An Additive type System for the linear-algebraic λ-calculus

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Motivation

- Algrebraic calculi:
  - Linear-algebraic λ-calculus [Arrighi, Dowek 2008]
  - Algebraic λ-calculus [Vaux 2009]
- CbV-CbN simulation [Díaz-Caro, Perdrix, Tasson, Valiron 2010]

Is it possible to find any interpretation of them in some well established theory?

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- Algrebraic calculi:
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Is it possible to find any interpretation of them in some well established theory?

- **In this work we try to answer this question for a simplified version of these calculi**
- We analyse the role of sums in the Linear-algebraic λ-calculus

Lineal [Arrighi, Dowek 2008]... without scalars

Higher-order computation

$\mathbf{t}, \mathbf{r}, \mathbf{s} ::= \quad x \mid \lambda x.\mathbf{t} \mid (\mathbf{t})\ \mathbf{r} \qquad\qquad \mid$

Lineal [Arrighi, Dowek 2008]... without scalars

Higher-order computation | **Linear algebra**

$\mathbf{t},\mathbf{r},\mathbf{s} ::= \quad x \mid \lambda x.\mathbf{t} \mid (\mathbf{t})\ \mathbf{r} \quad \mid \quad \mathbf{t}+\mathbf{r} \mid \not{\alpha.}\mathbf{t} \mid \mathbf{0}$

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Higher-order computation | **Linear algebra**

$\mathbf{t}, \mathbf{r}, \mathbf{s} ::= \quad x \mid \lambda x.\mathbf{t} \mid (\mathbf{t})\ \mathbf{r} \quad \mid \quad \mathbf{t} + \mathbf{r} \mid \not{\alpha}.\not{\mathbf{t}} \mid \mathbf{0}$

- $(\lambda x.\mathbf{t})\ \mathbf{b} \to \mathbf{t}[\mathbf{b}/x]$

b an abstraction or a variable.

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Higher-order computation | **Linear algebra**

$\mathbf{t}, \mathbf{r}, \mathbf{s} ::= \quad x \mid \lambda x.\mathbf{t} \mid (\mathbf{t})\ \mathbf{r} \quad \mid \quad \mathbf{t} + \mathbf{r} \mid \not{\alpha}.\not{\mathbf{t}} \mid \mathbf{0}$

- $(\lambda x.\mathbf{t})\ \mathbf{b} \to \mathbf{t}[\mathbf{b}/x]$

b an abstraction or a variable.

Restrictions lifted in the typed setting [Arrighi, Díaz-Caro 2009].

Several rules pruned without scalars.

- Elementary rule:
  $\mathbf{t} + \mathbf{0} \to \mathbf{t}$.
- Application rules:
  $(\mathbf{t} + \mathbf{r})\ \mathbf{s} \to (\mathbf{t})\ \mathbf{s} + (\mathbf{r})\ \mathbf{s}$
  $(\mathbf{t})\ (\mathbf{r} + \mathbf{s}) \to (\mathbf{t})\ \mathbf{r} + (\mathbf{t})\ \mathbf{s}$
  $(\mathbf{0})\ \mathbf{t} \to \mathbf{0}$
  $(\mathbf{t})\ \mathbf{0} \to \mathbf{0}$
- AC equivalences
  $\mathbf{t} + \mathbf{r} \equiv \mathbf{r} + \mathbf{t}$
  $\mathbf{t} + (\mathbf{r} + \mathbf{s}) \equiv (\mathbf{t} + \mathbf{r}) + \mathbf{s}$

Additive type System

Grammar

Types grammar:

$$T, R, S := U \mid T + R \mid \overline{0} \qquad \text{general types}$$

$$U, V, W := X \mid U \to T \mid \forall X.U \qquad \text{unit types}$$

Equivalences:

$$T + \overline{0} \equiv T$$

$$T + R \equiv R + T$$

$$T + (R + S) \equiv (T + R) + S$$

Additive type System

Typing rules

$$\frac{}{\Gamma, x : U \vdash x : U}\ ax \qquad \frac{}{\Gamma \vdash \mathbf{0} : \overline{0}}\ ax_{\overline{0}}$$

