Linearizing Higher-Order Processes

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Motivation

- Processes should be **reactive**:
  - Between any pair of visible actions, there should be only a **finite** number of invisible, internal actions.
  - There are type systems which guarantees this property on π-terms.
- Sometimes you want more than that, namely **bounded reactivity**:
  - A **bounded** number of invisible actions between any pair of visible actions.
  - Bounded by what?
    - Constant bounds? [Kobayashi03]
    - Parametric bounds, in a synchronous scenario [AmadioDabrowski07]

The Sequential, Functional Setting

- Reactivity is **termination**.
  - Or **productivity**.
- What is reactivity with parametric bounds?
  - Termination with bounded complexity.
  - Examples: polytime functions, linear time functions, exponential time functions, etc.
- Many different techniques for enforcing bounded termination in functional programming languages:
  - Type Systems [KraryWeirich00,Hofmann].
  - Static Analysis [MarionMoyen00,].
  - ICC [BellantoniCook92,Leivant93,Girard97,Terui01].

From Intuitionistic Logic to Soft Linear Logic

Logic	**Axioms**
Intuitionistic Logic	$\mathcal{CCC}$
(Intuitionistic) Multiplicative and Exponential Linear Logic	$\mathcal{SMCC}$ $!A \multimap !A \otimes !A$ $!A \multimap 1$ $!A \multimap !!A$ $!A \multimap A$
(Intuitionistic) Soft Linear Logic	$\mathcal{SMCC}$ $!A \multimap A \otimes \ldots \otimes A$ $!A \multimap 1$

Soft Linear Logic

- It is **polynomial time sound** [Lafont02]:
  - $\mathbb{B}(\pi)$ is the box depth of any proof π;

Theorem

There is a family of polynomials $\{p_n\}_n$ such that the normal form of any proof π can be computed in time $p_{\mathbb{B}(\pi)}(|\pi|)$

  - This holds for many notions of proofs: proof-nets, sequent-calculus, lambda-terms, etc.
- It is also **polynomial time complete** [Lafont02, MairsonTerui03]:
  - A function $f : \mathbb{N} \to \mathbb{N}$ can be represented in soft linear logic if a proof π_f rewrites to an encoding of $f(n)$ when cut against an encoding of n.

Theorem

Every polynomial time function can be represented in soft linear logic.

From Lambda Calculus to Soft Lambda Calculus

- Lambda calculus Λ:

$$M ::= x \mid \lambda x.M \mid MM$$

- Linear Lambda Calculus $\Lambda_!$

$$M ::= x \mid \lambda x.M \mid \lambda !x.M \mid MM \mid !M$$

where $\mathbf{NFO}(x, M) = 1$ and $\mathbf{LFO}(x, M) = \{0\}$ in $\lambda x.M$.

- Soft Lambda Calculus Λ_S

$$M ::= x \mid \lambda x.M \mid \lambda !x.M \mid MM \mid !M$$

where

- $\mathbf{NFO}(x, M) = 1$ and $\mathbf{LFO}(x, M) = 0$ in $\lambda x.M$.
- either $\mathbf{NFO}(x, M) = 1$ and $\mathbf{LFO}(x, M) = \{1\}$ or $\mathbf{LFO}(x, M) = \{0\}$ in $\lambda !x.M$.

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From Lambda Calculus to Soft Lambda Calculus

- $\Lambda \Longrightarrow \Lambda_!$ is a **Refinement**.
  - Whenever a term can be copied, it must be marked as such, with !.
  - Some results continue to hold
  - Λ can be embedded into $\Lambda_!$

$$\begin{aligned}\{x\} &= x\\ \{\lambda x.M\} &= \lambda !x.\{M\}\\ \{MN\} &= \{M\}!\{N\}\end{aligned}$$

- $\Lambda_! \Longrightarrow \Lambda_S$ is a **Restriction**.
  - Whenever you copy, you lose the possibility of copying.
  - Examples:

$$\begin{array}{cc}\lambda !x.yxx & \checkmark\\ \lambda !x.y!x & \checkmark\\ \lambda !x.y(!x)x & \lightning\end{array}$$

  - Some results about SLL continue to hold:
    - Polytime soundness
    - Polytime completeness

What About Processes?

- It has been showed that [EhrhardLaurent07]:
  - A **linear name-passing π-calculus** can be interpreted into...
  - ...**differential** interaction nets.
- Interesting Questions:
  - What is the expressive power of the encoded π-calculus?
  - Can we restrict differential interaction nets and capture interesting classes of processes?
- We here adopt a different strategy:
  - Forget about **logic**.
  - But keeping in mind the decomposition **copying-dispatching**.
  - Apply the decomposition to HOπ (higher-order π-calculus).

