The light side of Interval Temporal Logic: the Bernays-Schönfinkel's fragment of CDT

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Abstract

Decidability and complexity of the satisfiability problem for the logics of time intervals have been extensively studied in the last years. Even though most interval logics turn out to be undecidable, meaningful exceptions exist, such as the logics of temporal neighborhood and (some of) the logics of the subinterval relation.

In this paper, we explore a different path to decidability: instead of restricting the set of modalities or imposing suitable semantic restrictions, we take the most expressive interval temporal logic studied so far, namely, Venema's CDT, and we suitably limit the nesting degree of modalities.

The decidability of the satisfiability problem for the resulting CDT fragment is proved by embedding it into a well-known decidable prefix quantifier class of first-order logic, namely, the Bernays-Schönfinkel's class. In addition, we show that such a fragment is in fact NP-complete (the Bernays-Schönfinkel's class is NEXPTIME-complete), and that any natural extension of it is undecidable.

1. Introduction

In the last years, the study of temporal reasoning and logics via interval-based approaches has been very intensive. Since the seminal work by Halpern and Shoham on the interval logic HS [17], that features a modality for each Allen's relation [1] between pairs of intervals (over a linear order), and Venema's work on the very expressive interval logic CDT [28], a series of papers on interval temporal logics has been published, e.g., [2, 3, 6, 7, 9, 21, 22, 26]. They study and almost completely solve the problem of classifying all "natural", genuinely interval-based logics with respect to their expressive and computational power.

One of the most significant decidable fragments of HS is Propositional Neighborhood Logic (PNL for short), whose two modalities correspond to Allen's relations *meets* and *met by*. PNL has been introduced in [15], and further studied in [4], where it has been shown to be expressively complete with respect to the two-variable fragment of first-order logic interpreted over a number of classes of linearly ordered sets.

Recently, some decidable extensions of PNL have been identified. In [25], it has been shown that the pair of modalities corresponding to Allen's relations *starts* and *started by* (or, equivalently, *end* and *ended by*) can be added to PNL preserving decidability over finite linear orders, while a decidable metric extension of PNL over natural numbers has been investigated in [5]. Unfortunately, it is possible to show that the addition of quite simple hybrid or first-order constructs to PNL immediately leads to undecidability [10, 11].

The D fragment of HS featuring a single modality for the Allen's relation *during* is a meaningful example of how easy is to fall into undecidability: D is decidable over dense linear orders and undecidable over finite and (weakly) discrete linear orders. Moreover, the extension of D with modalities for the inverse relation *contains*, the pair of relations *starts* and *started by* (or, equivalently, *ends* and *ended by*), and the pair of relations *before* and *later* is still decidable over dense linear orders [24], but the extension of D with any of the PNL modalities turns out to be undecidable over all interesting classes of linear orders [8].

Classical first-order logic presents a situation somehow similar. Ever since it has been shown that the full language is undecidable, a great effort has been done in order to identify more and more expressive decidable fragments. At least three different strategies have been explored: (i) to limit the number of variables of the language, (ii) to limit the type of formulas allowed by relativizing the quantification (*guarded fragments*), and (iii) to limit the structure and the shape of the quantifiers prefix.

First-order logics with a limited number of variables have been already explored in connection with interval temporal logics; most notably, as recalled before, the equivalence in expressive power between the two-variable fragment over linear orders (shown to be NEXPTIME-complete in [27]) has made it possible to prove decidability of PNL before specific decision procedures were tailored for the latter.

Guarded fragments (see, e.g., [16]) of first-order logics have been shown to be extremely useful to justify and understand the good computational properties of modal logics, but, to the best of our knowledge, they turned out to be almost useless to tackle interval-based temporal logics, the main reason being the fact that transitive guards, necessary to force the linearity of the structures, preserve decidability only when at most two variables are allowed, while interval properties (when intervals are interpreted as pairs of points) are mostly three-variables.

In this paper, we explore a novel technique, based on the third strategy: we analyze the relationships between interval-based temporal logics and quantifier prefix decidable first-order logics. The decidability of the latter family of logics does not depend only on the shape of the quantifier prefixes, but also on the number and the arity of predicate and function symbols that are allowed in the formulas, and the presence/absence of equality. Seven, intrinsically different decidable classes have been identified in the literature (an up-to-date survey on prefix classes of first-order logic can be found in [12].

For the purpose of this paper, it is sufficient to focus on the prefix vocabulary class of fragments identified in 1928 by Bernays and Schönfinkel [12]. Such a class features all and only formulas in prenex form, where the quantifier prefix is of the form $\exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_m$ and the quantifier-free part of the formula can use any predicate symbol of any arity, but no function symbols, and, possibly, equality.

We will consider the most expressive (undecidable) interval temporal logic studied so far, namely, Venems's CDT [28], and we will tailor a syntactically-defined fragment of it, called CDT_{BS}, in such a way that its standard translation fits into the above-mentioned prefix vocabulary class. It is well known that Bernays-Schönfinkel's fragment of first-order logic is expressive enough to model a linear order without specific properties such as discreteness or denseness. Simpler frame properties commonly studied in interval temporal logic literature, such as unboundedness, can also be expressed.

Moreover, we take into consideration a well-known, non-terminating tableau-based deduction system for CDT developed in [14], and we show that, over this specific fragment, it actually terminates. As a side effect, we prove that the satisfiability problem for CDT_{BS} is NP-complete, in sharp contrast with that of the Bernays-Schönfinkel's fragment, which is NEXPTIME-complete. Finally, we show that any natural extension of CDT_{BS} immediately steps into undecidability.

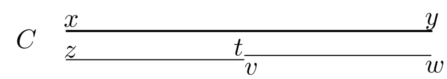

Figure 1. The ternary relation *chop*.

