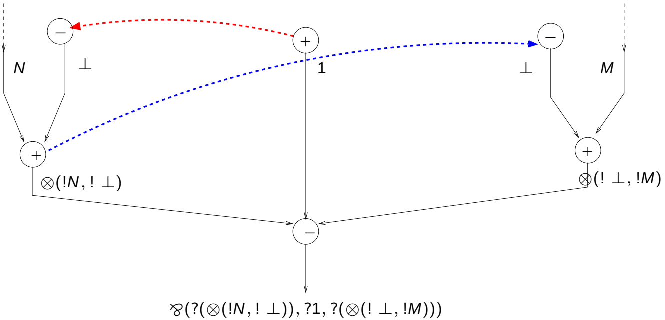
Given a cut-free J- net R, we denote by $\prec_R$ the strict partial order on the nodes of a J-net R obtained by taking the order associated to R as a d.a.g.
The *cone* of a negative edge a (denoted by C_R^a) conclusion of a node w is the set of nodes $\{b \in R; w \prec_R b\} \cup \{w\}$;

Given a negative edge a of R:

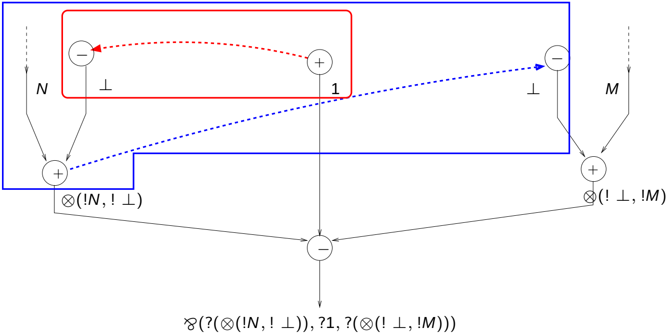
- there is no ambiguity in retrieving C_R^a;
- the conclusions of the links on the border of C_R^a (made exception for w) are all *positive*;
- moreover, if the order $\prec_R$ associated with R is arborescent, then given any other negative edge b of R, either C_R^b and C_R^a are included one into the other, either they are disjoint (*nesting condition*).

Built-in replacement of boxes!