Higher-Order π-Calculus

- Processes:

$$V ::= \star \mid \lambda x.P$$
$$P ::= \mathbf{0} \mid x \mid P \parallel P \mid a\langle x\rangle.P \mid \overline{a}\langle V\rangle.P \mid (\nu a)P \mid VV$$

- Reduction:

$$\frac{}{\overline{a}\langle V\rangle.P \parallel a\langle x\rangle.Q \rightarrow_{\mathsf{P}} P \parallel Q[x/V]} \qquad \frac{}{(\lambda x.P)V \rightarrow_{\mathsf{P}} P[x/V]}$$

$$\frac{P \rightarrow_{\mathsf{P}} Q}{P \parallel R \rightarrow_{\mathsf{P}} Q \parallel R} \qquad \frac{P \rightarrow_{\mathsf{P}} Q}{(\nu a)P \rightarrow_{\mathsf{P}} (\nu a)Q}$$

$$\frac{P \equiv Q \quad Q \rightarrow_{\mathsf{P}} R \quad R \equiv S}{P \rightarrow_{\mathsf{P}} S}$$

Higher-Order π-Calculus

- Nontermination:

$$P = \lambda y.a\langle x\rangle.(x \star \;||\; \overline{a}\langle x\rangle)$$
$$Q = P \star \;||\; \overline{a}\langle P\rangle$$

Indeed:

$$Q \to Q \to \dots$$

- More interesting example:

$$P = \lambda z.a\langle x\rangle.(b\langle y\rangle.\overline{c}\langle y\rangle.x \star \;||\; \overline{a}\langle x\rangle)$$
$$Q = P \star \;||\; \overline{a}\langle P\rangle$$

Linear Higher-Order π-Calculus: LHOπ

- Values and Processes:

$$V ::= \star \mid x \mid \lambda x.P \mid \lambda !x.P \mid !V$$
$$P ::= \mathbf{0} \mid P \parallel P \mid a\langle x\rangle.P \mid a\langle !x\rangle.P \mid \overline{a}\langle V\rangle.P \mid (\nu a)P \mid VV$$

  where $\mathbf{NFO}(x, P) = 1$ and $\mathbf{LFO}(x, P) = \{0\}$ in $a\langle x\rangle.P$ and $\lambda x.P$.

- Examples:

$$\begin{array}{cc} a\langle x\rangle.x\star & \checkmark \\ a\langle !x\rangle.(x\star \parallel !x\star) & \checkmark \\ a\langle !x\rangle.\overline{a}\langle x\rangle.\overline{b}\langle x\rangle.\mathbf{0} & \checkmark \\ a\langle !x\rangle.(b\langle y\rangle.\overline{c}\langle y\rangle.x\star \parallel \overline{a}\langle !x\rangle.\mathbf{0}) & \checkmark \\ a\langle x\rangle.(!x)\star & \lightning \end{array}$$

Linear Higher-Order π-Calculus : LHOπ

- Reduction:

$$\frac{}{\overline{a}\langle V\rangle.P \mid\mid a\langle x\rangle.Q \rightarrow_{\mathsf{L}} P \mid\mid Q[x/V]}$$

$$\frac{}{\overline{a}\langle !V\rangle.P \mid\mid a\langle !x\rangle.Q \rightarrow_{\mathsf{L}} P \mid\mid Q[x/V]}$$

$$\frac{}{(\lambda x.P)V \rightarrow_{\mathsf{L}} P[x/V]} \qquad \frac{}{(\lambda !x.P)!V \rightarrow_{\mathsf{L}} P[x/V]}$$

$$\frac{P \rightarrow_{\mathsf{L}} Q}{P \mid\mid R \rightarrow_{\mathsf{L}} Q \mid\mid R} \qquad \frac{P \rightarrow_{\mathsf{L}} Q}{(\nu a)P \rightarrow_{\mathsf{L}} (\nu a)Q}$$

$$\frac{P \equiv Q \quad Q \rightarrow_{\mathsf{L}} R \quad R \equiv S}{P \rightarrow_{\mathsf{L}} S}$$

Embedding LHOπ Into HOπ

$$
\begin{aligned}
[\star]_{\mathsf{V}} &= \star \\
[\lambda x.P]_{\mathsf{V}} &= \lambda !x.[P]_{\mathsf{P}} \\
[\mathbf{0}]_{\mathsf{P}} &= \mathbf{0} \\
[x]_{\mathsf{P}} &= x \\
[P \mid\mid Q]_{\mathsf{P}} &= [P]_{\mathsf{P}} \mid\mid [Q]_{\mathsf{P}} \\
[a\langle x\rangle.P]_{\mathsf{P}} &= a\langle !x\rangle.[P]_{\mathsf{P}} \\
[\overline{a}\langle V\rangle.P]_{\mathsf{P}} &= \overline{a}\langle ![V]_{\mathsf{V}}\rangle.[P]_{\mathsf{P}} \\
[(\nu a)P]_{\mathsf{P}} &= (\nu a)[P]_{\mathsf{P}} \\
[VV]_{\mathsf{P}} &= [V]_{\mathsf{V}}![V]_{\mathsf{V}}
\end{aligned}
$$