2. Preliminaries

The Bernays-Schönfinkel's prefix vocabulary class is defined by all and only those first-order formulas built with any relational symbol of any arity and such that they can be put in prenex form by using a quantifier prefix of the type $\exists\vec{x}\forall\vec{y}$, where $\vec{x} = x_1 \ldots x_n$ and $\vec{y} = y_1 \ldots y_m$ are (possibly empty) vectors of first-order variables. Throughout the paper we denote this class with FO_{BS}, and with $\text{FO}_{\text{BS}}[=]$ its extension with equality. It is well known that the satisfiability problem for both classes is NEXPTIME-complete [12]. It is worth to notice that, in general, the class is not closed by negation; nevertheless, since all our formulas can be thought of as sentences (free variables can be existentially quantified), the class is closed under conjunctions and disjunctions.

Interval temporal logics are usually interpreted over a linearly ordered set $\mathbb{D} = \langle D, < \rangle$. In this setting, an *interval* on $\mathbb{D}$ is any ordered pair $[d_i, d_j]$ such that $d_i \leq d_j$ (in recent literature, this is referred to as the *non-strict* semantics, in contrast with the *strict* one that excludes degenerate objects of the type $[d_i, d_i]$). The set of all intervals on $\mathbb{D}$ is denoted by $\mathbb{I}(\mathbb{D})$. The variety of all possible relations between any two intervals has been studied by Allen [1], that identified 12 different binary relations (plus the equality). Halpern and Shoham's HS is one of the first interval logic proposed in the literature, and it can be seen as the modal logic that features exactly one modal operator for each Allen's relation. HS is undecidable over most classes of linearly ordered sets [17]. In [28], the ternary relation *chop*, shown in Fig. 1, has been considered. The corresponding binary modal operator C, along with its two inverses D and T, and the modal constant π for point-intervals, gave rise to the interval logic CDT, that turned out to be undecidable whenever HS is, as the former is strictly more expressive than the latter. More recently, Hodkinson et al. studied in detail the properties of the three sub-fragments of CDT with only one binary modal operator, showing their undecidability [18].

3. The logic CDT_{BS} over all linear orders and its standard translation

Formulas of the logic CDT are built over a set of propositional letters $\mathcal{AP} = \{p, q, \ldots\}$, the classical connectives $\neg, \vee$, three binary modal operators C, D, and T, and one modal constant π, by the following abstract grammar [28]:

$$\varphi ::= p \mid \pi \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi\, C\, \varphi \mid \varphi\, D\, \varphi \mid \varphi\, T\, \varphi.$$

The other classical connectives can be thought of as shortcuts, as standard. Universal modalities do not have a special notation, and can be defined using the negation operation. The semantics of CDT-formulas can be expressed in terms of concrete *models* of the type $M = \langle \mathbb{I}(\mathbb{D}), V\rangle$, where $V : \mathcal{AP} \rightarrow 2^{\mathbb{I}(\mathbb{D})}$ is a *valuation function*:

- $M, [d_i, d_j] \Vdash p$ if and only if $[d_i, d_j] \in V(p)$,
- $M, [d_i, d_j] \Vdash \pi$ if and only if $d_i = d_j$,
- $M, [d_i, d_j] \Vdash \neg\varphi$ if and only if $M, [d_i, d_j] \not\Vdash \varphi$,
- $M, [d_i, d_j] \Vdash \varphi \vee \psi$ if and only if $M, [d_i, d_j] \Vdash \varphi$ or $M, [d_i, d_j] \Vdash \psi$,
- $M, [d_i, d_j] \Vdash \varphi\, C\, \psi$ if and only if there exists $d_i \leq d_k \leq d_j$ such that $M, [d_i, d_k] \Vdash \varphi$ and that $M, [d_k, d_j] \Vdash \psi$,
- $M, [d_i, d_j] \Vdash \varphi\, D\, \psi$ if and only if there exists $d_k \leq d_i$ such that $M, [d_k, d_i] \Vdash \varphi$ and that $M, [d_k, d_j] \Vdash \psi$,
- $M, [d_i, d_j] \Vdash \varphi\, T\, \psi$ if and only if there exists $d_k \geq d_j$ such that $M, [d_j, d_k] \Vdash \varphi$ and that $M, [d_i, d_k] \Vdash \psi$.

Standard translation is the classical tool used to express the semantics of a modal/temporal formula into first-order logic. The clauses for the standard translation $ST(\varphi)[x, y]$, over a pair of points x, y, are defined as follows:

$$\begin{aligned}
ST(p)[x, y] &= p(x, y)\\
ST(\pi)[x, y] &= (x = y)\\
ST(\neg\varphi)[x, y] &= \neg ST(\varphi)[x, y]\\
ST(\varphi \vee \psi)[x, y] &= ST(\varphi)[x, y] \vee ST(\psi)[x, y]\\
ST(\varphi\, C\, \psi)[x, y] &= \exists z(x \leq z \leq y \wedge ST(\varphi)[x, z] \wedge\\
&\qquad \wedge ST(\psi)[z, y])\\
ST(\varphi\, D\, \psi)[x, y] &= \exists z(z \leq x \wedge ST(\varphi)[z, x] \wedge\\
&\qquad \wedge ST(\psi)[z, y])\\
ST(\varphi\, T\, \psi)[x, y] &= \exists z(y \leq z \wedge ST(\varphi)[y, z] \wedge\\
&\qquad \wedge ST(\psi)[x, z])
\end{aligned}$$