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{\alpha}(U \to T_i) \qquad \Gamma \vdash \mathbf{r} : \sum_{j=1}^{\beta} U}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} T_i}\ \to_E \qquad \frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T}\ \to_I$$

$$\frac{\Gamma \vdash \mathbf{t} : \forall X.U}{\Gamma \vdash \mathbf{t} : U[V/X]}\ \forall_E \qquad \frac{\Gamma \vdash \mathbf{t} : U \qquad X \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t} : \forall X.U}\ \forall_I$$

$$\frac{\Gamma \vdash \mathbf{t} : T \qquad \Gamma \vdash \mathbf{r} : R}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T + R}\ +_I \qquad \frac{\Gamma \vdash \mathbf{t} : T \qquad T \equiv R}{\Gamma \vdash \mathbf{t} : R}\ \equiv$$

Additive type system

Example

Let

$$\Gamma \vdash \mathbf{b}_1 : U$$
$$\Gamma \vdash \mathbf{b}_2 : U$$
$$\Gamma \vdash \lambda x.\mathbf{t} : U \to T$$
$$\Gamma \vdash \lambda y.\mathbf{r} : U \to R$$

Then

$$\frac{\Gamma \vdash \lambda x.\mathbf{t} + \lambda y.\mathbf{r} : (U \to T) + (U \to R) \quad \Gamma \vdash \mathbf{b}_1 + \mathbf{b}_2 : U + U}{\Gamma \vdash (\lambda x.\mathbf{t} + \lambda y.\mathbf{r})\ (\mathbf{b}_1 + \mathbf{b}_2) : T + T + R + R} \to_E$$

Notice that

$$(\lambda x.\mathbf{t} + \lambda y.\mathbf{r})\ (\mathbf{b}_1 + \mathbf{b}_2) \to \underbrace{(\lambda x.\mathbf{t})\ \mathbf{b}_1}_{T} + \underbrace{(\lambda x.\mathbf{t})\ \mathbf{b}_2}_{T} + \underbrace{(\lambda y.\mathbf{r})\ \mathbf{b}_1}_{R} + \underbrace{(\lambda y.\mathbf{r})\ \mathbf{b}_2}_{R}$$

Subject Reduction

Theorem

Let $\boldsymbol{t} \longrightarrow^* \boldsymbol{t}'$*. Then* $\Gamma \vdash \boldsymbol{t}: T \Rightarrow \Gamma \vdash \boldsymbol{t}' : T$

Sketch of the proof:

- Extension of Barendregt's proof.
- Rule by rule in one-step reduction.
- Several generation lemmas.

Strong Normalisation

Theorem

If $\boldsymbol{t}$ *is typable in Additive, then it is strongly normalising*

Sketch of the proof:

- Extension of Girard's proof.
- Every type T is interpreted by a reducibility candidate $[T]$, *ie* a set of closed and strongly normalising terms.
- We prove that $\vdash \mathbf{t} : T$ implies $\mathbf{t} \in [T]$

System F with pairs

$$\begin{array}{rcl} t, u & := & x \mid \lambda x.t \mid tu \mid \star \mid \langle t, u\rangle \mid \pi_1(t) \mid \pi_2(t) \\ A, B & := & X \mid A \Rightarrow B \mid \forall X.A \mid \mathbf{1} \mid A \times B \end{array}$$

System F with pairs

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$$(\lambda x.t)\ u \to t[x := u] \qquad\qquad \pi_i(\langle t_1, t_2\rangle) \to t_i$$