J-nets and polarized proof nets

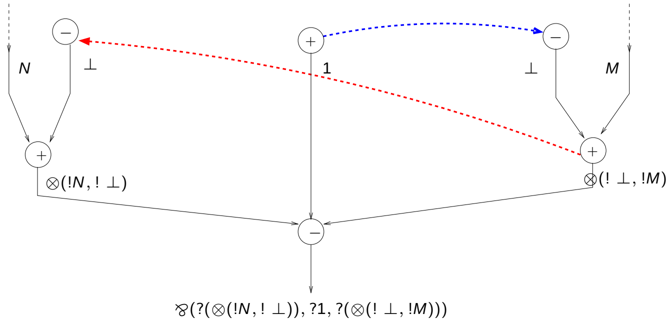


J-nets and polarized proof nets

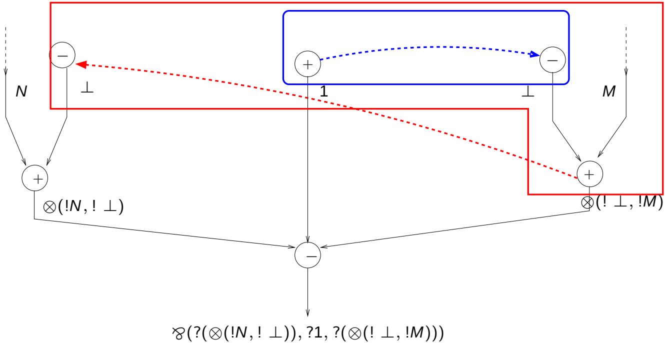


- Exactly one positive link at level 0

J-nets and polarized proof nets

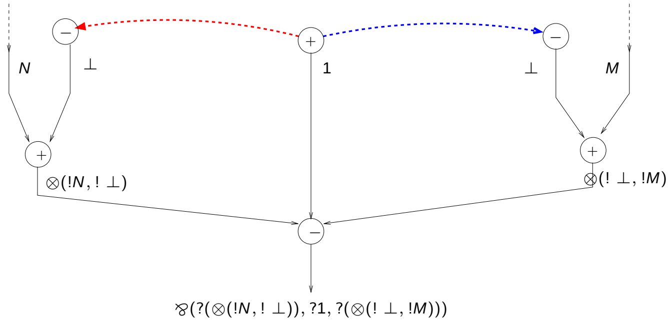


J-nets and polarized proof nets

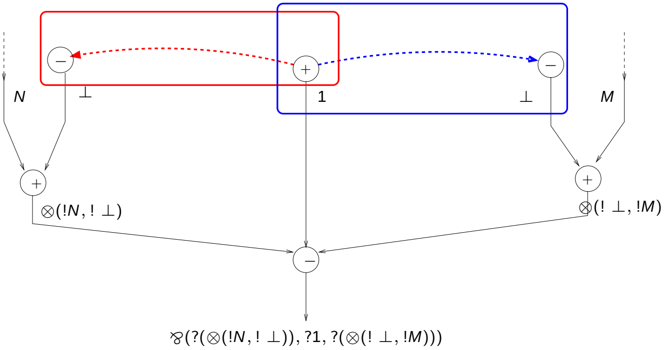


- exactly one positive link at level 0

Overlapping of cones



Overlapping of cones



Toward parallelism

- J-nets are a generalization of polarized proof nets where *superposition of cones* (i.e. boxes) is allowed;
- nevertheless, there is not any ambiguity when retrieving the cone of a negative edge;
- As a consequence when we define structural reductions in cut-elimination, we always know what is to duplicate and what is to erase.

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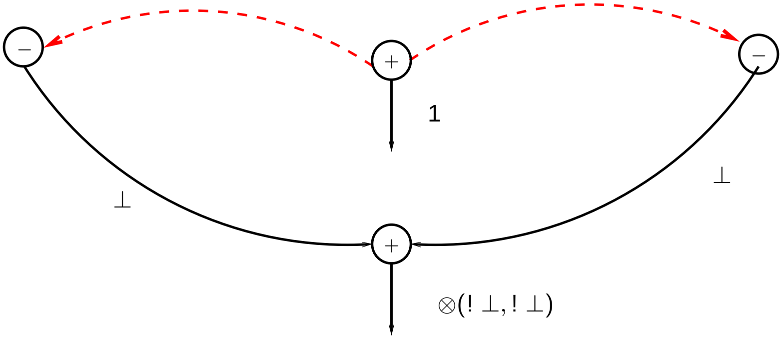
Cut-free J-nets are not necessarily correct!

Definition

A J-net is *acceptable* when is switching acyclic



Mix

We discard connectdness from the correctness criterion by accepting the following rule, called *Mix*

$$\frac{\vdash \Gamma_1 \ldots \qquad \ldots \vdash \Gamma_n}{\vdash \Gamma_1, \ldots, \Gamma_n} \; Mix$$

The 0-ary case of the Mix rule corresponds to the introduction of the empty sequent.

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Cut elimination

Definition

A J-net is *closed* when it has no positive conclusions.

Definition

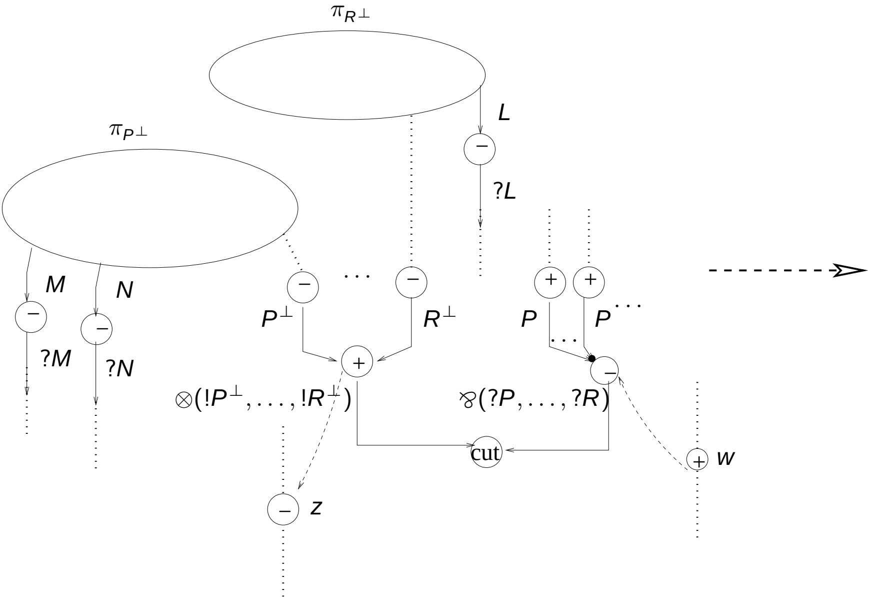
A J-proof net R is a J-net s.t. is acceptable and closed

Given two J-proof nets R_1, R_2, we define the relation $R_1 \xrightarrow{cut} R_2$ ('R reduces to R' in one step")

