Optimal Tableau Systems for Propositional Neighborhood Logic over All, Dense, and Discrete Linear Orders

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Abstract. In this paper, we focus our attention on tableau systems for the propositional interval logic of temporal neighborhood (Propositional Neighborhood Logic, PNL for short). PNL is the proper subset of Halpern and Shoham's modal logic of intervals whose modalities correspond to Allen's relations meets and met by. We first prove by a model-theoretic argument that the satisfiability problem for PNL over the class of all (resp., dense, discrete) linear orders is decidable (and NEXPTIME-complete). Then, we develop sound and complete tableau-based decision procedures for all the considered classes of orders, and we prove their optimality. (As a matter of fact, decidability with respect to the class of all linear orders had been already proved via a reduction to the decidable satisfiability problem for the two-variable fragment of first-order logic of binary relational structures over ordered domains.)

1 Introduction

Propositional interval temporal logics play a significant role in computer science, as they provide a natural framework for representing and reasoning about temporal properties [9]. This is the case, for instance, of natural language semantics, where significant interval-based logical formalisms have been developed to represent and reason about tenses and temporal prepositions, e.g., [12,16]. As another example, the possibility of encoding and reasoning about various constructs of imperative programming in interval temporal logic has been systematically explored by Moszkowski in [14]. Unfortunately, for a long time the computational complexity of most interval temporal logics has limited their systematic investigation and extensive use for practical applications: the two prominent ones, namely, Halpern and Shoham's HS [11] and Venema's CDT [17], are highly undecidable. A renewed interest in interval temporal logics has been stimulated by a number of recent positive results [13]. A general, non-terminating, tableau system for CDT, interpreted over partially ordered temporal domains, has been developed in [10]. It combines features of the classical tableau method for first-order logic with those of explicit tableau methods for modal logics with constraint label management, and it can be easily tailored to most propositional interval temporal logics proposed in the literature, including propositional temporal neighborhood logic. A tableau-based decision procedure for Moszkowski's ITL [14], interpreted over finite linearly ordered domains, has been devised by Bowman and Thompson [2]. As a matter of fact, decidability is achieved by introducing a simplifying hypothesis, called *locality* principle, that constrains the relation between the truth value of a formula over an interval and its truth values over the initial subintervals of that interval. Decidable tableau systems have been recently developed for some meaningful fragments of HS, interpreted over relevant classes of temporal structures, without resorting to any simplifying assumption. The most significant ones are the interval logics of the subinterval relation and the interval logics of temporal neighborhood.

In this paper, we focus our attention on the propositional logics of temporal neighborhood (PNL for short). PNL is the propositional fragment of Neighborhood Logic originally proposed in [7]. It can be viewed as the fragment of HS that features two modal operators $\langle A \rangle$ and $\langle \overline{A} \rangle$, that respectively correspond to Allen's relations *meets* and *met-by*. Basic logical properties of PNL have been investigated by Goranko et al. in [8]. The authors first introduce interval neighborhood frames and provide representation theorems for them; then, they develop complete axiomatic systems for PNL with respect to various classes of interval neighborhood frames. The satisfiability problem for PNL has been addressed by Bresolin et al. in [3]. NEXPTIME-completeness with respect to the classes of all linearly ordered domains, well-ordered domains, finite linearly ordered domains, and natural numbers has been proved via a reduction to the satisfiability problem for the two-variable fragment of first-order logic of binary relational structures over ordered domains [15].

Despite these significant achievements, the problem of devising decision procedures of practical interest for PNL has been only partially solved. In [6], a tableau system for its future fragment RPNL, interpreted over the natural numbers, has been developed; such a system has been later extended to full PNL over the integers [4]. In this paper, we develop a NEXPTIME tableau-based decision procedure for PNL interpreted over the class of all linear orders and then we show how to tailor it to the subclasses of dense linear orders and of (weakly) discrete linear orders. NEXPTIME-hardness can be proved exactly as in [6], and thus the proposed procedures turn out to be optimal. From a technical point of view, the proposed tableau systems are quite different from that for RPNL over the natural numbers [6]. While models for RPNL formulas over the natural numbers can be generated by simply adding future points (possibly infinitely many) to a given partial model, the construction of a model for an PNL formula over an arbitrary (resp., dense, discrete) linearly ordered domain may require the addition of points (possibly infinitely many) in between existing ones.

The paper is organized as follows. In Section 2 we introduce syntax and semantics of PNL. Then, in Section 3 we introduce the notion of labeled interval structure (LIS) and we show that PNL satisfiability can be reduced to the existence of a fulfilling LIS. In Section 4 we prove the decidability of PNL over different classes of linear orders by a model-theoretic argument. Next, in Section 5, by taking advantage of the results given in the previous section, we develop optimal tableau-based decision procedures for PNL over the considered classes of linear orders. Conclusions provide an assessment of the work and outline future research directions.

2 Propositional Neighborhood Logic

In this section, we give syntax and semantics of PNL interpreted over different classes of linear orders. Let D be a set of points and $\mathbb{D} = \langle D, < \rangle$ be a linear order on it. We say that $\mathbb{D}$ is (weakly) *discrete* if any point having a successor (resp., predecessor) has an immediate one and that $\mathbb{D}$ is *dense* if for every pair of points $d_i < d_j$ there exists a point d_k such that $d_i < d_k < d_j$. In the following, we will focus our attention on the representative classes of all linear orders, dense linear orders, and (weakly) discrete linear orders. In fact, similar results can be obtained for other classes of linear orders [3].

An *interval* on $\mathbb{D}$ is an ordered pair $[d_i, d_j]$ such that $d_i, d_j \in D$ and $d_i < d_j$ (strict semantics)[5]. The set of all intervals over $\mathbb{D}$ will be denoted by $\mathbb{I}(\mathbb{D})$. The pair $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ is called an *interval structure*. For every pair of intervals $[d_i, d_j], [d'_i, d'_j] \in \mathbb{I}(\mathbb{D})$, we say that $[d'_i, d'_j]$ is a *right* (resp., *left*) *neighbor* of $[d_i, d_j]$ if and only if $d_j = d'_i$ (resp., $d'_j = d_i$).

The language of PNL consists of a set AP of propositional letters, the connectives $\neg$ and $\vee$, and the modal operators $\langle A \rangle$ and $\langle \overline{A} \rangle$. The other connectives, as well as the logical constants $\top$ (true) and $\bot$ (false), can be defined as usual. *Formulae* of PNL, denoted by $\varphi, \psi, \ldots$, are recursively defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle A \rangle \varphi \mid \langle \overline{A} \rangle \varphi.$$

We denote by $|\varphi|$ the length of φ, that is, the number of symbols in φ (in the following, we shall use $|\ |$ to denote the cardinality of a set as well). A formula of the form $\langle A \rangle \psi$, $\neg\langle A \rangle \psi$, $\langle \overline{A} \rangle \psi$, or $\neg\langle \overline{A} \rangle \psi$ is called a *temporal formula* (from now on, we identify $\neg\langle A \rangle \neg\psi$ with $[A]\psi$ and $\neg\langle \overline{A} \rangle \neg\psi$ with $[\overline{A}]\psi$).

A *model* for a PNL formula is a pair $\mathbf{M} = \langle \langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle, \mathcal{V} \rangle$, where $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ is an interval structure and $\mathcal{V} : \mathbb{I}(\mathbb{D}) \longrightarrow 2^{AP}$ is a *valuation function* assigning to every interval the set of propositional letters true over it. Given a model $\mathbf{M} = \langle \langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle, \mathcal{V} \rangle$ and an interval $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$, the semantics of PNL is defined recursively by the *satisfiability relation* $\Vdash$ as follows:

[5] As an alternative, one may assume a non-strict semantics which admits point intervals, that is, intervals of the form $[d_i, d_i]$. It is not difficult to show that all results in the paper can be adapted to the case in which non-strict semantics is assumed.

- for every propositional letter $p \in AP$, $\mathbf{M}, [d_i, d_j] \Vdash p$ iff $p \in \mathcal{V}([d_i, d_j])$;
- $\mathbf{M}, [d_i, d_j] \Vdash \neg\psi$ iff $\mathbf{M}, [d_i, d_j] \nVdash \psi$;
- $\mathbf{M}, [d_i, d_j] \Vdash \psi_1 \vee \psi_2$ iff $\mathbf{M}, [d_i, d_j] \Vdash \psi_1$ or $\mathbf{M}, [d_i, d_j] \Vdash \psi_2$;
- $\mathbf{M}, [d_i, d_j] \Vdash \langle A\rangle\psi$ iff $\exists d_k \in D$ such that $d_k > d_j$ and $\mathbf{M}, [d_j, d_k] \Vdash \psi$;
- $\mathbf{M}, [d_i, d_j] \Vdash \langle \overline{A}\rangle\psi$ iff $\exists d_k \in D$ such that $d_k < d_i$ and $\mathbf{M}, [d_k, d_i] \Vdash \psi$.

We place ourselves in the most general (and difficult) setting and we do not impose any not constraint on the valuation function. As an example, given an interval $[d_i, d_j]$, it may happen that $p \in \mathcal{V}([d_i, d_j])$ and $p \notin \mathcal{V}([d'_i, d'_j])$ for all intervals $[d'_i, d'_j]$ (strictly) contained in $[d_i, d_j]$.
It can be shown that PNL is expressive enough to distinguish between satisfiability over the class of all linear orders and the class of discrete (resp., dense) linear orders. As a matter of fact, PNL also allows one to distinguish between satisfiability over the class of all (resp., dense, discrete) linear orders and over the integers. To this end, it suffices to exhibit a formula that is satisfiable over the former and unsatisfiable over the latter. The formulae are the following:

- Let $ImmediateSucc$ be the PNL formula $\langle A\rangle\langle A\rangle p \wedge [A][A][A]\neg p$. It is possible to show that $ImmediateSucc$ is satisfiable over the class of all (resp., discrete) linear orders, but it is not satisfiable over dense linear orders.
- Let $NoImmediateSucc$ be the PNL formula $(\langle\overline{A}\rangle\top \wedge [\overline{A}](p \wedge [A]\neg p \wedge [\overline{A}]p)) \wedge \langle A\rangle\langle A\rangle[\overline{A}]([\overline{A}]p \vee \langle\overline{A}\rangle\langle\overline{A}\rangle\neg p)$. It is possible to show that $NoImmediateSucc$ is satisfiable over the class of all (resp., dense) linear orders, but it is not satisfiable over discrete linear orders.
- Let $[G]$ be the *universally-in-the-future* operator defined as follows: $[G]\psi = \psi \wedge [A]\psi \wedge [A][A]\psi$ and let seq_p be a shorthand for $p \rightarrow \langle A\rangle p$. Consider the formula $AccPoints = \langle A\rangle p \wedge [G]seq_p \wedge \langle A\rangle[G]\neg p$. It is possible to show that $AccPoints$ is unsatisfiable over $\mathbb{Z}$, while it is satisfiable whenever the temporal structure in which it is interpreted has at least one *accumulation point*, that is, a point which is the right bound of an infinite (ascending) chain of points, thus including all, dense, and discrete linear orders.