Notice that the satisfiability problem for a generic modal logic is itself a first-order satisfiability problem: a formula φ is satisfiable if and only if there exists a model M and a pair of points x, y such that the standard translation of φ evaluated on x, y is (first-order) satisfiable. Now, we can ask ourselves the question: which CDT-formulas are such that their satisfiability problem is a first-order problem in Bernays-Schönfinkel's class? To answer this question we devise an abstract grammar that produces only CDT-formulas suitably limited in the nesting of modal operators:

$$\begin{aligned}
\varphi ::= {} & \pi \mid \neg\pi \mid p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid\\
& \varphi\, C\, \varphi \mid \varphi\, D\, \varphi \mid \varphi\, T\, \varphi \mid \qquad (1)\\
& \neg(\varphi_\exists\, C\, \varphi_\exists) \mid \neg(\varphi_\exists\, D\, \varphi_\exists) \mid \neg(\varphi_\exists\, T\, \varphi_\exists)\\
\varphi_\exists ::= {} & \pi \mid \neg\pi \mid p \mid \neg p \mid \varphi_\exists \wedge \varphi_\exists \mid \varphi_\exists \vee \varphi_\exists \mid \qquad (2)\\
& \varphi_\exists\, C\, \varphi_\exists \mid \varphi_\exists\, D\, \varphi_\exists \mid \varphi_\exists\, T\, \varphi_\exists
\end{aligned}$$

The above grammar generates a fragment of CDT that we call CDT_{BS} and that is characterized by the fact that the modal operators C, D, and T can occur in the scope of *at most one negation*. By exploiting this limitation we will show that the standard translation of every formula of CDT_{BS} is in Bernays-Schönfinkel's first-order class. It is easy to see that the syntactic limitations of CDT_{BS} do not prevent the logic to simulate every modal operator corresponding to an Allen's relation. For example, we have that: $\langle B\rangle\varphi \equiv \varphi\, C\, \neg\pi$, and similarly for the other cases [28].

In order to prove our main theorem, we need the following observation: the linear ordering relation $<$ can be axiomatized in Bernays-Schönfinkel's class with equality [12]. Indeed, consider the following classical properties:

1. $\forall x \neg(x < x)$;
2. $\forall x, y(x < y \rightarrow \neg y < x)$;
3. $\forall x, y, z(x < y \wedge y < z \rightarrow x < z)$;
4. $\forall x, y(x = y \vee x < y \vee y < x)$.

It is immediate to see that the conjunction Φ of the four axioms above is in $\text{FO}_{\text{BS}}[=]$.

Lemma 1. *For every formula φ of* CDT_{BS}*, its standard translation is of the type $\exists\vec{z}\forall\vec{w}\,\alpha(x, y)$, where $\alpha(x, y)$ is quantifier-free and $x, y \notin \vec{z}\vec{w}$.*

Proof. We proceed by structural induction. We start with the set of formulas generated by the sub-grammar for $\varphi_\exists$, and we show that their standard translations are of the form $\exists\vec{z}\,\alpha(x, y)$ with $x, y \notin \vec{z}$. As base case, assume $\varphi_\exists = p$ for some propositional letter p. Then $ST(p)[x, y] = p(x, y)$ and the claim holds trivially. The case $\neg p$ is similar, as well as the cases of π and $\neg\pi$. Consider now the case of $\varphi_\exists \wedge \varphi'_\exists$. By definition, $ST(\varphi_\exists \wedge \varphi'_\exists)[x, y] = ST(\varphi_\exists)[x, y] \wedge ST(\varphi'_\exists)[x, y]$. By inductive hypothesis, we have that $ST(\varphi_\exists)[x, y] = \exists\vec{z}\alpha(x, y)$ and $ST(\varphi'_\exists)[x, y] = \exists\vec{w}\beta(x, y)$, for some α and β quantifier-free and such that $x, y \notin \vec{z}\vec{w}$. We can assume that $\vec{z} \cap \vec{w} = \emptyset$ (otherwise we proceed to a suitable variables substitution), and therefore we have that $ST(\varphi_\exists \wedge \varphi'_\exists)[x, y] = \exists\vec{z}\vec{w}(\alpha(x, y) \wedge \beta(x, y))$. The case of disjunction is similar. Consider now the case of $\varphi_\exists C \varphi'_\exists$. By definition, $ST(\varphi_\exists\, C\, \varphi'_\exists)[x, y] = \exists z(x \leq z \leq y \wedge ST(\varphi_\exists)[x, z] \wedge ST(\varphi'_\exists)[z, y])$. By inductive hypothesis, we have that $ST(\varphi_\exists)[x, z] = \exists\vec{w}\alpha(x, z)$ and $ST(\varphi'_\exists)[z, y] = \exists\vec{t}\beta(z, y)$, with α and β quantifier-free and such that $x, y, z \notin \vec{w}\vec{t}$. As in the previous case, we assume $\vec{w} \cap \vec{t} = \emptyset$ and we can conclude that $ST(\varphi_\exists\, C\, \varphi'_\exists)[x, y] = \exists z(x \leq z \leq y \wedge \exists \vec{w}\alpha(x, y) \wedge \exists \vec{t}\beta(z, y)) = \exists z \exists \vec{w} \exists \vec{t}(x \leq z \leq y \wedge \alpha(x, y) \wedge \beta(z, y)$. The remaining two cases are similar.

We can now consider a generic formula generated by the grammar. The only interesting cases are those corresponding to negation of modalities. Therefore, consider the case of $\neg(\varphi_\exists\ C\ \varphi'_\exists)$. By definition, we have that $ST(\neg(\varphi_\exists C \varphi'_\exists))[x, y] = \neg\exists z(x \leq z \leq y \wedge ST(\varphi_\exists)[x, z] \wedge ST(\varphi'_\exists)[z, y])$. By the previous argument, we can assume that $ST(\neg(\varphi_\exists\ C\ \varphi'_\exists))[x, y] = \neg\exists z(x \leq z \leq y \wedge \exists \vec{w}\alpha(x, z) \wedge \exists \vec{t}\beta(z, y))$, that is equivalent to the formula in prenex form $\forall z \forall \vec{w} \forall \vec{t}(\neg(x \leq z \leq y) \vee \neg\alpha(x, z) \vee \neg\beta(z, y))$. The two remaining cases can be proved in a similar way. □

Theorem 1. *The satisfiability problem for* $\mathrm{CDT_{BS}}$ *in the class of all linear orders is decidable.*