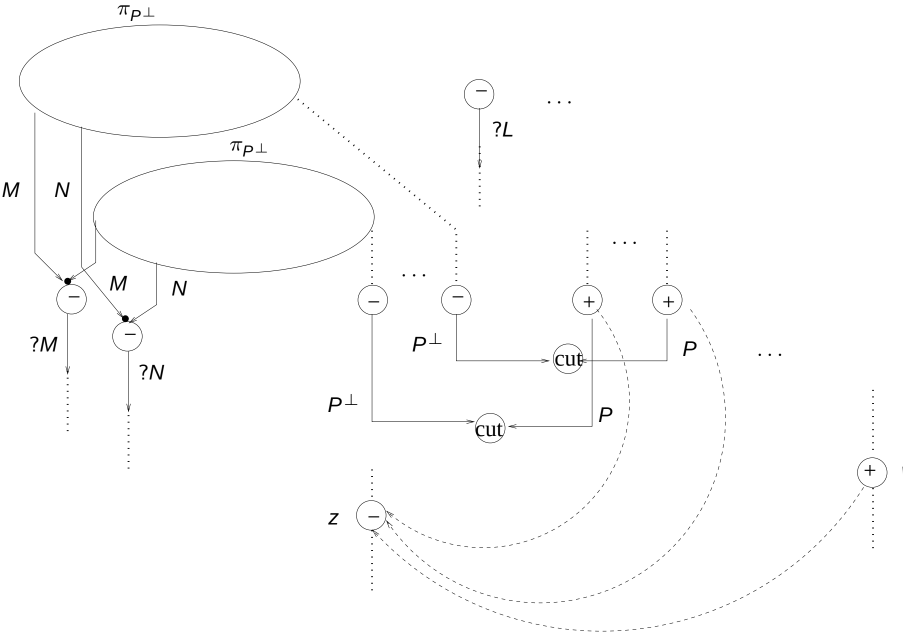
We remark that:

- There is only one big reduction rule composed by a multiplicative and a structural part;
- we can define the structural part of the reduction rule *by duplicating and erasing cones*;

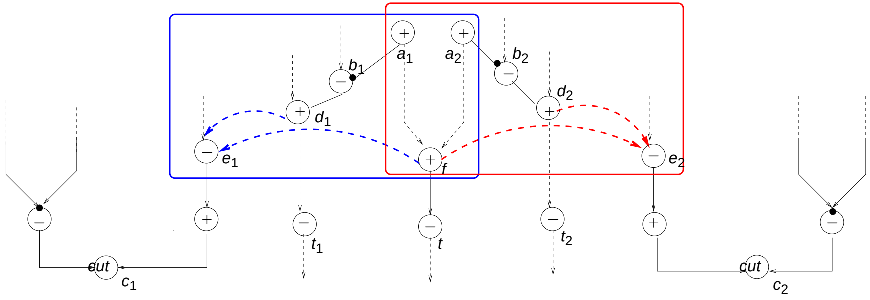
Reduction rule (redex)



Reduction rule (contractum)



Reduction example: contraction vs contraction



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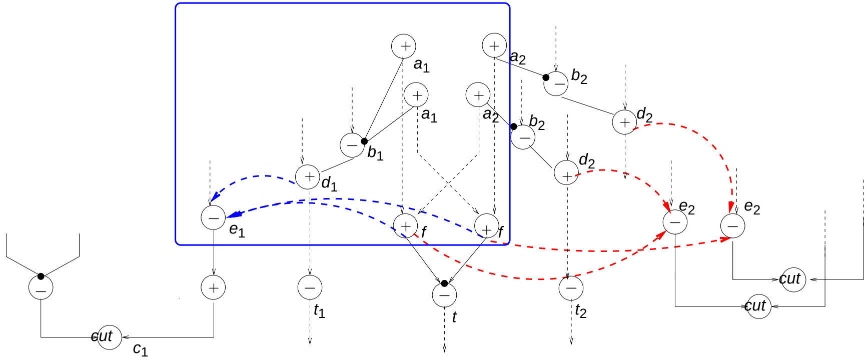

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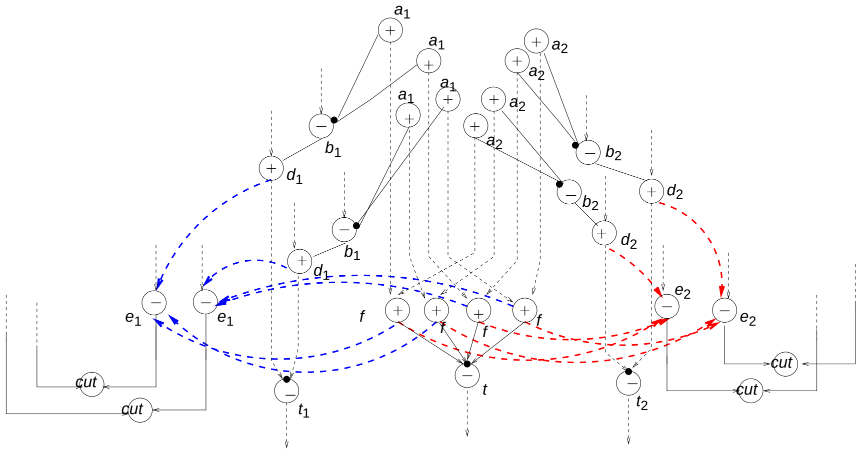

Properties of reduction

Theorem (Preservation of correctness)

Given a J-proof net R, if $R \xrightarrow{cut} R'$, then R' is a J-proof net.

Theorem

Reduction is strongly normalizing.

Theorem

Reduction is confluent.