Detailed proofs of these statements are given in the Appendix.

3 Labeled Interval Structures and Satisfiability

In this section, we introduce preliminary notions and we state basic results on which our tableau method for PNL relies. Let φ be a PNL formula to be checked for satisfiability and let AP be the set of its propositional letters. The *closure* $\mathrm{CL}(\varphi)$ of φ is the set of all subformulae of φ and of their negations (we identify $\neg\neg\psi$ with ψ). Moreover, the set of *temporal formulae* of φ is the set $\mathrm{TF}(\varphi) = \{\langle A\rangle\psi, [A]\psi, \langle\overline{A}\rangle\psi, [\overline{A}]\psi \in \mathrm{CL}(\varphi)\}$. Finally, a *maximal set of requests* for φ is a set $S \subseteq \mathrm{TF}(\varphi)$ that satisfies the following conditions: (i) for every $\langle A\rangle\psi \in \mathrm{TF}(\varphi)$, $\langle A\rangle\psi \in S$ iff $\neg\langle A\rangle\psi \notin S$; (ii) for every $\langle\overline{A}\rangle\psi \in \mathrm{TF}(\varphi)$, $\langle\overline{A}\rangle\psi \in S$ iff $\neg\langle\overline{A}\rangle\psi \notin S$. By induction on the structural complexity of φ, we can easily prove that, for every formula φ, $|\operatorname{CL}(\varphi)|$ is less than or equal to $2 \cdot (|\varphi| + 1)$, while $|\operatorname{TF}(\varphi)|$ is less than or equal to $2 \cdot |\varphi|$. We are now ready to introduce the notion of φ-atom.

Definition 1. *A* φ-atom *is a set $A \subseteq \operatorname{CL}(\varphi)$ such that (i) for every $\psi \in \operatorname{CL}(\varphi)$, $\psi \in A$ iff $\neg\psi \notin A$, and (ii) for every $\psi_1 \vee \psi_2 \in \operatorname{CL}(\varphi)$, $\psi_1 \vee \psi_2 \in A$ iff $\psi_1 \in A$ or $\psi_2 \in A$.*

We denote the set of all φ-atoms by A_φ. It can be easily checked that $|\mathcal{A}_\varphi| \leq 2^{|\varphi|+1}$. Atoms are connected by the following binary relation.

Definition 2. *Let LR_φ be a relation such that for every pair of atoms $A_1, A_2 \in \mathcal{A}_\varphi$, $A_1 \; LR_\varphi \; A_2$ if and only if (i) for every $[A]\psi \in \operatorname{CL}(\varphi)$, if $[A]\psi \in A_1$ then $\psi \in A_2$ and (ii) for every $[\overline{A}]\psi \in \operatorname{CL}(\varphi)$, if $[\overline{A}]\psi \in A_2$ then $\psi \in A_1$.*

We now introduce a suitable labeling of interval structures based on φ-atoms.

Definition 3. *A* φ-labeled interval structure *(*LIS *for short) is a pair $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$, where $\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle$ is an interval structure and $\mathcal{L} : \mathbb{I}(\mathbb{D}) \to \mathcal{A}_\varphi$ is a* labeling function *such that, for every pair of neighboring intervals $[d_i, d_j], [d_j, d_k] \in \mathbb{I}(\mathbb{D})$, $\mathcal{L}([d_i, d_j]) \; LR_\varphi \; \mathcal{L}([d_j, d_k])$.*

We say that a LIS $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ is *discrete* (resp., *dense*) if $\mathbb{D}$ is discrete (resp., dense). If we interpret the labeling function as a valuation function, LISs represent *candidate models* for φ. The truth of formulae devoid of temporal operators indeed follows from the definition of φ-atom; moreover, universal temporal conditions, imposed by $[A]/[\overline{A}]$ operators, are forced by the relation LR_φ. However, to obtain a model for φ, we must also guarantee the satisfaction of existential temporal conditions, imposed by $\langle A\rangle/\langle\overline{A}\rangle$ operators. To this end, we introduce the notion of fulfilling LIS.

Definition 4. *A* LIS $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ *is* fulfilling *iff (i) for every temporal formula $\langle A\rangle\psi \in \operatorname{TF}(\varphi)$ and every interval $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$, if $\langle A\rangle\psi \in \mathcal{L}([d_i, d_j])$, then there exists $d_k > d_j$ such that $\psi \in \mathcal{L}([d_j, d_k])$ and (ii) for every temporal formula $\langle\overline{A}\rangle\psi \in \operatorname{TF}(\varphi)$ and every interval $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$, if $\langle\overline{A}\rangle\psi \in \mathcal{L}([d_i, d_j])$, then there exists $d_k < d_i$ such that $\psi \in \mathcal{L}([d_k, d_i])$.*

The next theorem proves that for any given formula φ, the satisfiability of φ is equivalent to the existence of a fulfilling LIS with an interval labeled by φ.

Theorem 1. *A PNL formula φ is satisfiable iff there exists a fulfilling* LIS $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ *with $\varphi \in \mathcal{L}([d_i, d_j])$ for some $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$.*

The implication from left to right is straightforward; the opposite implication is proved by induction on the structural complexity of the formula (see the proof in the Appendix). From now on, we say that a fulfilling LIS $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ *satisfies* φ if and only if there exists an interval $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$ such that $\varphi \in \mathcal{L}([d_i, d_j])$.

Definition 5. *Given a* LIS $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ *and a point* $d \in D$*, we define the set of* future temporal requests of d *as the set* $\mathrm{REQ}_f^{\mathbf{L}}(d) = \{\langle A\rangle\xi/[A]\xi \in \mathrm{TF}(\varphi) : \exists d' \in D(\langle A\rangle\xi/[A]\xi \in \mathcal{L}([d', d]))\}$ *and the set of* past temporal requests of d *as the set* $\mathrm{REQ}_p^{\mathbf{L}}(d) = \{\langle\overline{A}\rangle\xi/[\overline{A}]\xi \in \mathrm{TF}(\varphi) : \exists d' \in D(\langle\overline{A}\rangle\xi/[\overline{A}]\xi \in \mathcal{L}([d, d']))\}$*. The set of* temporal requests of d *is defined as* $\mathrm{REQ}^{\mathbf{L}}(d) = \mathrm{REQ}_p^{\mathbf{L}}(d) \cup \mathrm{REQ}_f^{\mathbf{L}}(d)$*.*

Definition 6. *Given a* LIS $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ *for a PNL formula* φ*,* $d \in D$*, and* $\langle A\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}(d)$ *(resp.,* $\langle\overline{A}\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}(d)$*), we say that* $\langle A\rangle\psi$ *(resp.,* $\langle\overline{A}\rangle\psi$*) is* fulfilled for d in $\mathbf{L}$ *if there exists* $d' \in D$*, with* $d' > d$ *(resp.,* $d' < d$*), such that* $\psi \in \mathcal{L}([d, d'])$ *(resp.,* $\psi \in \mathcal{L}([d', d])$*). We say that* d *is fulfilled in* $\mathbf{L}$ *if for every* $\langle A\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}(d)$ *(resp.,* $\langle\overline{A}\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}(d)$*)* $\langle A\rangle\psi$ *(resp.,* $\langle\overline{A}\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}(d)$*) is fulfilled for* d *in* $\mathbf{L}$*.*

Definition 7. *Given a* LIS $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ *for a PNL formula* φ *and* $d \in D$*, we say that* d *(resp.,* $\mathrm{REQ}(d)$*) is* unique *in* $\mathbf{L}$ *if for every* $\tilde{d} \in D$*, with* $\tilde{d} \neq d$*,* $\mathrm{REQ}(\tilde{d}) \neq \mathrm{REQ}(d)$*.*

Given a formula φ, we denote by REQ_φ the set of all possible sets of requests. It is not difficult to show that $|\,\mathrm{REQ}_\varphi\,|$ is equal to $2^{\frac{|\,\mathrm{TF}(\varphi)|}{2}}$.

Definition 8. *Given a* LIS $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$*,* $D' \subseteq D$*, and* $\mathcal{R} \in \mathrm{REQ}_\varphi$*, we say that* $\mathcal{R}$ occurs n times in D' *iff there exist exactly* n *distinct points* $d_{i_1}, \ldots, d_{i_n} \in D'$ *such that* $\mathrm{REQ}^{\mathbf{L}}(d_{i_j}) = \mathcal{R}$*, for all* $1 \leq j \leq n$*.*

4 Decidability of PNL

In this section, we prove that the satisfiability problem for PNL over the classes of all linear orders is decidable. Moreover, we explain how to tailor the proof to the cases of dense and discrete linear orders.