Proof. By the above lemma, if φ is a $\mathrm{CDT_{BS}}$-formula, then $ST(\varphi)[x, y]$ is such that the formula $\exists x, y ST(\varphi)[x, y]$ is in the Bernays-Schönfinkel's class. Therefore, satisfiability of φ can be reduced to satisfiability of $\Phi \wedge \exists x, y ST(\varphi)[x, y]$, where, possibly, we have changed the variables in such a way that Φ and $ST(\varphi)[x, y]$ have no variables in common. Since the satisfiability problem for $\mathrm{FO_{BS}}[=]$ is decidable, decidability of $\mathrm{CDT_{BS}}$ trivially follows. □

4. A tableau method for $\mathrm{CDT_{BS}}$

In [14], Goranko et al. propose a tableau method for BCDT^+, a generalization of Venema's CDT logic to partial orders with linear interval property. Since the considered logic is undecidable, the method is not guaranteed to terminate, and it is only a semi-decision procedure. In this section we show how to tailor it to $\mathrm{CDT_{BS}}$, and how to exploit the syntactic restriction of this logic to guarantee termination and obtain an NP-complete decision procedure for it.

The tableau construction generates a tree, whose nodes are decorated with $\langle \psi, [d_i, d_j], \mathbb{D}, p, u \rangle$, where $\mathbb{D} = \langle D, < \rangle$ is a finite partial order (with linear interval property), $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$, $p \in \{0, 1\}$, and u is a *local flag function* which associates the values 0 or 1 with every branch B containing the node. Intuitively, the value 1 for a node n in a branch B means that n can be expanded on B. The auxiliary flag p distinguishes between formulas generated by rule (1) and formulas generated by rule (2) of the $\mathrm{CDT_{BS}}$ grammar, and it is added to simplify the termination and complexity proofs. If B is a branch, then $B \cdot n$ is the result of expanding B with the node n, while $B \cdot n_1 | \dots | n_k$ is the result of expanding B with k immediate successor nodes $n_1, \dots, n_k$. With $\mathbb{D}_B$ we denote the finite partial ordering in the leaf of B. Since in $\mathrm{CDT_{BS}}$ negation can occur only in front of propositional letters or modal operators, we need to introduce the notion of *dual formula* of a formula φ, denoted by $\overline{\varphi}$ and inductively defined as follows:

- $\overline{p} = \neg p$ and $\overline{\neg p} = p$, for every $p \in \mathcal{AP} \cup \{\pi\}$;
- $\overline{\varphi \vee \psi} = \overline{\varphi} \wedge \overline{\psi}$;
- $\overline{\varphi \wedge \psi} = \overline{\varphi} \vee \overline{\psi}$;
- $\overline{\varphi\, R\, \psi} = \neg(\varphi\, R\, \psi)$, for $R \in \{C, D, T\}$;
- $\overline{\neg(\varphi\, R\, \psi)} = \varphi\, R\, \psi$, for $R \in \{C, D, T\}$.

Notice that the dual of a generic formula of $\mathrm{CDT_{BS}}$ does not necessarily belong to $\mathrm{CDT_{BS}}$: this is the case, for instance, of the formula $pC\neg(qCr)$. However, it can be easily proved (by induction on the above production rules) the dual of a formula generated by the sub-grammar for $\varphi_\exists$ is always a formula of $\mathrm{CDT_{BS}}$. This observation will be crucial for the correctness of the tableau method.

The construction of a tableau for $\mathrm{CDT_{BS}}$ starts from a three-node *initial tree* built up from an empty-decorated root and two leaves with decorations $\langle \varphi, [d_0, d_0], \{d_0\}, 1, 1 \rangle$ and $\langle \varphi, [d_0, d_1], \{d_0 < d_1\}, 1, 1 \rangle$, respectively, where φ is the formula to check for satisfiability. The procedure exploits a set of expansion rules to add new nodes to the tree. Notice that the rules and other concepts are very similar to [14], which we refer to for further explanations.

Definition 1. *Given a tree* $\mathcal{T}$*, a branch* B *in* $\mathcal{T}$*, and a node* $n \in B$ *decorated with* $\langle \psi, [d_i, d_j], \mathbb{D}, p_n, u_n \rangle$ *such that* $u_n(B) = 1$*, the* branch expansion rule *for* B *and* n *is defined as follows. In all considered cases,* $u_{n'}(B') = 1$ *for all new nodes* n' *and branches* B'*.*

R1 *If* $\psi = \xi_0 \wedge \xi_1$*, then expand* B *to* $B \cdot n_0 \cdot n_1$*, where* n_0 *is decorated with* $\langle \xi_0, [d_i, d_j], \mathbb{D}_B, p_n, u_{n_0} \rangle$ *and* n_1 *is decorated with* $\langle \xi_1, [d_i, d_j], \mathbb{D}_B, p_n, u_{n_1} \rangle$*.*

R2 *If* $\psi = \xi_0 \vee \xi_1$*, expand* B *as in [14].*

R3 *If* $\psi = \neg(\xi_0\, C\, \xi_1)$ *and* d *is an element of* $\mathbb{D}_B$ *such that* $d_i \leq d \leq d_j$ *and* d *has not been used yet to expand* n *in* B*, then expand* B *to* $B \cdot n_0 | n_1$*, where* n_0 *is decorated with* $\langle \overline{\xi_0}, [d_i, d], \mathbb{D}_B, 0, u_{n_0} \rangle$ *and* n_1 *is decorated with* $\langle \overline{\xi_1}, [d, d_j], \mathbb{D}_B, 0, u_{n_1} \rangle$*.*