Proposition (Simulation)

If $P \rightarrow_{\mathsf{P}} Q$, then $[P]_{\mathsf{P}} \rightarrow_{\mathsf{L}} [Q]_{\mathsf{P}}$

Soft Processes: SHOπ

Processes:

$$V ::= \star \mid x \mid \lambda x.P \mid \lambda !x.P \mid !V$$
$$P ::= \mathbf{0} \mid P \mid\mid P \mid a\langle x\rangle.P \mid a\langle !x\rangle.P \mid \overline{a}\langle V\rangle.P \mid (\nu a)P \mid VV$$

where

- $\mathbf{NFO}(x, P) = 1$ and $\mathbf{LFO}(x, P) = \{0\}$ in $a\langle x\rangle.P$ and $\lambda x.P$.
- Either

  $$\mathbf{NFO}(x, P) = 1 \wedge \mathbf{LFO}(x, P) = \{1\}$$

  or

  $$\mathbf{LFO}(x, P) = \{0\}$$

  in $x\langle !P\rangle.$ and $\lambda !x.P$.

Soft Processes: Examples

$a\langle x\rangle.x\star$ ✓

$a\langle !x\rangle.(x\star \parallel (!x)\star)$ $\lightning$

$a\langle !x\rangle.\overline{a}\langle x\rangle.\overline{b}\langle x\rangle.\mathbf{0}$ ✓

$a\langle !x\rangle.(b\langle y\rangle.\overline{c}\langle y\rangle.x \parallel \overline{a}\langle !x\rangle)$ $\lightning$

Soft Processes: Examples

$$a\langle x\rangle.x\star \qquad \checkmark$$

$$a\langle !x\rangle.(x\star \mid\mid (!x)\star) \qquad \lightning$$

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Soft Processes: Examples

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$$a\langle !x\rangle.(b\langle y\rangle.\overline{c}\langle y\rangle.x \;||\; \overline{a}\langle !x\rangle) \qquad \lightning$$

Polytime Soundness

Definition

Given a process P, define:

- $\mathbb{B}(P)$: the maximum !-nesting depth of P;
- $\mathbb{D}(P) = \max\{\mathbf{NFO}(x, Q) \mid Q \text{ is a subprocess of } P\}$;
- $\mathbb{W}_n(P)$: like $|P|$, but processes inside a ! counts for n;
- $\mathbb{W}(P) = \mathbb{W}_{\mathbb{D}(P)}(P)$.

Similarly for values.

Examples:

$$\mathbb{B}(!!x) = 2$$
$$\mathbb{D}(P) = \mathbb{D}(a\langle !x\rangle.((x\star) \mid\mid (x\star) \mid\mid (x\star))) = 3$$
$$\mathbb{W}(!(\lambda y.P)) = 1 + 3 \cdot 5 = 16$$

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Polytime Soundness

Lemma

For every P, $\mathbb{W}(P) \geq |P|$.

Proposition

If $\emptyset \vdash_{\mathsf{P}} Q$ *and* $Q \rightarrow_{\mathsf{L}} P$, *then* $\mathbb{W}(Q) > \mathbb{W}(P)$, $\mathbb{D}(Q) \geq \mathbb{D}(P)$ *and* $\mathbb{B}(Q) \geq \mathbb{B}(P)$.

Proposition

For every process P, $\mathbb{W}(P) \leq |P|^{\mathbb{B}(P)+1}$. *Similarly, for every value* V, $\mathbb{W}(V) \leq |V|^{\mathbb{B}(V)+1}$.

Theorem

There is a family of polynomials $\{p_n\}_n$ *such that for every process* P *and for every* m, *if* $P \rightarrow_{\mathsf{L}}^{m} Q$, *then* $m, |Q| \leq p_{\mathbb{B}(P)}(|P|)$.

Capturing Interesting Examples?

- Can we relax the well-formedness discipline to capture examples like the following?

$$\lambda !z.a\langle !x\rangle.(b\langle y\rangle.\overline{c}\langle y\rangle.x\star \;||\; \overline{a}\langle !x\rangle)$$

- The answer is **yes**, but we need another operator $\Box$, similar to !:

$$\lambda !z.a\langle \Box x\rangle.(b\langle y\rangle.\overline{c}\langle y\rangle.x\star \;||\; \overline{a}\langle \Box x\rangle)$$

- Notice that
  - x appears twice in the body P;
  - $\mathbf{LFO}(x, P) = \{0, 1\}$;
  - The occurrence of x at level 0 is in the "scope" on an input on b, on which we never do outputs.
- Polynomial soundness still holds.

Ongoing and Future Works

- Turn SHOπ into a type system for ordinary HOπ.
- Completeness?
- Find the "modal discipline" corresponding to existing type systems guarateeing termination of HOπ (e.g. [DemangeonHirshkoffSangiorgi09]).

Questions?