Definition 9. *Let* φ *be a PNL formula,* A *be a* φ*-atom, and* $S_1, S_2 \subseteq \mathrm{TF}(\varphi)$ *be two maximal sets of requests. The triplet* $\langle S_1, A, S_2\rangle$ *is an* interval-tuple *iff (i) for every* $[A]\psi \in S_1$*,* $\psi \in A$*, (ii) for every* $[\overline{A}]\psi \in S_2$*,* $\psi \in A$*, (iii) for every* $\langle A\rangle\psi \in \mathrm{TF}(\varphi)$ *(resp.,* $\langle\overline{A}\rangle\psi \in \mathrm{TF}(\varphi)$*),* $\langle A\rangle\psi \in A$ *(resp.,* $\langle\overline{A}\rangle\psi \in A$*) iff* $\langle A\rangle\psi \in S_2$ *(resp.,* $\langle\overline{A}\rangle\psi \in S_1$*), and (iv) for every* $\psi \in A$ *such that* $\langle A\rangle\psi \in \mathrm{TF}(\varphi)$ *(resp.,* $\langle\overline{A}\rangle\psi \in \mathrm{TF}(\varphi)$*),* $\langle A\rangle\psi \in S_1$ *(resp.,* $\langle\overline{A}\rangle\psi \in S_2$*).*

Proposition 1. *Let* $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ *be a* LIS *for a PNL formula* φ*. For every* $d, d' \in D$*, the triplet* $\langle\mathrm{REQ}^{\mathbf{L}}(d), \mathcal{L}([d, d']), \mathrm{REQ}^{\mathbf{L}}(d')\rangle$ *is an interval-tuple.*

Proof. It easily follows from Definition 2, Definition 3, and Definition 5. □

Definition 10. *Let* $\langle R, A, R'\rangle$ *be an interval-tuple. We say that* $\langle R, A, R'\rangle$ occurs *in* $\mathbf{L}$ *if there exists* $[d, d'] \in \mathbb{I}(\mathbb{D})$ *such that* $\mathcal{L}([d, d']) = A$*,* $\mathrm{REQ}^{\mathbf{L}}(d) = R$*, and* $\mathrm{REQ}^{\mathbf{L}}(d') = R'$*. If* $\langle R, A, R'\rangle$ *occurs in* $\mathbf{L}$ *and there exists* $[d, d']$ *such that* $\mathcal{L}([d, d']) = A$*,* $\mathrm{REQ}^{\mathbf{L}}(d) = R$*,* $\mathrm{REQ}^{\mathbf{L}}(d') = R'$*, and both* d *and* d' *are fulfilled in* $\mathbf{L}$*, then we say that* $\langle R, A, R'\rangle$ *is* fulfilled *in* $\mathbf{L}$ *(via* $[d, d']$*).*

Definition 11. *Given a finite* LIS $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ *for a PNL formula* φ*, we say that* $\mathbf{L}$ *is a* pseudo-model for φ *if every interval-tuple* $\langle R, A, R'\rangle$ *that occurs in* $\mathbf{L}$ *is fulfilled.*

From the fact that all interval-tuples are fulfilled in $\mathbf{L}$, that is, $\mathbf{L}$ is a pseudo-model for φ, it does not follow that $\mathbf{L}$ is fulfilling, since in $\mathbf{L}$ there can be multiple occurrences of the same interval-tuple, associated with different intervals. Thus, to turn a pseudo-model into a fulfilling LIS (for φ) some additional effort is needed. The next definition introduces an important ingredient of such a process.

Definition 12. *Let* φ *be a PNL formula and* $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ *be a fulfilling* LIS *that satisfies it. For any* $d \in D$*, we say that:*

(future) a set $ES^d_f \subseteq D$ *is a* future essential set *for* d *if (i) for every* $\langle A\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}(d)$*, there exists* $d' \in ES^d_f$ *such that* $\psi \in \mathcal{L}([d, d'])$ *(fulfilling condition) and (ii) for every* $d' \in ES^d_f$ *there exists a formula* $\langle A\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}(d)$ *such that, for every* $d'' \in (ES^d_f \setminus \{d'\})$*,* $\neg\psi \in \mathcal{L}([d, d''])$ *(minimality);*

(past) a set $ES^d_p \subseteq D$ *is a* past essential set *for* d *if (i) for every* $\langle\overline{A}\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}(d)$*, there exists* $d' \in ES^d_p$ *such that* $\psi \in \mathcal{L}([d', d])$ *(fulfilling condition) and (ii) for every* $d' \in ES^d_p$ *there exists a formula* $\langle\overline{A}\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}(d)$ *such that, for every* $d'' \in (ES^d_p \setminus \{d'\})$*,* $\neg\psi \in \mathcal{L}([d'', d])$ *(minimality).*

Let $d \in D$. By Definition 12, we have that for all $d' \in ES^d_f$ (resp., $d' \in ES^d_p$), there exists at least one formula ψ belonging to $\mathcal{L}([d, d'])$ (resp., $\mathcal{L}([d', d])$ only. On the contrary, we cannot exclude the existence of formulas ψ that belong to the labeling of more than one interval $[d, d']$ (resp., $[d', d]$), with $d' \in ES^d_f$ (resp., $d' \in ES^d_p$).

Definition 13. *Given a PNL formula* φ*, a fulfilling* LIS $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ *that satisfies it, and* $d \in D$*, we define the sets* $\mathrm{Future}^{\mathbf{L}}(d) = \{\mathrm{REQ}^{\mathbf{L}}(d') \mid d' > d\}$ *and* $\mathrm{Past}^{\mathbf{L}}(d) = \{\mathrm{REQ}^{\mathbf{L}}(d') \mid d' < d\}$*.*

The decidability of the satisfiability problem for PNL over the class of all linear orders rests on the following lemma.

Lemma 1. *Given a pseudo-model* $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ *for a* PNL *formula* φ*, there exists a fulfilling LIS* $\mathbf{L}'$ *that satisfies* φ*.*

Proof. We show how to obtain a fulfilling LIS $\mathbf{L}'$ starting from the pseudo-model $\mathbf{L}$ as the limit of a possibly infinite sequence of pseudo-models $\mathbf{L}_0(= \mathbf{L}), \mathbf{L}_1, \mathbf{L}_2, \ldots$. In the following, we describe how to obtain the pseudo-model $\mathbf{L}_{i+1}$ from the pseudo-model $\mathbf{L}_i$, for any $i \geq 0$. Let Q_i be the queue of all points $d \in D_i$ that must be checked for fulfillment (for $i = 0$, Q_i consists of all and only the points $d \in D$ such that d is not fulfilled in $\mathbf{L}$). If Q_i is empty, then we stop the procedure by putting $\mathbf{L}' = \mathbf{L}_i$. Otherwise, $\mathbf{L}_{i+1}$ is built as follows. Let d be the first element of the queue Q_i. If d is fulfilled, we remove it from the queue and put $\mathbf{L}_{i+1} = \mathbf{L}_i$ (every point in the queue is not fulfilled at insertion time; however, it

may happen that subsequent expansions of the domain make it fulfilled before the time at which it is taken into consideration). Otherwise, either there exists $\langle A\rangle\psi \in REQ^{\mathbf{L}_i}(d)$ which is not fulfilled, or there exists $\langle \overline{A}\rangle\psi \in REQ^{\mathbf{L}_i}(d)$ which is not fulfilled, or both.
Suppose that there exists a $\langle A\rangle$-formula in $REQ^{\mathbf{L}_i}(d)$ which is not fulfilled. Two cases may arise:

1) There exists $d' > d$ such that $\mathrm{REQ}^{\mathbf{L}_i}(d') = \mathrm{REQ}^{\mathbf{L}_i}(d)$ and d' is fulfilled. Let $ES_f^{d'} = \{d_1, ..., d_k\}$. For $j = 1, \ldots, k$, we proceed as follows:
   a) If d_j is unique, then we put $\mathcal{L}_{i+1}([d, d_j]) = \mathcal{L}_i([d', d_j])$. We prove that such a replacement does not introduce new defects for d_j. Suppose by contradiction that it is not the case. Then, there must exists a formula $\langle \overline{A}\rangle\theta \in REQ^{\mathbf{L}_i}(d_j)$ that is fulfilled only by the interval $[d, d_j]$ (in $\mathbf{L}_i$). Since the interval-tuple $\langle REQ^{\mathbf{L}_i}(d),\ \mathcal{L}_i([d, d_j]), \mathrm{REQ}^{\mathbf{L}_i}(d_j)\rangle$ is fulfilled in $\mathbf{L}_i$, there exists an interval $[d'', d''']$ such that $\langle REQ^{\mathbf{L}_i}(d), \mathcal{L}_i([d, d_j]), \mathrm{REQ}^{\mathbf{L}_i}(d_j)\rangle$ is fulfilled in $\mathbf{L}_i$ via $[d'', d''']$. Since d_j is unique, $d''' = d_j$. However, since d in not fulfilled in $\mathbf{L}_i$, $d'' \neq d$, and thus the interval $[d'', d_j]$ fulfills $\langle \overline{A}\rangle\theta$, in contradiction with the hypothesis that $\langle \overline{A}\rangle\theta$ causes a defect for d_j. This case is depicted in Figure 1.