R4 *If* $\psi = \neg(\xi_0\, D\, \xi_1)$*, the rule is analogous to* ***R3****.*

R5 *If* $\psi = \neg(\xi_0\, T\, \xi_1)$*, the rule is analogous to* ***R3****.*

R6 *If* $\psi = \xi_0\, C\, \xi_1$*, then expand the branch* B *to* $B \cdot (n_i \cdot m_i) | \dots | (n_j \cdot m_j) | (n'_i \cdot m'_i) | \dots | (n'_{j-1} \cdot m'_{j-1})$*, where:*

- *(a) for all* $i \leq k \leq j$*,* n_k *is decorated with* $\langle \xi_0, [d_i, d_k], \mathbb{D}_B, p_n, u_{n_k} \rangle$ *and* m_k *is decorated with* $\langle \xi_1, [d_k, d_j], \mathbb{D}_B, p_n, u_{m_k} \rangle$*;*
- *(b) for all* $i \leq k \leq j - 1$*,* $\mathbb{D}_k$ *is the linear ordering obtained by inserting a new element* d *between* d_k *and* d_{k+1}*,* n'_k *is decorated with* $\langle \xi_0, [d_i, d], \mathbb{D}_k, p_n, u_{n'_k} \rangle$ *and* m'_k *is decorated with* $\langle \xi_1, [d, d_j], \mathbb{D}_k, p_n, u_{m'_k} \rangle$*.*

R7 *If $\psi = \xi_0\ D\ \xi_1$, the rule is analogous to* ***R6****.*
R8 *If $\psi = \xi_0\ T\ \xi_1$, the rule is analogous to* ***R6****.*

Finally, let $u_n(B) = 0$ and, for every node $m \neq n$ in B and any branch B' extending B, let $u_m(B') = u_m(B)$, while for every branch B' extending B, $u_n(B') = 0$, unless $\psi = \neg(\xi_0 C \xi_1)$, $\psi = \neg(\xi_0 D \xi_1)$, or $\psi = \neg(\xi_0 T \xi_1)$ (in such cases $u_n(B') = 1$).

We briefly explain the expansion rules for $\xi_0\ C\ \xi_1$ and $\neg(\xi_0 C \xi_1)$ (similar considerations can be made for the cases of the temporal operators D and T). The rule for the formula $\xi_0\ C\ \xi_1$ deals with two possible cases: either there exists $d_k \in \mathbb{D}_B$ such that ξ_0 holds over $[d_i, d_k]$ and ξ_1 holds over $[d_k, d_j]$, or such an element must be added to $\mathbb{D}_B$. On the converse, the formula $\neg(\xi_0\ C\ \xi_1)$ states that, for all $d_i \leq d \leq d_j$, either $\overline{\xi_0}$ holds over $[d_i, d]$ or $\overline{\xi_1}$ holds over $[d, d_j]$. The expansion rule imposes such a condition for a single element d and keeps the flag equal to 1. In this way, all elements of $\mathbb{D}_B$ are eventually considered, including those elements that will be added in some subsequent steps of the tableau construction.

Intuitively, a branch is *closed* if there are two nodes n, n' in B such that n is decorated with $\langle p, [d_i, d_j], \mathbb{D}, p_n, u_n \rangle$ and n' is decorated with $\langle \neg p, [d_i, d_j], \mathbb{D}', p_{n'}, u_{n'} \rangle$, for some $p \in \mathcal{AP}$, or if π (resp., $\neg\pi$) is in the decoration of a node where $d_i \neq d_j$ (resp., $d_i = d_j$). Otherwise, the branch is *open*. The expansion strategy for the tableau expands a branch B only if it is open and it applies the branch expansion rule to the closest-to-the-root node for which the branch expansion rule is applicable and generates at least one node with a new decoration. It is easy to adapt the result in [14] to obtain the following theorem.

Theorem 2. *The tableau method for* $\mathrm{CDT_{BS}}$ *is sound and complete.*

To prove that the method is terminating, and to establish its computational complexity, we need to fix some preliminary results. First of all, we define the *counting function* on B as follows:

$$\mathrm{Count}(B) = \sum_{n \in B} |\psi_n| \cdot p_n \cdot u_n(B),$$

where ψ_n and p_n are the formula and the p-flag in the decoration of n, respectively. The following lemma proves that $\mathrm{Count}(B)$ is non-increasing with respect to the expansion strategy.

Lemma 2. *Let B be a branch in a tableau for φ, and let B' be an expansion of B that respects the expansion strategy. Then $\mathrm{Count}(B') \leq \mathrm{Count}(B)$. Moreover, if the expansion strategy applied either* ***R1****,* ***R2****,* ***R6****,* ***R7****, or* ***R8*** *rule to a node n such that $p_n = 1$, then $\mathrm{Count}(B') < \mathrm{Count}(B)$.*

Proof. Let $\mathcal{T}$ be a tableau for φ, B a branch on it and n the closest to the root node for which the branch expansion rule is applicable. Let B' a branch obtained by applying the expansion strategy on B. We proceed by induction on the expansion rule applied to n. The missing rules are similar to other cases, and skipped.