System F with pairs

$$\begin{array}{rcl} t, u & := & x \mid \lambda x.t \mid tu \mid \star \mid \langle t, u \rangle \mid \pi_1(t) \mid \pi_2(t) \\ A, B & := & X \mid A \Rightarrow B \mid \forall X.A \mid \mathbf{1} \mid A \times B \end{array}$$

$$(\lambda x.t)\ u \to t[x := u] \qquad \pi_i(\langle t_1, t_2 \rangle) \to t_i$$

$$\frac{}{\Delta, x : A \vdash_F x : A}{Ax} \qquad \frac{}{\Delta \vdash_F \star : \mathbf{1}}{\mathbf{1}} \qquad \frac{\Delta \vdash_F t : A \times B}{\Delta \vdash_F \pi_1(t) : A}{\times E_\ell}$$

$$\frac{\Delta \vdash_F t : A \qquad \Delta \vdash_F u : B}{\Delta \vdash_F \langle t, u \rangle : A \times B}{\times I} \qquad \frac{\Delta \vdash_F t : A \times B}{\Delta \vdash_F \pi_2(t) : B}{\times E_r}$$

$$\frac{\Delta, x : A \vdash_F t : B}{\Delta \vdash_F \lambda x.t : A \Rightarrow B}{\Rightarrow I} \qquad \frac{\Delta \vdash_F t : A \Rightarrow B \qquad \Delta \vdash_F u : A}{\Delta \vdash_F tu : B}{\Rightarrow E}$$

$$\frac{\Delta \vdash_F t : A \qquad X \notin FV(\Delta)}{\Delta \vdash_F t : \forall X.A}{\forall I} \qquad \frac{\Delta \vdash_F t : \forall X.A}{\Delta \vdash_F t : A[X := B]}{\forall E}$$

Translation of types

Additive		**System F with pairs**
X	$\rightsquigarrow$	X
$U \rightarrow T$	$\rightsquigarrow$	$\lvert U \rvert \Rightarrow \lvert T \rvert$
$\forall X.U$	$\rightsquigarrow$	$\forall X.\lvert U \rvert$
$\bar{0}$	$\rightsquigarrow$	$\mathbf{1}$
$T + S$	$\rightsquigarrow$	$\lvert T \rvert \times \lvert S \rvert$

Sums as Pairs

$$+\ ,\ \overline{0} \qquad \rightsquigarrow \qquad \times\ ,\ \mathbf{1}$$

$$\begin{aligned} T + S &\equiv S + T & A \times B &\neq B \times A \\ T + (S + R) &\equiv (T + S) + R & A \times (B \times C) &\neq (A \times B) \times C \\ T + \overline{0} &\equiv T & A \times \mathbf{1} &\neq A \end{aligned}$$

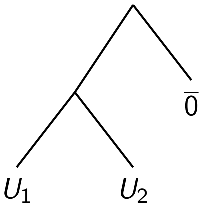
Sums as Pairs

$$+\ ,\ \overline{0} \qquad \rightsquigarrow \qquad \times\ ,\ \mathbf{1}$$

$$\begin{aligned} T + S &\equiv S + T \\ T + (S + R) &\equiv (T + S) + R \\ T + \overline{0} &\equiv T \end{aligned} \qquad \begin{aligned} A \times B &\neq B \times A \\ A \times (B \times C) &\neq (A \times B) \times C \\ A \times \mathbf{1} &\neq A \end{aligned}$$

$$\begin{aligned} T, R, S &:= U \mid T + R \mid \overline{0} \\ U, V, W &:= X \mid U \to T \mid \forall X.U \end{aligned}$$

Type $\rightsquigarrow$ Binary tree (leaf: U or $\overline{0}$)
Example: $T = (U_1 + U_2) + \overline{0}$

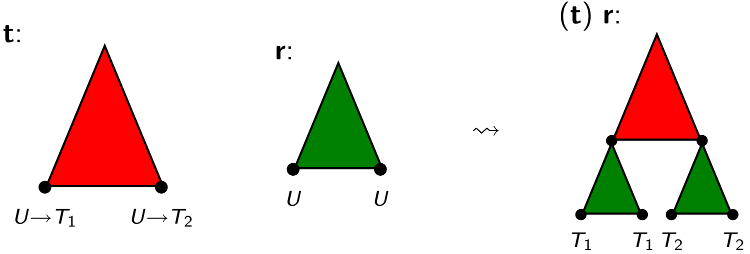


We keep structured sum types

Structured Arrow-elimination

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{\alpha}(U \to T_i) \qquad \Gamma \vdash \mathbf{r} : \sum_{j=1}^{\beta} U}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} T_i}$$