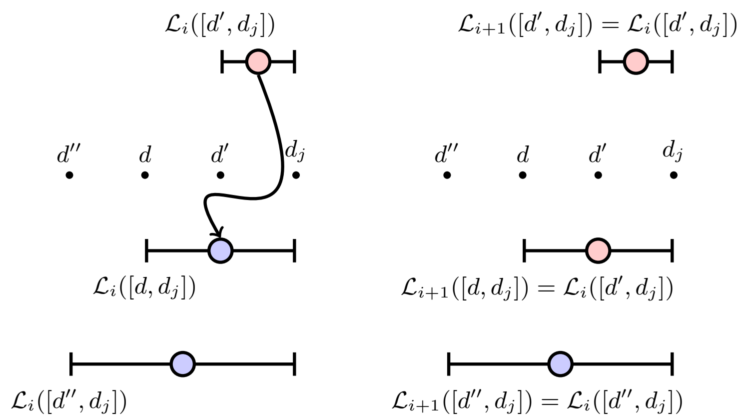

Fig. 1. Relabeling of the interval $[d, d_j]$ in Case 1a.

   b) If d_j is not unique, then there exists $\overline{d} \neq d_j$, with $\mathrm{REQ}^{\mathbf{L}_i}(\overline{d}) = \mathrm{REQ}^{\mathbf{L}_i}(d_j)$. In such a case, we introduce a new point $\widehat{d}$ immediately after d_j with the same requests as d_j, that is, we put $D_{i+1} = D_i \cup \{\widehat{d}\}$, with $d_j < \widehat{d}$ and for all $\tilde{d}$, if $\tilde{d} > d_j$, then $\tilde{d} > \widehat{d}$, and we force $\mathrm{REQ}^{\mathbf{L}_{i+1}}(\widehat{d})$ to be equal to $\mathrm{REQ}^{\mathbf{L}_i}(d_j)$. To this end, for every $d'' \notin \{d, d_j, d'\}$, we put $\mathcal{L}_{i+1}([d'', \widehat{d}]) = \mathcal{L}_i([d'', d_j])$ (when $d'' < \widehat{d}$) and $\mathcal{L}_{i+1}([\widehat{d}, d'']) = \mathcal{L}_i([d_j, d''])$ (when $d'' > \widehat{d}$). Moreover, we put $\mathcal{L}_{i+1}([d, \widehat{d}]) = \mathcal{L}_i([d', d_j])$ and $\mathcal{L}_{i+1}([d', \widehat{d}]) = \mathcal{L}_i([d, d_j])$, as depicted in Figure 2. In such a way, d satisfies over $[d, \widehat{d}]$ the request that d' satisfies over $[d', d_j]$. At the same time, we guarantee that $\widehat{d}$ satisfies the same past requests that d_j satisfies: $\widehat{d}$ satisfies over $[d, \widehat{d}]$ (resp., $[d', \widehat{d}]$) the request that d_j satisfies over $[d', d_j]$ (resp., $[d, d_j]$) and it satisfies the remaining past requests over intervals that start at the same point where the intervals over which d_j satisfies them start. Finally, if $\overline{d} > d_j$, we put $\mathcal{L}_{i+1}([d_j, \widehat{d}]) = \mathcal{L}_i([d_j, \overline{d}])$, and $\mathcal{L}_{i+1}([d_j, \widehat{d}]) = \mathcal{L}_i([\overline{d}, d_j])$ otherwise. For all the remaining pairs d_r, d_s the labeling remains unchanged, that is, $\mathcal{L}_{i+1}([d_r, d_s]) = \mathcal{L}_i([d_r, d_s])$. Now, we observe that, by definition of $\mathcal{L}_{i+1}$, if d_j is fulfilled (in $\mathbf{L}_i$), then $\widehat{d}$ is fulfilled (in $\mathbf{L}_{i+1}$), while if d_j is not fulfilled (in $\mathbf{L}_i$), being $\widehat{d}$ fulfilled or not (in $\mathbf{L}_{i+1}$) depends on the labeling of the interval $[d_j, \widehat{d}]$. If $\widehat{d}$ is not fulfilled (in $\mathbf{L}_{i+1}$), we insert it into the queue Q_{i+1}.

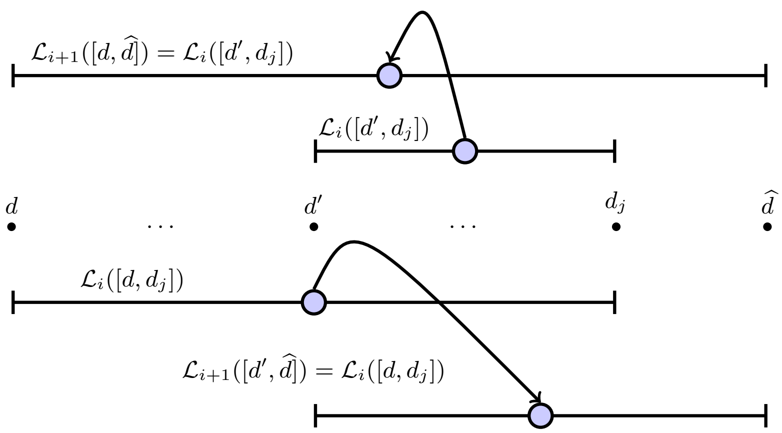

Fig. 2. Labeling of the intervals $[d, \widehat{d}]$ and $[d', \widehat{d}]$ in Case 1b.

2) For every $d' > d$, with $\mathrm{REQ}^{\mathbf{L}_i}(d') = \mathrm{REQ}^{\mathbf{L}_i}(d)$, d' is not fulfilled. Let $d' < d$ such that $\mathrm{REQ}^{\mathbf{L}_i}(d') = \mathrm{REQ}^{\mathbf{L}_i}(d)$, d' is fulfilled, and, for every $d' < d'' < d$, if $\mathrm{REQ}^{\mathbf{L}_i}(d'') = \mathrm{REQ}^{\mathbf{L}_i}(d)$, then d'' is not fulfilled.
   As a preliminary step, we prove that $\mathrm{Past}^{\mathbf{L}_i}(d') = \mathrm{Past}^{\mathbf{L}_i}(d)$. Suppose, by contradiction, that there exists $d' < d'' < d$ such that $\mathrm{REQ}^{\mathbf{L}_i}(d'') \notin \mathrm{Past}^{\mathbf{L}_i}(d')$. Since $\mathbf{L}^i$ is a pseudo-model, there exist $\overline{d}, \overline{d}' \in D_i$ such that the interval-tuple $\langle \mathrm{REQ}^{\mathbf{L}_i}(d''), \mathcal{L}_i([d'', d]), \mathrm{REQ}^{\mathbf{L}_i}(d) \rangle$ is fulfilled in $\mathbf{L}^i$ via $[\overline{d}, \overline{d}']$. By definition, both $\overline{d}$ and $\overline{d}'$ are fulfilled; moreover, $\mathrm{REQ}^{\mathbf{L}_i}(\overline{d}) = \mathrm{REQ}^{\mathbf{L}_i}(d'')$, $\mathrm{REQ}^{\mathbf{L}_i}(\overline{d}') = \mathrm{REQ}^{\mathbf{L}_i}(d)$, and $\mathcal{L}_i([\overline{d}, \overline{d}']) = \mathcal{L}_i([d'', d])$. Since $\mathrm{REQ}^{\mathbf{L}_i}(d'') \notin \mathrm{Past}^{\mathbf{L}_i}(d')$, $d' < \overline{d}$ and thus $d' < \overline{d}'$. However, since d' is the largest fulfilled element in D_i with $\mathrm{REQ}^{\mathbf{L}_i}(d') = \mathrm{REQ}^{\mathbf{L}_i}(d)$, $\overline{d}'$ cannot be greater than d' (contradiction). Hence, $\mathrm{Past}^{\mathbf{L}_i}(d') = \mathrm{Past}^{\mathbf{L}_i}(d)$.
   Let $ES_f^{d'} = \{d_1, ..., d_k\}$. For every $j = 1, ..., k$, we proceed as follows:
   a) If d_j is unique, then $d_j > d$, since $\mathrm{Past}^{\mathbf{L}_i}(d') = \mathrm{Past}^{\mathbf{L}_i}(d)$. We proceed as in Case 1a.
   b) If d_j is not unique and $d_j > d$, then we proceed as in Case 1b.

c) If d_j is not unique and $d' < d_j < d$, then we introduce a new point $\widehat{d}$ immediately after d with the same requests as d_j, that is, we put $D_{i+1} = D_i \cup \{\widehat{d}\}$, with $d < \widehat{d}$ and for all $\tilde{d}$, if $\tilde{d} > d$, then $\tilde{d} > \widehat{d}$, and we force $\mathrm{REQ}^{\mathbf{L}_{i+1}}(\widehat{d})$ to be equal to $\mathrm{REQ}^{\mathbf{L}_i}(d_j)$. To this end, for every d'', with $d'' < d_j$ (resp., $d'' > \widehat{d}$), we put $\mathcal{L}_{i+1}([d'', \widehat{d}]) = \mathcal{L}_i([d'', d_j])$ (resp., $\mathcal{L}_{i+1}([\widehat{d}, d'']) = \mathcal{L}_i([d_j, d''])$). Since $\mathrm{Past}^{\mathbf{L}_i}(d') = \mathrm{Past}^{\mathbf{L}_i}(d)$, for all $d_j \leq d'' < d$ there exists $d''' < d'$ such that $\mathrm{REQ}^{\mathbf{L}_i}(d''') = \mathrm{REQ}^{\mathbf{L}_i}(d'')$. Hence, we put $\mathcal{L}_{i+1}([d'', \widehat{d}]) = \mathcal{L}_i([d''', d_j])$. Moreover, we put $\mathcal{L}_{i+1}([d, \widehat{d}]) = \mathcal{L}_i([d', d_j])$. Finally, if $\widehat{d}$ is not fulfilled, we insert it into the queue Q_{i+1}. This case is depicted in Figure 3.

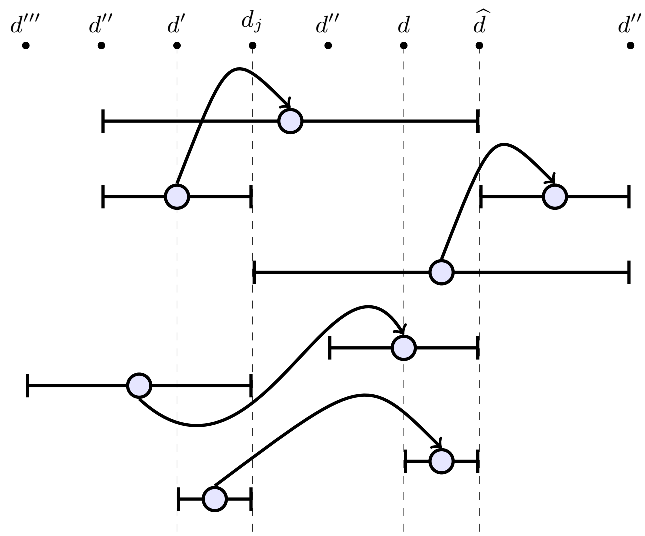

Fig. 3. Labeling of intervals starting/ending in $\widehat{d}$ in Case 2c.