Such results are proved following the work on strong normalization for *LL* from Pagani-Tortora

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Mix and confluence

A J-proof net R is *s-connected* iff :
- There are no maximal negative links;
- choosing a incident edge $s(n)$ for all negative link n of R and erasing all the others yields a connected graph for all choices of s

Theorem

For any arborescent J-proof net R, if R is s-connected and $R \xrightarrow{cut} R'$, then R' is s-connected.

Preservation of s-connectdness under reduction doesn't hold in the general case of (not arborescent) J-proof nets: forcing s-connectdness (by modifying the reduction rule) brings to the loss of *confluence*.

Work in progress

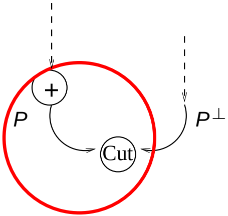
- Interpretation in the relational model, using the notion of *thick subtree* (joint work with Pierre Boudes, LIPN);
- Including additives (already existing) in the picture;
- Relation with concurrent game semantics (L-nets and exponential ludics, asynchronous games, etc.);
- Relation with the Λ nets of Accatoli-Guerrini;
- Correspondance with linear π-calculus?
- Proof nets for classical logic?
- Generalization to the not-polarized case (*LL*)?
- ...

THANKS!

Order associated with a J-net (in presence of *cut*-links)

We extend the definition of $\prec_R$ to a J-net R (possibly containing *cut*-links) in the following way:

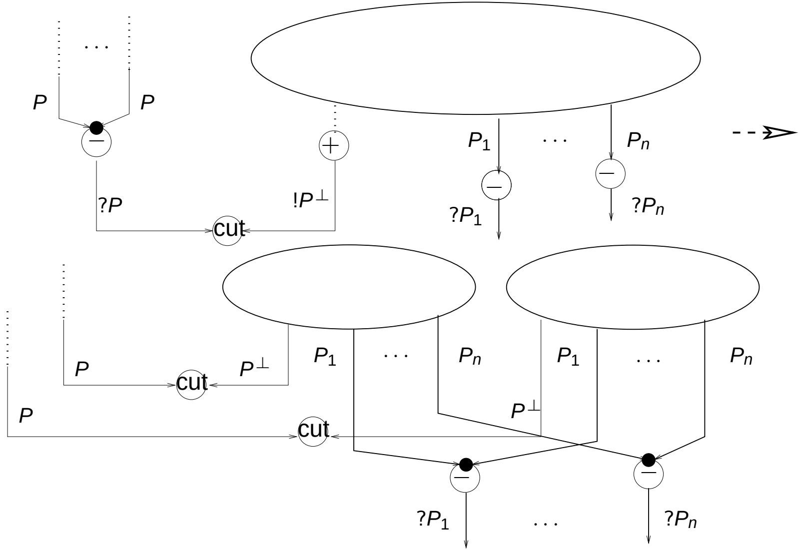
- we take the order associated to R as a d.a.g;
- we identify any cut link c with the link whose conclusion is the positive premise of c, as below:



In this way we can tell if a *cut*-link is inside a given cone.

Special cases of reduction

- "Contraction" step:



Saturation and sequentialization

Definition

An acceptable J-net R is saturated, when for every negative link n and for every positive link p of R adding a jump between n and p creates a switching cycle or doesn't increase the order $\prec_R$.

Lemma (Arborisation)

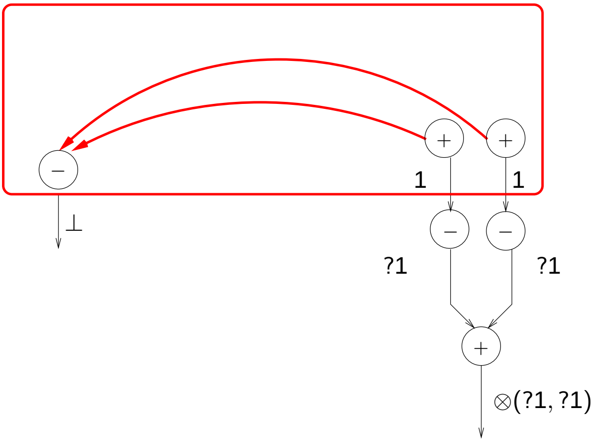
Given a acceptable J-net R, if R is saturated, then $\prec_R$ is arborescent.

For any acceptable J-net R, we can make its associated order arborescent by gradually adding jumps.