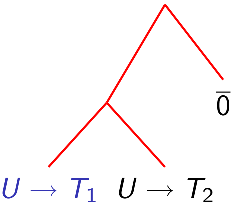
$$\frac{\Gamma \vdash \mathbf{t} : \mathcal{A}[\ell \mapsto (U \to T_\ell)] \qquad \Gamma \vdash \mathbf{r} : \mathcal{A}'[\ell' \mapsto U]}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \mathcal{A} \circ \mathcal{A}'[\ell\ell' \mapsto T_\ell]}$$



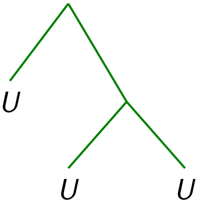
Structured Arrow-elimination

An example

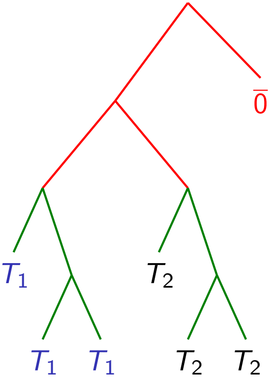
$\mathbf{t} = (\mathbf{t}_1 + \mathbf{t}_2) + 0$



$\mathbf{r} = \mathbf{r}_1 + (\mathbf{r}_2 + \mathbf{r}_3)$



$$
\begin{aligned}
(\mathbf{t})\ \mathbf{r} \to^* &\ (\mathbf{t}_1)\ (\mathbf{r}_1 + (\mathbf{r}_2 + \mathbf{r}_3)) + \\
&\ (\mathbf{t}_2)\ (\mathbf{r}_1 + (\mathbf{r}_2 + \mathbf{r}_3)) + 0 \\
\to^* &\ (\mathbf{t}_1)\ \mathbf{r}_1 + ((\mathbf{t}_1)\ \mathbf{r}_2 + (\mathbf{t}_1)\ \mathbf{r}_3) + \\
&\ (\mathbf{t}_2)\ \mathbf{r}_1 + ((\mathbf{t}_2)\ \mathbf{r}_2 + (\mathbf{t}_2)\ \mathbf{r}_3) + 0
\end{aligned}
$$



Equivalence in System F

What about associativity, commutativity and neutral element in system F with pairs?

Theorem

$$T \equiv T' \qquad \textit{implies} \qquad |T| \Leftrightarrow |T'|$$

Where $A \Leftrightarrow B$ means

$$\vdash_F \varepsilon_{A,B} : A \Rightarrow B \qquad \text{and} \qquad \vdash_F \varepsilon_{B,A} : B \Rightarrow A$$

for some terms $\varepsilon_{A,B}, \varepsilon_{B,A}$ s.t.

$$\varepsilon_{A,B} \circ \varepsilon_{B,A} \approx id_A \qquad \text{and} \qquad \varepsilon_{B,A} \circ \varepsilon_{A,B} \approx id_B$$

Translation of terms

What happens with linearity?

$$\mathbf{t} + \mathbf{r} \quad \rightsquigarrow \quad \langle [\mathbf{t}], [\mathbf{r}] \rangle$$

$$(\mathbf{t}_1 + \mathbf{t}_2)\ (\mathbf{r}_1 + \mathbf{r}_2) \to^* \mathbf{t}_1\ \mathbf{r}_1 + \mathbf{t}_1\ \mathbf{r}_2 + \mathbf{t}_2\ \mathbf{r}_1 + \mathbf{t}_2\ \mathbf{r}_2 \quad \text{but} \quad \langle t_1, t_2 \rangle\ \langle r_1, r_2 \rangle \not\to \langle \langle t_1 r_1, t_1 r_2 \rangle, \langle t_2 r_1, t_2 r_2 \rangle \rangle$$

No linearity in System F with pairs

Translation of terms

What happens with linearity?