The case in which there exists a $\langle \overline{A} \rangle$-formula in $REQ^{\mathbf{L}_i}(d)$ which is not fulfilled is completely symmetric, and thus its description is omitted. This concludes the construction of $\mathbf{L}_{i+1}$. Since all points which are fulfilled in $\mathbf{L}_i$ remain fulfilled in $\mathbf{L}_{i+1}$, it is immediate to conclude that $\mathbf{L}_{i+1}$ is a pseudo-model. Moreover, as d is fulfilled in $\mathbf{L}_{i+1}$, it can be safely removed from the queue. As it can be easily checked, the proposed construction does not remove any point, but it can introduce new ones, possibly infinitely many. However, the use of a queue to manage points which are (possibly) not fulfilled guarantees that the defects of each of them sooner or later will be fixed.

To complete the proof, it suffices to show that the fulfilling LIS $\mathbf{L}'$ for φ we were looking for is the limit of this (possibly infinite) construction. Let $\mathbf{L}_i^-$ be equal to $\mathbf{L}_i$ devoid of the labeling of all intervals consisting of a (non-unique) point in Q_i and a unique point (in $D_i \setminus Q_i$). We define $\mathbf{L}'$ as the limit of $\cup_{i \geq 0} \mathbf{L}_i^-$ when i tends to infinity (if Q_i turns out to be empty for some i, then $\mathbf{L}'$ is simply equal to $\cup_{i \geq 0} \mathbf{L}_i^-$ ($= \mathbf{L}_i$). It is trivial to check that for every pair D_i, D_{i+1},

$D_i \subseteq D_{i+1}$. To prove that for every pair $\mathcal{L}_i^-$, $\mathcal{L}_{i+1}^-$, it holds that $\mathcal{L}_i^- \subseteq \mathcal{L}_{i+1}^-$, we observe that: (i) the labeling of intervals whose endpoints are both non-unique points (resp., unique points) never changes, that is, it is fixed once and for all, and (ii) for every pair of point $d, d' \in D_i \setminus Q_i$ such that d is a non-unique point and d' is a unique one, if $d < d'$ (resp., $d' < d$), then $\mathcal{L}_j([d, d']) = \mathcal{L}_i([d, d'])$ (resp., $\mathcal{L}_j([d', d]) = \mathcal{L}_i([d', d])$) for all $j \geq i$, that is, the labeling of an interval consisting of a non-unique point and a unique one may possibly change when the non-unique point is removed from the queue and then it remains unchanged forever (notice that non-unique points which are fulfilled from the beginning never change "their labeling"). Finally, to prove that all points are fulfilled in $\cup_{i \geq 0} \mathbf{L}_i^-$, it is sufficient to observe that: (i) all unique points belong to D_0 and are fulfilled in the restriction of $\mathbf{L}_0$ to $D_0 \setminus Q_0$ (and thus in $\mathbf{L}_0^-$), and (ii) for every $i \geq 0$, all points in $D_i \setminus Q_i$ are fulfilled in $\mathbf{L}_i^-$ and the first element of Q_i may be not fulfilled in $\mathbf{L}_i$ (and thus in $\mathbf{L}_i^-$), but it is fulfilled in $\mathbf{L}_{i+1}^-$. Every point is indeed either directly inserted into $D_i \setminus Q_i$ or added to Q_i (and thus it becomes the first element of Q_j for some $j > i$) for some $i \geq 0$. □

Lemma 2. *Given a PNL formula φ and a fulfilling* LIS $\mathbf{L} = \langle\langle \mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ *that satisfies it, there exists a pseudo-model $\mathbf{L}'$ for φ, with $|D'| \leq 2 \cdot |\varphi| \cdot 2^{3 \cdot |\varphi| + 1}$.*

Decidability of PNL over the class of all linear orders immediately follows.

Theorem 2. *The satisfiability problem for PNL over the class of all linear orders is decidable.*

We conclude the section by explaining how to tailor the above proofs to the cases of dense and discrete linear orders (details can be found in [5]).

To cope with dense linear orders, we introduce the notion of covering.

Definition 14. *Let $\mathbf{L} = \langle\langle \mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ be a pseudo-model for a PNL formula φ and $d \in D$. We say that d is* covered *if either d is not unique or (d is unique and) both its immediate predecessor (if any) and successor (if any) are not unique. We say that $\mathbf{L}$ is* covered *if every $d \in D$ is covered.*

The construction of Lemma 1 is then revised to force each point in a pseudo-model for φ to be *covered* so that we can always insert a point in between any pair of consecutive points, thus producing a dense model for φ.

To deal with discrete orders, we make us of the notion of *safe pseudo-model*.

Definition 15. *Let $\mathbf{L} = \langle\langle \mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$ be a pseudo-model for a PNL formula φ and $d \in D$. We say that d is* safe *if either d is not unique or (d is unique and) both its immediate predecessor (if any) and successor (if any) are fulfilled. We say that $\mathbf{L}$ is* safe *if every $d \in D$ is safe.*

Such a safety condition guarantees that the building procedure of Lemma 1 can be done in such a way that all points added during the construction get their (definitive) immediate successor and immediate predecessor in at most one step.

As for complexity, it is possible to show that forcing covering (resp., safety) does not cause any exponential blow-up in the maximum size of a pseudo-model.

More formally, by suitably adapting Lemma 2, we can prove that if φ is satisfiable over dense (resp., discrete) linear orders, then there exists a covered (resp., safe) pseudo-model $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle, \mathcal{L}\rangle$l for it with $|D| \leq 4 \cdot |\varphi| \cdot 2^{3\cdot|\varphi|+1}$ (resp., $|D| \leq 2 \cdot |\varphi| \cdot 2^{4\cdot|\varphi|+1}$). It easily follows that the satisfiability problem for PNL over all (resp., dense, discrete) linear orders belongs to the NEXPTIME complexity class. NEXPTIME-hardness immediately follows from [6], where a reduction of the exponential tiling problem, which is known to be NEXPTIME-complete [1], to the satisfiability problem for the future fragment of PNL is provided (as it can be easily verified, the reduction is completely independent from the considered linear order). This allows us to conclude that the satisfiability problem for PNL over all (resp., dense, discrete) linear orders is NEXPTIME-complete.

5 Tableau systems for PNL

In this section, we develop tableau-based decision procedures for PNL over all, dense, and discrete linear orders. We describe in detail the tableau system for the general case (all linear orders), and then we briefly explain how to specialize it to deal with the dense and discrete cases. The presentation is organized as follows. First, we give the rules of the tableau system; then, we describe expansion strategies and blocking conditions; finally, we state termination, soundness, and completeness of the method. We conclude the section by proving the optimality of all the proposed tableau-based decision procedures.

We preliminarily introduce basic concepts and notation. A tableau for a PNL formula φ is a special *decorated tree* $\mathcal{T}$. We associate a finite linear order $\mathbb{D}_B = \langle D_B, <\rangle$ and a *request function* $\mathrm{REQ}_B : D_B \mapsto \mathrm{REQ}_\varphi$ with every branch B of $\mathcal{T}$. Every node n in B is labeled with a pair $\langle [d_i, d_j], A_n\rangle$ such that the triple $\langle \mathrm{REQ}_B(d_i), A_n, \mathrm{REQ}_B(d_j)\rangle$ is an interval-tuple. The *initial tableau* for φ consists of a single node (and thus of a single branch B) labeled with the pair $\langle [d_0, d_1], A_\varphi\rangle$, where $\mathbb{D}_B = \{d_0 < d_1\}$ and $\varphi \in A_\varphi$.

Given a point $d \in D_B$ and a formula $\langle A\rangle\psi \in \mathrm{REQ}_B(d)$, we say that $\langle A\rangle\psi$ is *fulfilled in* B *for* d if there exists a node $n' \in B$ such that n' is labeled with $\langle [d, d'], A_{n'}\rangle$ and $\psi \in A_{n'}$. Similarly, given a point $d \in D_B$ and a formula $\langle \overline{A}\rangle\psi \in \mathrm{REQ}_B(d)$, we say that $\langle \overline{A}\rangle\psi$ is *fulfilled in* B *for* d if there exists a node $n' \in B$ such that n' is labeled with $\langle [d', d], A_{n'}\rangle$ and $\psi \in A_{n'}$. Given a point $d \in D_B$, we say that d *is fulfilled in* B if every $\langle A\rangle\psi$ (resp., $\langle \overline{A}\rangle\psi$) in $\mathrm{REQ}_B(d)$ is fulfilled in B for d.

Let $\mathcal{T}$ be a tableau and B be a branch of $\mathcal{T}$, with $\mathbb{D}_B = \{d_0 < \ldots < d_k\}$. We denote by $B \cdot n$ the expansion of B with an immediate successor node n and by $B \cdot n_1 | \ldots | n_h$ the expansion of B with h immediate successor nodes $n_1, \ldots, n_h$. To possibly expand B, we apply one of the following *expansion rules*:

1. $\langle A\rangle$*-rule*. If there exist $d_j \in D_B$ and $\langle A\rangle\psi \in \mathrm{REQ}_B(d_j)$ such that $\langle A\rangle\psi$ is not fulfilled in B for d_j, we proceed as follows. If there is not an interval-tuple $\langle \mathrm{REQ}_B(d_j),\ A_\psi, S\rangle$, with $\psi \in A_\psi$, we *close* the branch B. Otherwise, let $\langle \mathrm{REQ}_B(d_j), A_\psi, S\rangle$ be such an interval-tuple. We take a new point d and we expand B with $h = k-j+1$ immediate successor nodes $n_1, \ldots, n_h$ such that,

for every $1 \leq l \leq h$, $\mathbb{D}_{B \cdot n_l} = \mathbb{D}_B \cup \{d_{j+l-1} < d < d_{j+l}\}$ (for $l = h$, we simply add a new point d, with $d > d_k$, to the linear order), $n_l = \langle [d_j, d], A_\psi \rangle$, with $\psi \in A_\psi$, $\mathrm{REQ}_{B \cdot n_l}(d) = S$, and $\mathrm{REQ}_{B \cdot n_l}(d') = \mathrm{REQ}_B(d')$ for every $d' \in D_B$.