- Rule **R1** is applied to n. Then, n is decorated with $\langle \xi_0 \wedge \xi_1, [d_i, d_j], \mathbb{D}, p_n, u_n \rangle$ and $B' = B \cdot n' \cdot m'$ is such that n' is decorated with ξ_0 and m' is decorated with ξ_1. Since $p_{n'} = p_{m'} = p_n$, $u_{n'}(B') = u_{m'}(B') = 1$, $u_n(B') = 0$, and $u_m(B') = u_m(B)$ for every $m \notin \{n, n', m'\}$, we have that $\mathrm{Count}(B') = \mathrm{Count}(B) - |\xi_0 \wedge \xi_1| + |\xi_0| + |\xi_1| < \mathrm{Count}(B)$, when $p_n = 1$, and that $\mathrm{Count}(B') = \mathrm{Count}(B)$ when $p_n = 0$.
- Rule **R3** is applied to B. Then, n is decorated with $\langle \neg(\xi_0\ C\ \xi_1), [d_i, d_j], \mathbb{D}, p_n, u_n \rangle$ and $B' = B \cdot n'$ is such that either n' is decorated with $\overline{\xi_0}$ or with $\overline{\xi_1}$. In both cases n' is decorated with $p_{n'} = 0$. This implies that $\mathrm{Count}(B') = \mathrm{Count}(B)$.
- Rule **R6** is applied to B. Then, n is decorated with $\langle \xi_0 C \xi_1, [d_i, d_j], \mathbb{D}, p_n, u_n \rangle$ and $B' = B \cdot n' \cdot m'$ where n' is decorated with ξ_0 and m' with ξ_1. Since $p_{n'} = p_{m'} = p_n$, $u_{n'}(B') = u_{m'}(B') = 1$, $u_n(B') = 0$, and $u_m(B') = u_m(B)$ for every $m \notin \{n, n', m'\}$, we have that $\mathrm{Count}(B') = \mathrm{Count}(B) - |\xi_0\ C\ \xi_1| + |\xi_0| + |\xi_1| < \mathrm{Count}(B)$ when $p_n = 1$, and that $\mathrm{Count}(B') = \mathrm{Count}(B)$ when $p_n = 0$.

In every case, we have that $\mathrm{Count}(B') \leq \mathrm{Count}(B)$. Moreover, in the cases of rules **R1**, **R2**, **R6**, **R7**, and **R8**, applied to a node n such that $p_n = 1$, we have that $\mathrm{Count}(B') < \mathrm{Count}(B)$. Hence, the thesis is proved. ☐

Let $\mathcal{L}(\varphi_\exists)$ be the language of all formulas that can be generated by the grammar rule (2) for $\mathrm{CDT_{BS}}$. Another crucial property of the tableau method is that the p flag in the decoration correctly marks formulas that belong to $\mathcal{L}(\varphi_\exists)$.

Lemma 3. *Let $\mathcal{T}$ be a tableau for φ and let n be a node in $\mathcal{T}$ decorated with $\langle \psi, [d_i, d_j], \mathbb{D}, p_n, u_n \rangle$. Then, $p_n = 0$ implies that $\overline{\psi} \in \mathcal{L}(\varphi_\exists)$.*

Proof. Let n be a node in $\mathcal{T}$ decorated with $\langle \psi, [d_i, d_j], \mathbb{D}, p_n, u_n \rangle$, and let B be the branch connecting the root of the tableau with n. We prove the claim by induction on the length of B. If $|B| \leq 2$ then n is either the root or one of the leaves of the initial tableau. In both cases the claim follows trivially. Now, let $|B| > 2$ and suppose by inductive hypothesis that the claim holds for every ancestor of n in B. Consider the node n' to which the expansion rule has been applied in the construction of $\mathcal{T}$ to obtain the node n; as before, only conceptually different cases are shown.

- Rule **R1** was applied to n'. Then, n' is decorated with $\langle \xi_0 \wedge \xi_1, [d_i, d_j], \mathbb{D}, p_{n'}, u_{n'} \rangle$ and n is decorated either with ξ_0 or with ξ_1. Suppose n decorated with ξ_0 (the opposite case is analogous) and $p_n = 0$. By rule **R1** we have that $p_{n'} = 0$. By inductive hypothesis we have that $\overline{\xi_0 \wedge \xi_1} = \overline{\xi_0} \vee \overline{\xi_1} \in \mathcal{L}(\varphi_\exists)$. From the grammar rules it follows that $\overline{\xi_0} \in \mathcal{L}(\varphi_\exists)$.
- Rule **R3** is applied to B. Then, n' is decorated with $\langle \neg(\xi_0 C \xi_1), [d_i, d_j], \mathbb{D}, p_{n'}, u_{n'} \rangle$ and n is decorated either with $\overline{\xi_0}$ or with $\overline{\xi_1}$. Suppose n decorated with $\overline{\xi_0}$ (the opposite case is analogous). By rule **R3** we have that $p_n = 0$. From the grammar for $\mathrm{CDT}_{\mathrm{BS}}$ it follows that $\xi_0 \in \mathcal{L}(\varphi_\exists)$. Since $\overline{\overline{\xi_0}} = \xi_0$, the claim is proved.
- Rule **R6** is applied to B. Then, n' is decorated with $\langle \xi_0 \ C \ \xi_1, [d_i, d_j], \mathbb{D}, p_{n'}, u_{n'} \rangle$ and n is decorated either with ξ_0 or with ξ_1. Suppose $p_{n'} = 0$: by inductive hypothesis this implies that $\overline{\xi_0 C \xi_1} = \neg(\xi_0 C \xi_1) \in \mathcal{L}(\varphi_\exists)$, a contradiction. Hence, $p_{n'} = 1$ and thus, by rule **R6**, $p_n = 1$. □

By exploiting Lemma 2 and Lemma 3 we can prove that the length of any branch B of any tableau for φ is polynomially bounded by the length of the formula.

Theorem 3. *Let $\mathcal{T}$ be a tableau for φ, and let B be a branch in $\mathcal{T}$. Then $|B| \leq 2|\varphi|^3 + 8|\varphi|^2 + 8|\varphi|$.*

Proof. Let B be a branch in a tableau $\mathcal{T}$ for φ. By the expansion rules and the expansion strategy, we have that there cannot be two nodes in B decorated with the same formula and the same interval. Since the formula in the decoration of a node is either a subformula of φ or the dual of a subformula of φ, we have that $|B| \leq 2 \cdot |\varphi| \cdot (|\mathbb{D}_B|)^2$.

To give a bound on the number of points in $\mathbb{D}_B$, it is sufficient to notice that:

1. only rules **R6**, **R7**, and **R8** adds new points to $\mathbb{D}_B$,
2. by Lemma 3, they can be applied only to nodes where the p flag is equal to 1, and
3. by Lemma 2, every application of them strictly decreases the value of $\mathrm{Count}(B)$.