Theorem

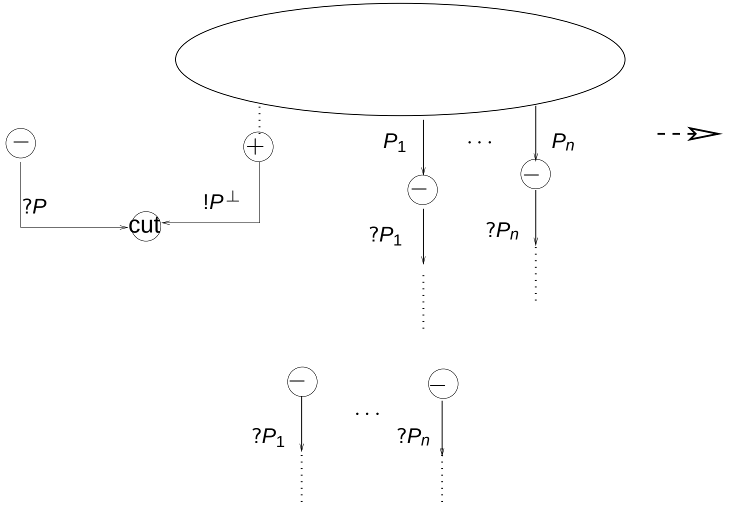
A J-net R whose associated order is arborescent correspond to a unique proof π of $MELL_{pol}$ (+ Mix).

Blackbox principle



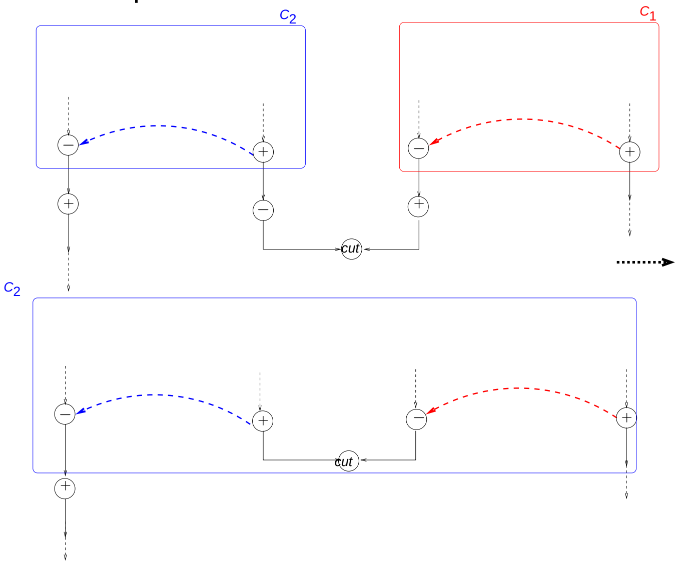
Special cases of reduction

- “Weakening” step:



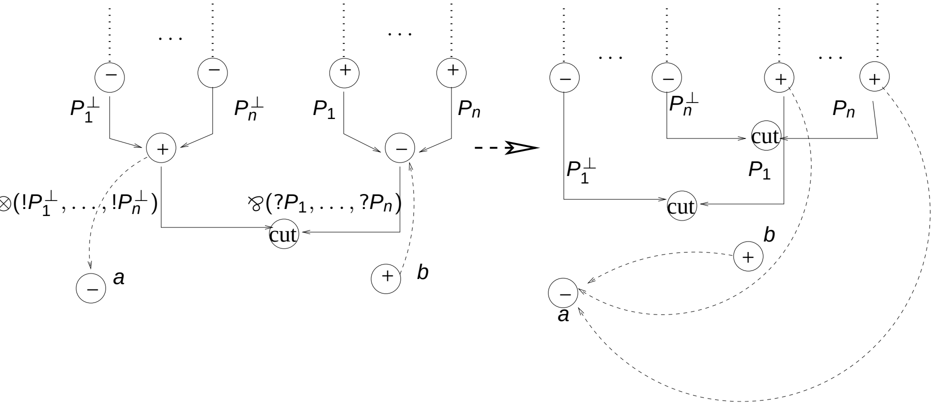
Special cases of reduction

- “box” step:



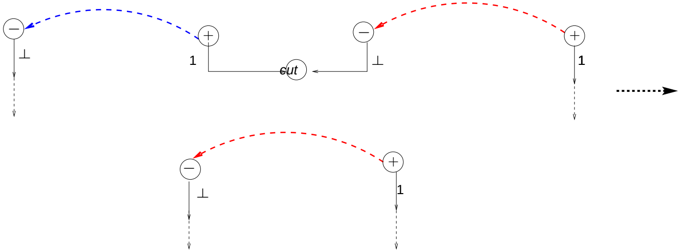
Special cases of reduction

- "Multiplicative" step:

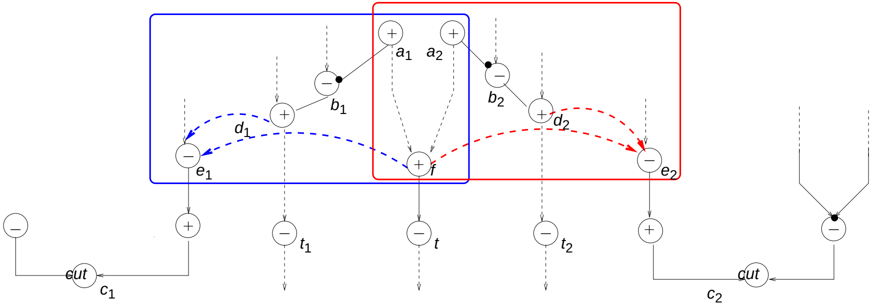


Special cases of reduction

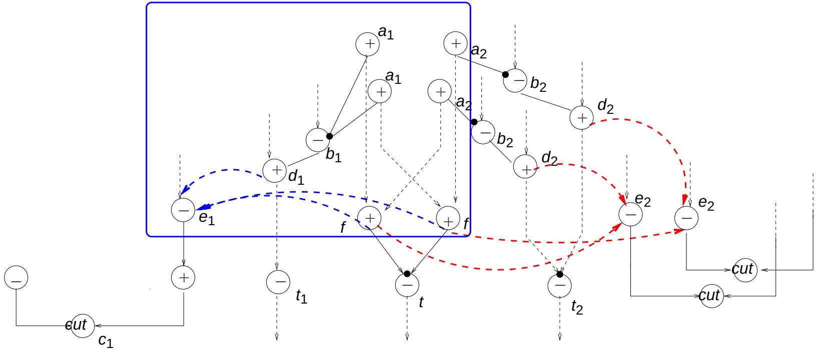
- "Axiom" step:



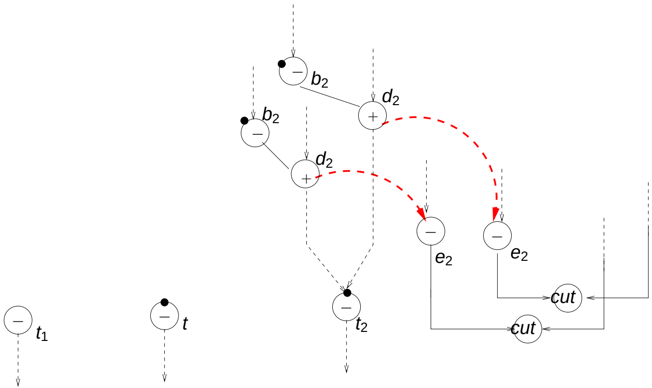
Reduction example: weakening vs contraction



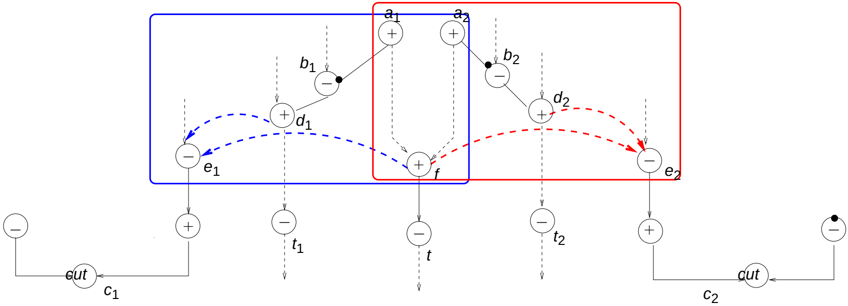
Reduction example: weakening vs contraction



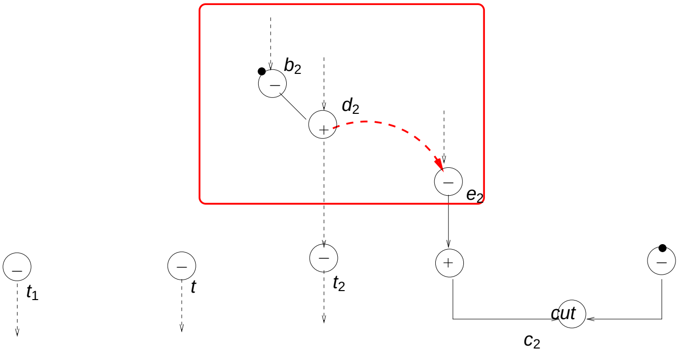
Reduction example: weakening vs contraction