2. $\langle \overline{A} \rangle$*-rule*. It is symmetric to the $\langle A \rangle$-rule and thus its description is omitted.
3. *Fill-in rule*. If there exist two points d_i, d_j, with $d_i < d_j$, such that there is not a node in B decorated with the interval $[d_i, d_j]$ and there exists an interval-tuple $\langle \mathrm{REQ}_B(d_i), A, \mathrm{REQ}_B(d_j) \rangle$, then we expand B with a node $n = \langle [d_i, d_j], A \rangle$ If such an interval-tuple does not exist, then we *close* the branch B.

The application of any of the above rules may result in the replacement of the branch B with one or more new branches, each one featuring one new node n. However, while the *Fill-in rule* decorates such a node with a new interval whose endpoints already belong to D_B, the $\langle A \rangle$-rule (resp., $\langle \overline{A} \rangle$-rule) adds a new point d to D_B which becomes the ending (resp., beginning) point of the interval associated with the new node.

We say that a node $n = \langle [d_i, d_j], A \rangle$ in a branch B is *active* if for every predecessor $n' = \langle [d, d'], A' \rangle$ of n in B, the interval-tuples $\langle \mathrm{REQ}_B(d_i), A, \mathrm{REQ}_B(d_j) \rangle$ and $\langle \mathrm{REQ}_B(d), A', \mathrm{REQ}_B(d') \rangle$ are different. Moreover, we say that a point $d \in D_B$ is *active* if and only if there exists an active node n in B such that $n = \langle [d, d'], A \rangle$ or $n = \langle [d', d], A \rangle$, for some $d' \in D_B$ and some atom A. Given a non-closed branch B, we say that B is *complete* if for every $d_i, d_j \in D_B$, with $d_i < d_j$, there exists a node n in B labeled with $n = \langle [d_i, d_j], A \rangle$, for some atom A. It can be easily seen that if B is complete, then the tuple $\langle \mathbb{D}_B, \mathbb{I}(\mathbb{D}_B), \mathcal{L}_B \rangle$ such that, for every $[d_i, d_j] \in \mathbb{I}(\mathbb{D}_B)$, $\mathcal{L}_B([d_i, d_j]) = A$ if and only if there exists a node n in B labeled with $\langle [d_i, d_j], A \rangle$, is a LIS. Given a non-closed branch B, we say that B is *blocked* if B is complete and for every active point $d \in B$ we have that d is fulfilled in B.

We start from an initial tableau for φ and we apply the expansion rules to all the non-blocked and non-closed branches B. The expansion strategy is the following one:

1. Apply the *Fill-in rule* until it generates no new nodes in B.
2. If there exist an active point $d \in D_B$ and a formula $\langle A \rangle \psi \in \mathrm{REQ}_B(d)$ such that $\langle A \rangle \psi$ is not fulfilled in B for d, then apply the $\langle A \rangle$*-rule* on d. Go back to step 1.
3. If there exist an active point $d \in D_B$ and a formula $\langle \overline{A} \rangle \psi \in \mathrm{REQ}_B(d)$ such that $\langle \overline{A} \rangle \psi$ is not fulfilled in B for d, then apply the $\langle \overline{A} \rangle$*-rule* on d. Go back to step 1.

A tableau $\mathcal{T}$ for φ is *final* if and only if every branch B of $\mathcal{T}$ is closed or blocked.

Theorem 3 (Termination). *Let $\mathcal{T}$ be a final tableau for a PNL formula φ and B be a branch of $\mathcal{T}$. We have that* $|B| \leq (2 \cdot |\varphi| \cdot 2^{3 \cdot |\varphi| + 1}) \cdot (2 \cdot |\varphi| \cdot 2^{3 \cdot |\varphi| + 1} - 1)/2$.

Theorem 4 (Soundness). *Let $\mathcal{T}$ be a final tableau for a PNL formula φ. If $\mathcal{T}$ features one blocked branch, then φ is satisfiable over all linear orders.*

Theorem 5 (Completeness). *Let φ be a PNL formula which is satisfiable over the class of all linear orders. Then, there exists a final tableau for φ with at least one blocked branch.*

The above tableau system can be tailored to the dense and discrete cases. As for the dense case, it suffices to apply the following rule immediately after the $\langle\overline{A}\rangle/\langle A\rangle$-*rules*:

Dense rule: If there exist two consecutive non-covered points d_i, d_{i+1}, we proceed as follows. If there is not an interval-tuple $\langle \mathrm{REQ}_B(d_i), A, S\rangle$ for some $S \in \mathrm{REQ}_\varphi$ and $A \in A_\varphi$, we *close* the branch B. Otherwise, let $\langle \mathrm{REQ}_B(d_i), A, S\rangle$ be such an interval-tuple. We expand B with a node n, labeled with $\langle [d_i, d], A\rangle$, such that $\mathrm{REQ}_{B\cdot n}(d) = S$ and $\mathbb{D}_{B\cdot n} = \mathbb{D}_B \cup \{d_i < d < d_{i+1}\}$.

The discrete case is more complex. First, we partition nodes (intervals) in two classes, namely, *free* and *unit* nodes. Free nodes are labeled with triples of the form $\langle [d, d'], A, free\rangle$, meaning that a point can be added in between d and d'; unit nodes, labeled with triples of the form $\langle [d, d'], A, unit\rangle$, denote unit intervals (insertions are forbidden). The set of expansion rules is then updated as follows. The *Fill-in* rule remains unchanged. The $\langle A\rangle/\langle\overline{A}\rangle$-rules are revised to prevent the insertion of points inside *unit*-intervals. The introduction of unit intervals is managed by two additional rules, (*Predecessor* and *Successor* rules) to be applied immediately after the $\langle\overline{A}\rangle/\langle A\rangle$-*rules*.

Successor rule. If there exists $d_j \in D_B$ such that d_j is unique in D_B, its immediate successor d_{j+1} in D_B is not fulfilled, there exists a node n labeled by $\langle [d_j, d_{j+1}], A_n, free\rangle$, for some atom A_n, in B, and there exists no node n' labeled by $\langle [d_j, d_{j+1}], A_{n'}, unit\rangle$, for some atom $A_{n'}$, in B, then we proceed as follows. We expand B with 2 immediate successor nodes n_1, n_2 such that $n_1 = \langle [d_j, d_{j+1}], A_n, unit\rangle$ and $n_2 = \langle [d_j, d], A', unit\rangle$, with $d_j < d < d_{j+1}$ and there exists an interval-tuple $\langle \mathrm{REQ}_B(d_j), A', S\rangle$, for some A' and S (the existence of such an interval tuple is guaranteed by the existence of a node n with label $\langle [d_j, d_{j+1}], A_n, free\rangle$). We have that $\mathbb{D}_{B\cdot n_1} = \mathbb{D}_B$ and $\mathbb{D}_{B\cdot n_2} = \mathbb{D}_B \cup \{d_j < d < d_{j+1}\}$. Moreover, $\mathrm{REQ}_{B\cdot n_2}(d) = S$ and $\mathrm{REQ}_{B\cdot n_2}(d') = \mathrm{REQ}_B(d')$ for every $d' \in D_B$.

Predecessor rule. Symmetric to the successor rule and thus omitted.

As for the complexity, in both cases (dense and discrete), no exponential blow-up in the maximum length of a branch B (with respect to the general case) occurs. More formally, following the reasoning path of Theorem 3, we can prove that the maximum length of a branch B in the dense (resp., discrete) case is $|B| \leq (4 \cdot |\varphi| \cdot 2^{3\cdot|\varphi|+1} - 1) \cdot (4 \cdot |\varphi| \cdot 2^{3\cdot|\varphi|+1} - 2)/2$ (resp., $|B| \leq (2 \cdot (3 \cdot |\varphi| + 1) \cdot 2^{3\cdot|\varphi|+1}) \cdot (2 \cdot (3 \cdot |\varphi| + 1) \cdot 2^{3\cdot|\varphi|+1} - 1)/2$). Optimality easily follows.

6 Conclusions and future work

In this paper, we have developed an optimal tableau system for PNL interpreted over the class of all linear orders, and we have shown how to adapt it to deal with the subclasses of dense and (weakly) discrete linear orders. We are currently working at the implementation of the three tableau systems.

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Appendix

Proposition 2. *The PNL formula $ImmediateSucc$ is satisfiable over the class of all (resp., discrete) linear orders, as well as over $\mathbb{Z}$, but it is not satisfiable over dense linear orders.*

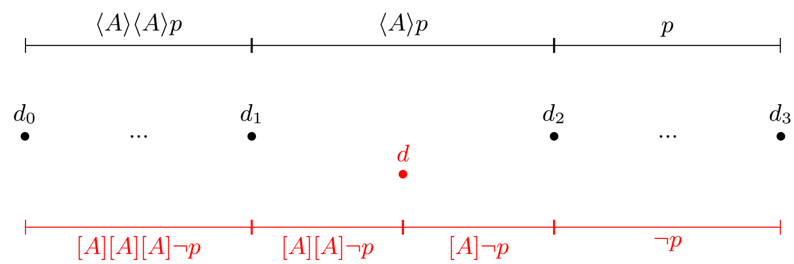

Fig. 4. A model for $ImmediateSucc$ (top), which is unsatisfiable over dense linear orders (bottom).