Now, let B_0 be the two-node prefix of B made by the root and one of its successor labelled with φ. Since $\mathrm{Count}(B_0) = |\varphi|$, we have that $|\mathbb{D}_B| \leq |\varphi| + 2$ and thus we can conclude that $|B| \leq 2 \cdot |\varphi| \cdot (|\varphi| + 2)^2 \leq 2|\varphi|^3 + 8|\varphi|^2 + 8|\varphi|$. □

From Theorem 3 it follows that the tableau method for $\mathrm{CDT}_{\mathrm{BS}}$ is terminating and that its computational complexity is in NP. Since satisfiability for propositional logic is NP-hard, the following result trivially holds.

Corollary 1. *The satisfiability problem for* $\mathrm{CDT}_{\mathrm{BS}}$ *is NP-complete.*

5. Undecidable extensions of $\mathrm{CDT}_{\mathrm{BS}}$

In the previous sections we have shown that the satisfiability problem for $\mathrm{CDT}_{\mathrm{BS}}$ is decidable and, more precisely, it is NP-complete. Since it fits into the Bernays-Schönfinkel's class, which is NEXPTIME-complete, one may ask whether we can extend $\mathrm{CDT}_{\mathrm{BS}}$ preserving decidability. In this section, we show that the most natural extension of $\mathrm{CDT}_{\mathrm{BS}}$ is already undecidable.

As we have seen, $\mathrm{CDT}_{\mathrm{BS}}$ allows one to build formulas in modal prenex form and such that modal operators can occur in the scope of at most one negation. Therefore, the most natural extension is to allow one more nesting of negations and modal operators, obtaining formulas of the type $\neg(\neg(pCq)Cq)$ or $\neg(pC\neg(qCr))$. In [18] it has been shown that CDT is undecidable over the class of all linearly ordered sets even if only one binary modal operator and π are allowed in the formulas. Undecidability has been proven by reducing the problem of finding a solution to the *Octant Tiling Problem* to the satisfiability problem of the logic. The following theorem is based on the simple observation that the entire construction exploits formulas where modal operators occur in the scope of at most *two* negations.

Theorem 4. *The satisfiability problem for any extension of* $\mathrm{CDT}_{\mathrm{BS}}$ *where modal operators occur in the scope of two negations is undecidable.*

sketch. The octant tiling problem is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile the second octant of the integer plane $\mathcal{O} = \{(i,j) : i,j \in \mathbb{N} \wedge 0 \leq i \leq j\}$. For every tile type $t_i \in \mathcal{T}$, let $ri(t_i)$ (resp., $le(t_i)$, $up(t_i)$, $do(t_i)$) be the color of the right (resp., left, up, down) side of t_i. To solve the problem, one must find a function $f : \mathcal{O} \rightarrow \mathcal{T}$ such that $ri(f(n,m)) = le(f(n+1,m))$ and $up(f(n,m)) = do(f(n,m+1))$. Given an instance $\mathcal{T} = \{t_1, \ldots, t_k\}$ of the octant tiling problem, we will assume that $\mathcal{AP}$ contains the following propositional letters: u, t_1,..., t_k. G is the *universal* operator, defined in such a way that $G\varphi$ is true over an interval $[d_i, d_j]$ if and only if φ is true over every interval $[d_k, d_l]$, with $d_k \geq d_j$. It is defined as follows:

$$G\varphi ::= \neg(\top T(\neg\varphi T \top)). \tag{3}$$

Consider now the following formulas:

$$uT\top \wedge G(u \rightarrow uT\neg u), \tag{4}$$

$$G(u \rightarrow \bigvee_{t_i \in \mathcal{T}} t_i), \tag{5}$$

$$G \bigwedge_{t_i \neq t_j} \neg(t_i \wedge t_j), \tag{6}$$

$$G \bigwedge_{t_i \in \mathcal{T}} (t_i \rightarrow \neg(uT\neg \bigvee_{t_j \in \mathcal{T}, up(t_i)=do(t_j)} t_j)), \quad (7)$$

$$G\Big(u \rightarrow \bigwedge_{t_i,t_j \in \mathcal{T},\, ri(t_j) \neq le(t_i)} \neg(t_i T t_j)\Big). \quad (8)$$

It is easy to see that, in formulas (4), (5), and (6), modal operators occur in the scope of at most two negations. Also, formulas (7) and (8) can be rewritten in such a way that modal operators occur in the scope of at most two negations. In [18] it has been shown that $\varphi_{\mathcal{T}}$ is satisfiable if and only if $\mathcal{T}$ tiles the second octant. Using König's lemma, one can prove that a tiling system tiles the second octant if and only if it tiles arbitrarily large squares if and only if it tiles $\mathbb{N} \times \mathbb{N}$ if and only if it tiles $\mathbb{Z} \times \mathbb{Z}$. Undecidability of the first problem immediately follows from that of the last one [12]. □

6. Conclusions and future work

In this paper, we studied a syntactic fragment of Venema's CDT logic, that we called $\mathrm{CDT_{BS}}$, whose standard translation to first-order logic fits into the Bernays-Schönfinkel's class of prefix quantified formulas. Decidability of $\mathrm{CDT_{BS}}$ is thus a direct consequence of the one of the Bernays-Schönfinkel's class. We analyzed the computational complexity of the logic by developing a terminating tableau method and proving its NP-completeness. Finally, we showed that any natural relaxation of the syntactic restrictions we imposed on $\mathrm{CDT_{BS}}$ leads to an interval logic which is expressive enough to encode the octant tiling problem, and thus turns out to be undecidable.