Local confluence: weakening vs weakening



Local confluence: weakening vs weakening



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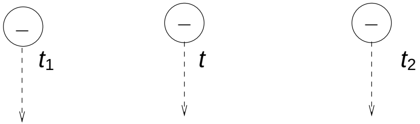
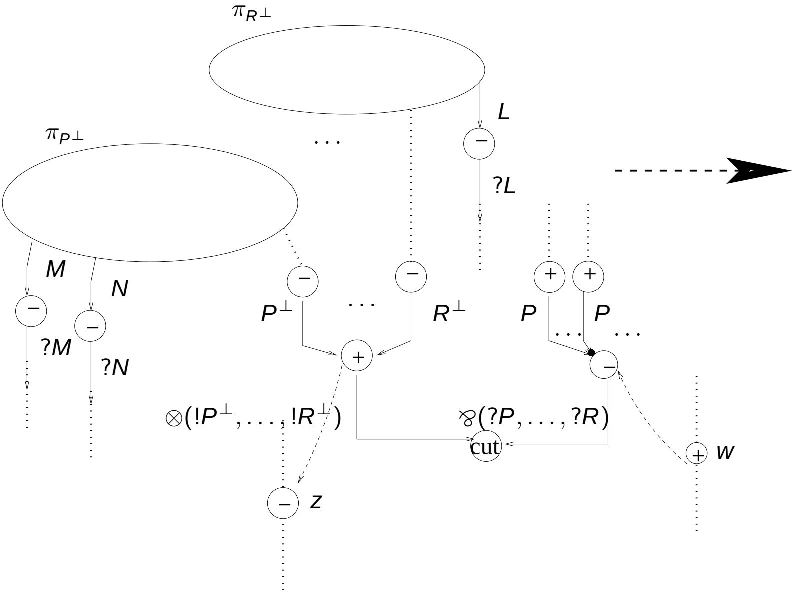
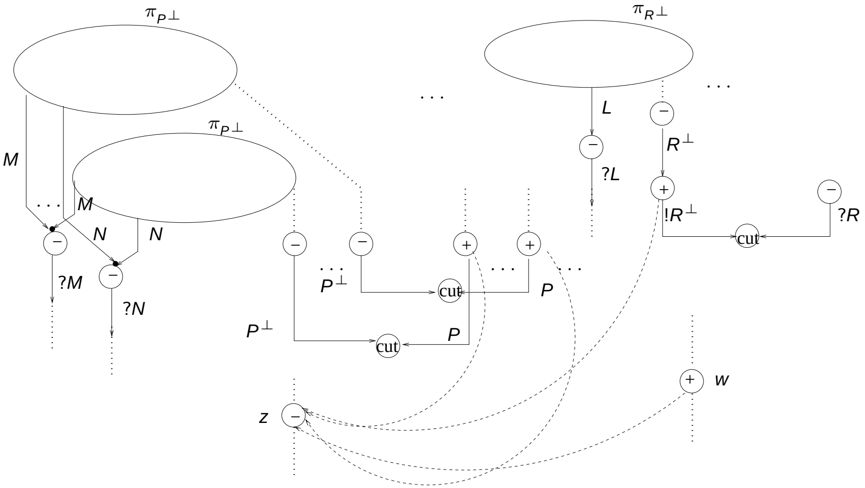


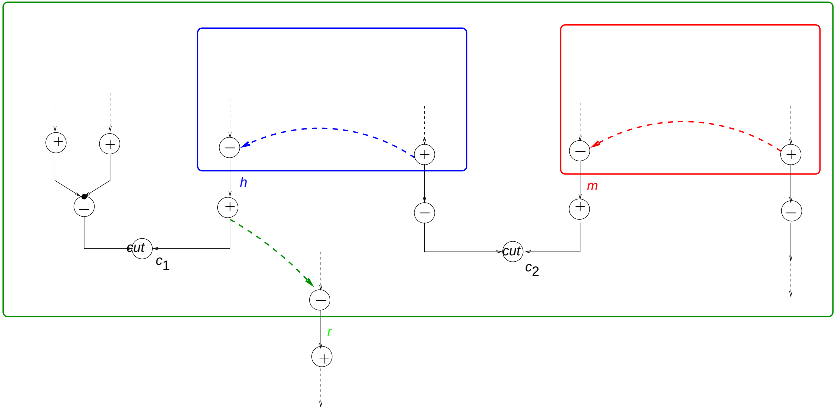
Not erasing reduction (redex)



Not erasing reduction (contractum)



$\xrightarrow{\neg e}$ is increasing



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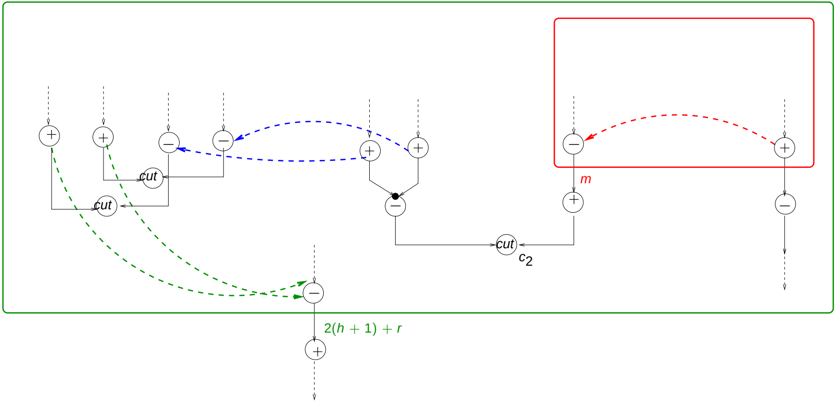
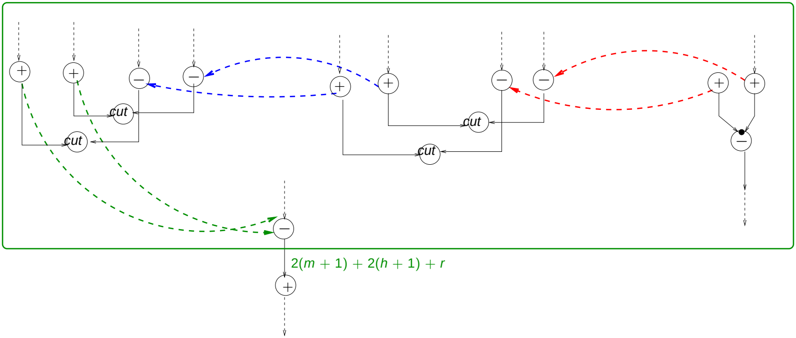