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$(\mathbf{t}_1 + \mathbf{t}_2)\ (\mathbf{r}_1 + \mathbf{r}_2) \to^*$ $\mathbf{t}_1\ \mathbf{r}_1 + \mathbf{t}_1\ \mathbf{r}_2 + \mathbf{t}_2\ \mathbf{r}_1 + \mathbf{t}_2\ \mathbf{r}_2$ but $\langle t_1, t_2 \rangle \langle r_1, r_2 \rangle \not\to \langle \langle t_1 r_1, t_1 r_2 \rangle, \langle t_2 r_1, t_2 r_2 \rangle \rangle$

No linearity in System F with pairs

Main ideas:

- The sum is distributed during the translation of application

$$[(\mathbf{t})\ \mathbf{r}] = \langle\ \langle [\mathbf{t}_1][\mathbf{r}_1], [\mathbf{t}_1][\mathbf{r}_2] \rangle\ ,\ \langle [\mathbf{t}_2][\mathbf{r}_1], [\mathbf{t}_2][\mathbf{r}_2] \rangle\ \rangle$$

if $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$ and $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$

Translation of terms

What happens with linearity?

$$\mathbf{t} + \mathbf{r} \qquad \rightsquigarrow \qquad \langle [\mathbf{t}], [\mathbf{r}] \rangle$$

$(\mathbf{t}_1 + \mathbf{t}_2)\ (\mathbf{r}_1 + \mathbf{r}_2) \to^*$
$\mathbf{t}_1\ \mathbf{r}_1 + \mathbf{t}_1\ \mathbf{r}_2 + \mathbf{t}_2\ \mathbf{r}_1 + \mathbf{t}_2\ \mathbf{r}_2$

but $\langle t_1, t_2 \rangle \langle r_1, r_2 \rangle \not\to$
$\langle \langle t_1 r_1, t_1 r_2 \rangle, \langle t_2 r_1, t_2 r_2 \rangle \rangle$

No linearity in System F with pairs

Main ideas:

- The sum is distributed during the translation of application
$$[(\mathbf{t})\ \mathbf{r}] = \langle\ \langle [\mathbf{t}_1][\mathbf{r}_1], [\mathbf{t}_1][\mathbf{r}_2] \rangle\ ,\ \langle [\mathbf{t}_2][\mathbf{r}_1], [\mathbf{t}_2][\mathbf{r}_2] \rangle\ \rangle$$
if $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$ and $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$
- The "sum structure" of a term is known thanks to its type
$$\Gamma \vdash \mathbf{t} : (T_1 + T_2) + T_3 \qquad \rightsquigarrow \quad \mathbf{t} \text{ “}\simeq\text{” } (\mathbf{t}_1 + \mathbf{t}_2) + \mathbf{t}_3$$
with $\Gamma \vdash \mathbf{t}_i : T_i$

Translation of terms

$$\Gamma \;\vdash\; \mathbf{t} : T \qquad \rightsquigarrow \qquad |\Gamma| \;\vdash_F\; [\mathbf{t}]_{\mathcal{D}} : |T|$$

$$\Gamma, x : T \vdash x : T \qquad \rightsquigarrow \qquad [x]_{\mathcal{D}} = x$$

$$\Gamma \vdash \mathbf{0} : \overline{0} \qquad \rightsquigarrow \qquad [\mathbf{0}]_{\mathcal{D}} = \star$$

Translation of terms

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$$\frac{\Gamma \vdash \mathbf{t} : T \quad T \equiv T'}{\Gamma \vdash \mathbf{t} : T'} \qquad \rightsquigarrow \qquad [\mathbf{t}]_{\mathcal{D}} = \varepsilon_{|T|,|T'|}[\mathbf{t}]_{\mathcal{D}'}$$

Translation of terms

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$$\frac{\Gamma \vdash \mathbf{t} : T \quad \Gamma \vdash \mathbf{r} : S}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T + S} \quad \rightsquigarrow \quad [\mathbf{t}+\mathbf{r}]_{\mathcal{D}} = \langle [\mathbf{t}]_{\mathcal{D}_1}, [\mathbf{r}]_{\mathcal{D}_2} \rangle$$

$$\frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T} \quad \rightsquigarrow \quad [\lambda x.\mathbf{t}]_{\mathcal{D}} = \lambda x.[\mathbf{t}]_{\mathcal{D}'}$$