Proof. We first show that the formula $ImmediateSucc$ is unsatisfiable over dense linear orders. The proof is by contradiction (a graphical account of the argument is given in Figure 4 - bottom). Let us assume that there exists an interpretation $\mathbf{M}$, based on a dense linear order, such that $\mathbf{M}, [d_0, d_1] \models ImmediateSucc$ for some d_0, d_1. From $\mathbf{M}, [d_0, d_1] \models \langle A\rangle\langle A\rangle p$, it follows that there exist two points d_1 and d_2, with $d_1 < d_2 < d_3$, such that $\mathbf{M}, [d_1, d_2] \models \langle A\rangle p$ and $\mathbf{M}, [d_2, d_3] \models p$. Since $\mathbf{M}$ is based on a dense linear order, there exists a point d between d_1 and d_2. Hence, from $\mathbf{M}, [d_0, d_1] \models [A][A][A]\neg p$, it follows that $\mathbf{M}, [d_1, d] \models [A][A]p$, $\mathbf{M}, [d, d_2] \models [A]\neg p$ and $\mathbf{M}, [d_2, d_3] \models \neg p$ (contradiction). Let us consider now the class of all linearly ordered domains. A model satisfying $ImmediateSucc$ can be built as follows. We take a model $\mathbf{M}$ whose domain contains four points $d_0 < d_1 < d_2 < d_3$ such that d_2 is an immediate successor of d_1. Then, we impose $\mathbf{M}, [d_2, d_3] \models p$ and $\mathbf{M}, [d, d'] \models \neg p$ for every interval $[d, d'] \neq [d_2, d_3]$. It easily follows that $\mathbf{M}, [d_0, d_1] \models ImmediateSucc$. A pictorial representation of the model is given in Figure 4 - top. Exactly the same argument can be applied in the case of (weakly) discrete linear orders and $\mathbb{Z}$. □

Proposition 3. *The PNL formula $NoImmediateSucc$ is satisfiable over the class of all (resp., dense) linear orders, but it is not satisfiable over discrete linear orders and integers.*

Proof. We first show that the formula $NoImmediateSucc$ is unsatisfiable over discrete linear orders (the very same same argument applies to the integers). The proof is by contradiction. Let us assume that there exists a discrete model

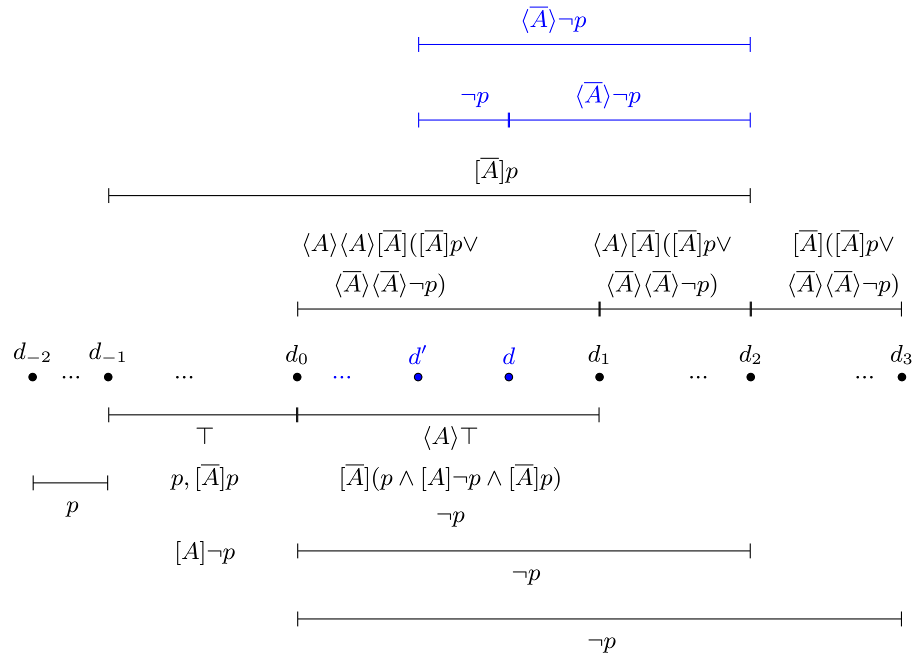

Fig. 5. A model for $NoImmediateSucc$: d_0 does not have any immediate successor.

$\mathbf{M}$ for $NoImmediateSucc$, that is, $\mathbf{M}, [d_0, d_1] \models NoImmediateSucc$ for some d_0, d_1. Let d be the immediate successor of d_0. The first tow conjuncts $\langle\overline{A}\rangle\top$ and $[\overline{A}](p \wedge [A]\neg p \wedge [\overline{A}]p)$ constrain p to hold everywhere to the left of d_0, including the intervals ending in d_0, and $\neg p$ to hold over all intervals starting from d_0, including $[d_0, d]$. The third conjunct $\langle A\rangle\langle A\rangle[\overline{A}]([\overline{A}]p \vee \langle\overline{A}\rangle\langle\overline{A}\rangle\neg p)$ forces the existence of an interval $[d_2, d_3]$, with $d_1 < d_2$, satisfying $[\overline{A}]([\overline{A}]p \vee \langle\overline{A}\rangle\langle\overline{A}\rangle\neg p)$. By the second conjunct we have that $\mathbf{M}, [d_0, d] \models \neg p$ and that $\mathbf{M}, [d, d_2] \not\models [\overline{A}]p$. Hence, $\langle\overline{A}\rangle\langle\overline{A}\rangle\neg p$ must hold over $[d, d_2]$. The formula $\langle\overline{A}\rangle\langle\overline{A}\rangle\neg p$ forces the existence of two points d'', d', with $d'' < d' < d$, such that $\mathbf{M}, [d'', d'] \models \neg p$. Two cases are possible: either $d' \leq d_0$ or $d_0 < d'$. In the former case, by the first conjunct of $NoImmediateSucc$, we have that $\mathbf{M}, [d'', d'] \models p$; in the latter case, it turns out that d is not the immediate successor of d_0. Thus, in both cases we get a contradiction. Let us consider now the class of all linearly ordered domains. A model satisfying $NoImmediateSucc$ can be built as follows. We take a model $\mathbf{M}$ that contains two points d_0, d_1 such that d_0 has at least one predecessor, d_1 has at least two successors, and d_0 has not an immediate successor, whose valuation function is defined as follows: $\mathbf{M}, [d_b, d_e] \models p$ for every ordered pair of points d_b, d_e, with $d_e \leq d_0$, and $\mathbf{M}, [d_b, d_e] \models \neg p$ for every ordered pair of points d_b, d_e, with $d_b \geq d_0$ (the truth value of p over the intervals $[d_b, d_e]$, with $d_b < d_0 < d_e$, does not matter, and thus it can be assigned in an arbitrary way). From $\mathbf{M}, [d_b, d_e] \models p$ for every ordered pair of points d_b, d_e, with $d_e \leq d_0$, and $\mathbf{M}, [d_0, d_e] \models \neg p$ for every d_e, it follows that $\mathbf{M}, [d_0, d_1] \models \langle \overline{A} \rangle \top \wedge [\overline{A}](p \wedge [A]\neg p \wedge [\overline{A}]p)$. Now, let d_2, d_3, with $d_2 < d_3$, be two successors of d_1, whose existence is guaranteed by definition of $\mathbf{M}$. We show that $\mathbf{M}, [d_2, d_3] \models [\overline{A}]([\overline{A}]p \vee \langle \overline{A} \rangle \langle \overline{A} \rangle \neg p)$. Let d be a predecessor of d_2. We distinguish two cases: either $d \leq d_0$ or $d > d_0$. In the former case, $\mathbf{M}, [d, d_2] \models [\overline{A}]p$ as $\mathbf{M}, [d_b, d_e] \models p$ for every ordered pair of points d_b, d_e, with $d_e \leq d_0$. In the latter case, since d is not an immediate successor of d_0 (d_0 has not an immediate successor), there exists d' in between d_0 and d such that $\mathbf{M}, [d_0, d'] \models \neg p$, as $\mathbf{M}, [d_0, d_e] \models \neg p$ for every d_e, and thus $\mathbf{M}, [d, d_2] \models \langle \overline{A} \rangle \langle \overline{A} \rangle \neg p$. A pictorial representation of the model is given in Figure 5. □

Proposition 4. *The PNL formula AccPoints is satisfiable over the class of all (resp., dense, discrete) linear orders, while it is not satisfiable over* $\mathbb{Z}$.

Proof. We first show that *AccPoints* is not satisfiable over $\mathbb{Z}$. Suppose, by contradiction, that there exists an interpretation $\mathbf{M}$, based on $\mathbb{Z}$, such that $\mathbf{M}, [d_0, d_1] \models AccPoints$. From $\mathbf{M}, [d_0, d_1] \models \langle A \rangle p \wedge [G]seq_p$, it follows that there exists a sequence of points $d_1 < d_{j_1} < d_{j_2} \ldots$ such that $\mathbf{M}, [d_1, d_{j_1}] \models p$ and $\mathbf{M}, [d_{j_i}, d_{j_{i+1}}] \models p$, for all $i \geq 1$. Moreover, from $\mathbf{M}, [d_0, d_1] \models \langle A \rangle [G] \neg p$, it follows that there exists a point d_i such that $\mathbf{M}, [d_1, d_i] \models [G]\neg p$. Two cases may arise.

Case (1). Suppose $d_i \leq d_{j_1}$. From $\mathbf{M}, [d_1, d_i] \models [G]\neg p$, it follows that $\mathbf{M}, [d, d_{j_1}] \models [A]\neg p$ and thus $\mathbf{M}, [d_{j_1}, d_{j_2}] \models \neg p$. This allows us to conclude that both p and $\neg p$ hold over $[d_{j_1}, d_{j_2}]$, as shown in Figure 6, and a contradiction is found.

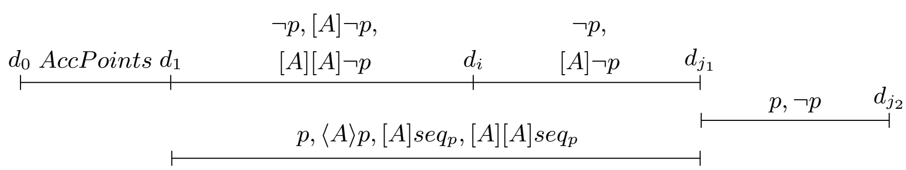

Fig. 6. Unsatisfiability of *AccPoints* over $\mathbb{Z}$: case (1).