$\xrightarrow{\neg e}$ is increasing

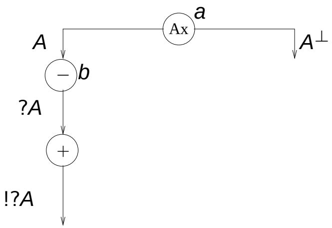


$SN^{\neg e} \Rightarrow SN^{cut}$

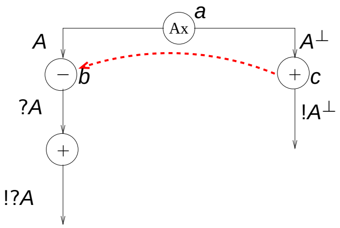
- Any reduction $R \xrightarrow{cut} *R'$ can be simulated with a sequence of not erasing steps followed by a sequence of "weakening" steps;
- "weakening" steps can always be postponed with respect to not erasing steps;
- from this and strong normalization of $\xrightarrow{\neg e}$ follows that there cannot be an infinite reduction composed by alternating sequences of not erasing steps and "weakening steps", so $\xrightarrow{cut}$ is strongly normalizing.

Adding axioms to the picture: balancedness

A J-net with R with axiom links is *balanced* if for every $-$ link b of R, such that a premise of b is a conclusion of an *ax* link a, there exists a positive link $c \prec_R a$ which jumps on b in R.

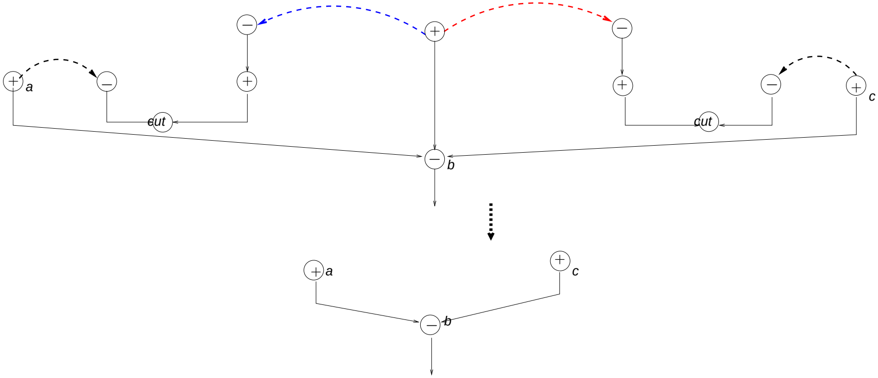


Adding axioms to the picture: balancedness



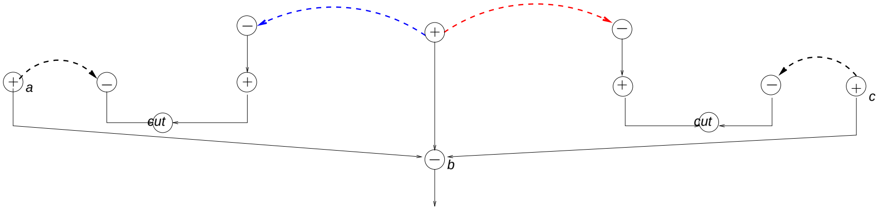
- "Boxing" of axioms.

Mix and confluence



Mix and confluence

We can try to modify the definition of reduction step, in a natural way, in order to preserve s-connectdness:



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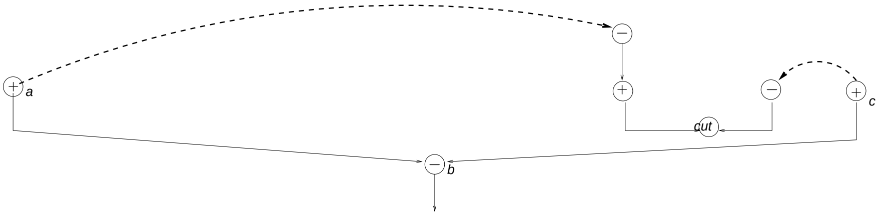
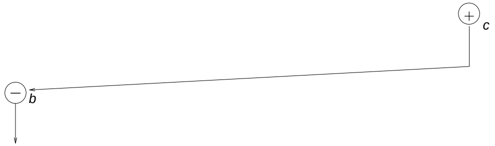
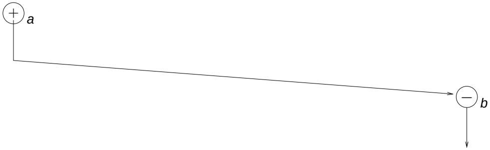


Mix and confluence

But then we lose confluence:



Mix and confluence



For the general case of (not-saturated) J-proof nets, we have to allow Mix in order to gain confluence.