Translation of terms

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$$\frac{\Gamma \vdash \mathbf{t} : T \quad \Gamma \vdash \mathbf{r} : S}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T + S} \quad \rightsquigarrow [\mathbf{t}{+}\mathbf{r}]_{\mathcal{D}} = \langle [\mathbf{t}]_{\mathcal{D}_1}, [\mathbf{r}]_{\mathcal{D}_2} \rangle$$

$$\frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T} \quad \rightsquigarrow \quad [\lambda x.\mathbf{t}]_{\mathcal{D}} = \lambda x.[\mathbf{t}]_{\mathcal{D}'}$$

$$\frac{\Gamma \vdash \mathbf{t} : \mathcal{A}[w \mapsto (U \to T_w)] \quad \Gamma \vdash \mathbf{r} : \mathcal{A}'[v \mapsto U]}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \mathcal{A} \circ \mathcal{A}'[wv \mapsto T_w]}$$

$$\rightsquigarrow [(\mathbf{t})\ \mathbf{r}]_{\mathcal{D}} = \mathcal{A} \circ \mathcal{A}'[wv \mapsto \pi_{\overline{w}}([\mathbf{t}]_{\mathcal{D}_1})\pi_{\overline{v}}([\mathbf{u}]_{\mathcal{D}_2})]$$

Translation of terms

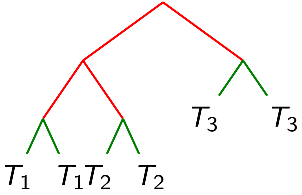
An example

$$\frac{\Gamma \vdash \mathbf{t} : \Big((U \to T_1) + (U \to T_2)\Big) \;+\; (U \to T_3) \qquad \Gamma \vdash \mathbf{r} : U + U}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \Big((T_1 + T_1) + (T_2 + T_2)\Big) + (T_3 + T_3)}$$

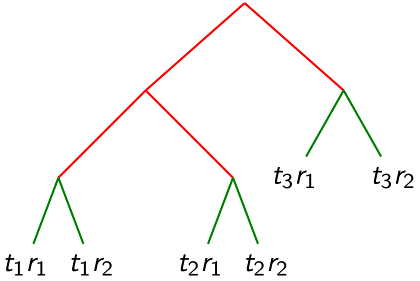
$t_1 = \pi_{11}([\mathbf{t}]); t_2 = \pi_{12}([\mathbf{t}]); t_3 = \pi_2([\mathbf{t}]);$

$r_1 = \pi_1([\mathbf{r}]); r_2 = \pi_2([\mathbf{r}]);$

$(\mathbf{t})\ \mathbf{r}$:



$[(\mathbf{t})\ \mathbf{r}]_{\mathcal{D}} =$



Correctness

Correctness of typing:

Theorem

$$\Gamma \vdash \boldsymbol{t} : T \quad \Longrightarrow \quad |\Gamma| \vdash_F [\boldsymbol{t}]_{\mathcal{D}} : |T|$$

Correctness of reduction (work-in-progress):

$\Gamma \vdash \mathbf{t} : T$ and $\mathbf{t} \rightarrow \mathbf{t}'$ implies $[\mathbf{t}]_{\mathcal{D}} \leftrightarrow^* [\mathbf{t}']_{\mathcal{D}'}$ for some $\mathcal{D}'$

Conclusions and Future work

- Sums in algebraic calculi $\rightarrow$ AC pairs + application ✓
- Can we find an interpretation for the scalars?

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- Can we find an interpretation for the scalars?
  - Natural numbers $\rightarrow$ sums ✓
  - Positive reals: Taking the floor (or ceiling), possible abstract interpretation using this result
  - Any general ring?[1] no clue

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- Can we find an interpretation for the scalars?
  - Natural numbers $\rightarrow$ sums ✓
  - Positive reals: Taking the floor (or ceiling), possible abstract interpretation using this result
  - Any general ring?[1] no clue
- Denotational semantic

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