Given that the proposed translation uses binary predicates only, the following question naturally arises: can every formula in Bernays-Schönfinkel's class of first-order logic, interpreted over linear orders and limited to binary predicates, be turned into a $\mathrm{CDT_{BS}}$-formula? Expressive completeness issues have been dealt with in [13, 19, 20] (for point-based temporal logics) and in [4, 28] (for interval-based temporal logics). We conjecture that an analogous result can be given for $\mathrm{CDT_{BS}}$ with respect to the Bernays-Schönfinkel's class of first-order logic, interpreted over linear orders, by limiting the language to binary predicates.

References

[1] J. Allen. Maintaining knowledge about temporal intervals. *Communications of the Association for Computing Machinery*, 26(11):832–843, 1983.

[2] D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco. Decidable and undecidable fragments of Halpern and Shoham's interval temporal logic: towards a complete classification. In *Proc. of LPAR'08*, volume 5330 of *LNCS*, pages 590–604, 2008.

[3] D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco. Undecidability of interval temporal logics with the overlap modality. In *Proc. TIME'09*, pages 88–95. IEEE Comp. Society, 2009.

[4] D. Bresolin, V. Goranko, A. Montanari, and G. Sciavicco. Propositional interval neighborhood logics: expressiveness, decidability, and undecidable extensions. *Annals of Pure and Applied Logic*, 161(3):289–304, 2009.

[5] D. Bresolin, D. D. Monica, V. Goranko, A. Montanari, and G. Sciavicco. Metric Propositional Neighborhood Logics. *Journal of Software and System Modeling*, 2011. DOI: 10.1007/s10270-011-0195-y.

[6] D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco. Optimal tableaux for right propositional neighborhood logic over linear orders. In *Proc. of JELIA'08*, volume 5293 of *LNAI*, pages 62–75. Springer, 2008.

[7] D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco. What's decidable about Halpern and Shoham's interval logic? The maximal fragment $\mathsf{AB\overline{B}\overline{L}}$. Proc. of LICS'11, 2011.

[8] D. Della Monica. *Expressiveness, decidability, and undecidability of Interval Temporal Logic*. PhD thesis, University of Udine, 2011. Available on: http://www.dia.unisa.it/dottorandi/dario.dellamonica/pubs/thesis/phd_thesis.pdf.

[9] D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco. Expressiveness of the Interval Logics of Allen's Relations on the Class of all Linear Orders: Complete Classification. Proc. of IJCAI'11, 2011.

[10] D. Della Monica, V. Goranko, and G. Sciavicco. Hybrid Metric Propositional Neighborhood Logics with Interval Length Binders. In *Proc. of the International Workshop on Hybrid Logic and Applications (HyLo 2010)*, Edinburgh, Scotland, UK, July 2010. To appear in ENTCS.

[11] D. Della Monica and G. Sciavicco. On First-Order Propositional Neighborhood Logics: a First Attempt. In *Proc. of the ECAI 2010 Workshop on Spatio-Temporal Dynamics (STeDY 2010)*, pages 43–48, Lisbon, Portugal, August 2010.

[12] Y. G. E. Börger, E. Grädel. *The Classical Decision Problem*. Springer, 1999.

[13] K. Etessami, M. Y. Vardi, and T. Wilke. First-order logic with two variables and unary temporal logic. *Information and Computation*, 179(2):279–295, 2002.

[14] V. Goranko, A. Montanari, P. Sala, and G. Sciavicco. A general tableau method for propositional interval temporal logics: theory and implementation. *Journal of Applied Logic*, 4(3):305–330, 2006.

[15] V. Goranko, A. Montanari, and G. Sciavicco. Propositional interval neighborhood temporal logics. *Journal of Universal Computer Science*, 9(9):1137–1167, 2003.

[16] E. Grädel. Why are modal logics so robustly decidable? *Bulletin EATCS*, 68:90–103, 1999.

[17] J. Halpern and Y. Shoham. A propositional modal logic of time intervals (short version). In *Proc. of LICS'86*, pages 279–292. IEEE Comp. Society, 1986.

[18] I. Hodkinson, A. Montanari, and G. Sciavicco. Non-finite axiomatizability and undecidability of interval temporal logics with C, D, and T. In *Proc. of CSL'08*, volume 5213 of *LNCS*, pages 308–322, 2008.

[19] N. Immerman and D. Kozen. Definability with bounded number of bound variables. *Information and Computation*, 83(2):121–139, 1989.

[20] H. Kamp. Events, instants and temporal reference. In *Semantics from different points of view*, pages 376–417. de Gruyter, 1979.

[21] J. Marcinkowski and J. Michaliszyn. The ultimate undecidability result for the Halpern-Shoham logic. Proc. of LICS'11, 2011.

[22] J. Marcinkowski, J. Michaliszyn, and E. Kieronski. B and D are enough to make the Halpern-Shoham logic undecidable. In *Proc. of ICALP'10 - Part II*, volume 6199 of *LNCS*, pages 357–368, 2010.

[23] A. Montanari, G. Puppis, and P. Sala. A decidable spatial logic with cone-shaped cardinal directions. In *Proc. of CSL'09*, volume 5771 of *LNCS*, pages 394–408, 2009.

[24] A. Montanari, G. Puppis, and P. Sala. A decidable spatial logic with cone-shape cardinal directions (extended version of [23]). Research Report 3, Dipartimento di Matematica ed Informatica, Università di Udine, 2010.

[25] A. Montanari, G. Puppis, and P. Sala. Maximal decidable fragments of Halpern and Shoham's modal logic of intervals. In *Proc. of ICALP'10 - Part II*, volume 6199 of *LNCS*, pages 345–356, 2010.

[26] A. Montanari, G. Puppis, P. Sala, and G. Sciavicco. Decidability of the interval temporal logic $AB\bar{B}$ on natural numbers. In *Proc. of STACS'10*, pages 597–608, 2010.

[27] M. Otto. Two variable first-order logic over ordered domains. *Journal of Symbolic Logic*, 66(2):685–702, 2001.

[28] Y. Venema. A modal logic for chopping intervals. *Journal of Logic and Computation*, 1(4):453–476, 1991.