Case (2). Suppose $d_{j_1} < d_i$. From $\mathbf{M}, [d_1, d_i] \models [A][A]\neg p$, it follows that, for any point $d_k > d_i$, $\mathbf{M}, [d_i, d_k] \models [A]\neg p$ and, for any point $d_m > d_k$, $\mathbf{M}, [d_k, d_m] \models \neg p$. Since *AccPoints* is interpreted over $\mathbb{Z}$, there exists a point $d_{j_h} > d_i$ that belongs to the sequence. This implies that p holds over $[d_{j_h}, d_{j_{h+1}}]$. Hence, both p and $\neg p$ hold over $[d_{j_h}, d_{j_{h+1}}]$, as shown in Figure 7, and a contradiction is found.

Let us consider now the class of all linearly ordered domains. A model satisfying *AccPoints* can be built as follows: we take an infinite sequence of points $d_{j_1} < d_{j_2} < d_{j_3} < \ldots$ such that $\mathbf{M}, [d_{j_i}, d_{j_{i+1}}] \models p$, for every $i \geq 1$, and then we add an accumulation point d_ω greater than d_{j_i}, for every $i \geq 1$, such that $\mathbf{M}, [d_1, d_\omega] \models [G]\neg p$. The definition of the valuation function can be easily completed without introducing any contradiction, thus showing that *AccPoints* is

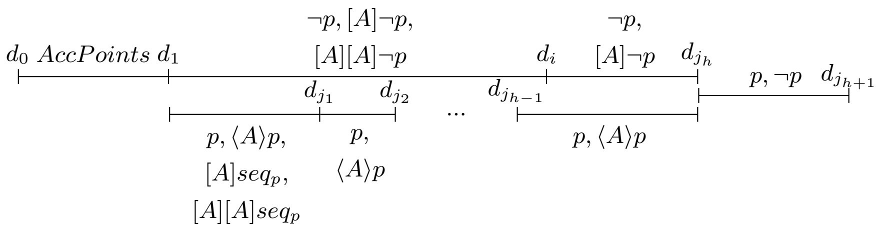

Fig. 7. Unsatisfiability of *AccPoints* over $\mathbb{Z}$: case (2).

satisfiable (see Figure 8). Exactly the same argument holds for dense linear orders. As for (weakly) discrete linear orders, it is sufficient to consider the domain consisting of the concatenation of two copies of $\mathbb{Z}$. $\square$

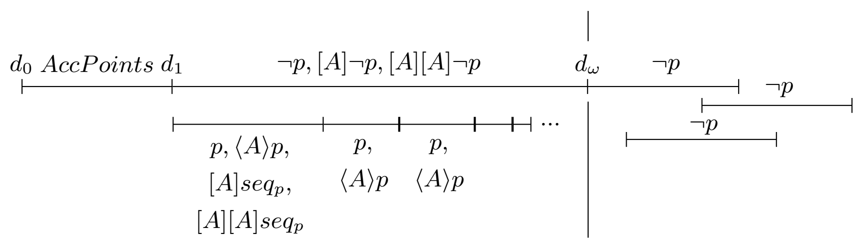

Fig. 8. A model for *AccPoints* over the class of linearly ordered domains.

Theorem 1. *A PNL formula φ is satisfiable ifif there exists a fulfilling* LIS $\mathbf{L} = \langle \langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle, \mathcal{L} \rangle$ *with* $\varphi \in \mathcal{L}([d_i, d_j])$ *for some* $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$.

Proof. We first prove the left-to-right direction. Let φ be a satisfiable PNL formula. Hence, there exist a model $\mathbf{M} = \langle \langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle, \mathcal{V} \rangle$ and an interval $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$ such that $\mathbf{M}, [d_i, d_j] \Vdash \varphi$. We show that $\mathbf{L} = \langle \langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle, \mathcal{L} \rangle$, where, for every $[d, d'] \in \mathbb{I}(\mathbb{D})$, $\mathcal{L}([d, d']) = \{\psi \in \mathrm{CL}(\varphi) \mid \mathbf{M}, [d, d'] \Vdash \psi\}$, is a fulfilling LIS for φ. We first prove that for every $[d, d'] \in \mathbb{I}(\mathbb{D})$, $\mathcal{L}([d, d'])$ is a φ-atom. For every $[d, d'] \in \mathbb{I}(\mathbb{D})$ and $\psi \in \mathrm{CL}(\varphi)$, we have that:

– by definition of $\Vdash$, $\mathbf{M}, [d, d'] \Vdash \psi$ if and only if $\mathbf{M}, [d_i, d_j] \nVdash \neg\psi$ and thus, by definition of $\mathcal{L}$, $\psi \in \mathcal{L}([d, d'])$ if and only $\neg\psi \notin \mathcal{L}([d, d'])$;
– by definition of $\Vdash$, $\mathbf{M}, [d, d'] \Vdash \psi_1 \vee \psi_2$ if and only if $\mathbf{M}, [d, d'] \Vdash \psi_1$ or $\mathbf{M}, [d, d'] \Vdash \psi_2$ and thus, by definition of $\mathcal{L}$, $\psi \in \mathcal{L}([d, d'])$ if and only if $\psi_1 \in \mathcal{L}([d, d'])$ or $\psi_2 \in \mathcal{L}([d, d'])$.

Next, we prove that for every d, d', d'' in D, if $d < d' < d''$, then $\mathcal{L}([d, d'])$ LR_φ $\mathcal{L}([d', d''])$. Suppose, by contradiction, that there exist d, d', d'' in D such that

$d < d' < d''$ and $\mathcal{L}([d,d'])\ LR_\varphi\ \mathcal{L}([d',d''])$ does not hold. By definition of LR_φ, this means that there exists $[A]\psi \in \mathrm{CL}(\varphi)$ such that $[A]\psi \in \mathcal{L}([d,d'])$ and $\psi \notin \mathcal{L}([d',d''])$ (and thus $\neg\psi \in \mathcal{L}([d',d''])$ or there exists $[\overline{A}]\psi \in \mathrm{CL}(\varphi)$ such that $[\overline{A}]\psi \in \mathcal{L}([d',d''])$ and $\psi \notin \mathcal{L}([d,d'])$ (and thus $\neg\psi \in \mathcal{L}([d,d'])$). Let us consider the first case (the second one is completely symmetric, and thus omitted). By definition of $\mathcal{L}$, we have that $\mathbf{M},[d,d'] \models [A]\psi$ and $\mathbf{M},[d',d''] \models \neg\psi$. By definition of $\Vdash$, $\mathbf{M},[d,d'] \models [A]\psi$ implies that $\mathbf{M},[d',\overline{d}] \models \psi$ for all $\overline{d} > d'$. Since, by hypothesis, $d'' > d'$, we have that $\mathbf{M},[d',d''] \models \psi$, which contradicts $\mathbf{M},[d',d''] \models \neg\psi$. Finally, to prove that $\mathbf{L}$ is fulfilling, we must show that for every $[d,d'] \in \mathbb{I}(\mathbb{D})$ and every $\langle A\rangle\psi \in \mathcal{L}([d,d'])$ (resp., $\langle\overline{A}\rangle\psi \in \mathcal{L}([d,d'])$) there exists $d'' > d'$ (resp., $d'' < d$) such that $\psi \in \mathcal{L}([d',d''])$ (resp., $\psi \in \mathcal{L}([d'',d])$). Let $\langle A\rangle\psi \in \mathcal{L}([d,d'])$ (the case in which $\langle\overline{A}\rangle\psi \in \mathcal{L}([d,d'])$ is completely symmetric, and thus omitted). By definition of $\mathcal{L}$, we have that $\mathbf{M},[d,d'] \models \langle A\rangle\psi$. Since $\mathbf{M}$ is a model, we have that there exists $d'' \in \mathbb{D}$ with $d'' > d'$ for which $\mathbf{M},[d',d''] \models \psi$ and, by definition of $\mathcal{L}$, we have $\psi \in \mathcal{L}([d',d''])$.

Let us consider now the right-to-left. Let $\mathbf{L} = \langle\langle\mathbb{D},\mathbb{I}(\mathbb{D})\rangle,\mathcal{L}\rangle$ a fulfilling LIS for a PNL formula φ we define $\mathbf{M} = \mathbf{M} = \langle\langle\mathbb{D},\mathbb{I}(\mathbb{D})\rangle,\mathcal{V}\rangle$ with $\mathcal{V}([d,d']) = \mathcal{L}([d,d']) \cap AP$. For every $[d,d'] \in \mathbb{I}(\mathbb{D})$ we prove by induction on the structure of ψ that $\mathbf{M},[d,d'] \models \psi$ if and only if $\psi \in \mathcal{L}([d,d'])$.

- $\psi = p$ with $p \in AP$. By construction $p \in \mathcal{V}([d,d'])$ if and only if $p \in \mathcal{L}([d,d'])$.
- $\psi = \neg\psi_1$. By inductive hypothesis, $\mathbf{M},[d,d'] \models \psi_1$ if and only if $\psi_1 \in \mathcal{L}([d,d'])$. Since $\mathcal{L}([d,d'])$ is an atom we have $\psi_1 \in \mathcal{L}([d,d'])$ if and only if $\neg\psi_1 \notin \mathcal{L}([d,d'])$. Hence, $\mathbf{M},[d,d'] \models \neg\psi_1$ if and only if $\neg\psi_1 \in \mathcal{L}([d,d'])$.
- $\psi = \psi_1 \vee \psi_2$. By inductive hypothesis we have that $\mathbf{M},[d,d'] \models \psi_1$ if and only if $\psi_1 \in \mathcal{L}([d,d'])$ and $\mathbf{M},[d,d'] \models \psi_2$ if and only if $\psi_2 \in \mathcal{L}([d,d'])$. By definition of atom we have that $\psi_1 \vee \psi_2 \in \mathcal{L}([d,d'])$ if and only if $\psi_1 \in \mathcal{L}([d,d'])$ or $\psi_2 \in \mathcal{L}([d,d'])$. Hence, $\mathbf{M},[d,d'] \Vdash \psi_1 \vee \psi_2$ if and only if $\psi_1 \vee \psi_2 \in \mathcal{L}([d,d'])$.
- $\psi = \langle A\rangle\psi_1$. Since L is a fulfilling LIS we have that there exists $d'' > d'$ with $\psi_1 \in \mathcal{L}([d',d''])$ if and only if $\langle A\rangle\psi_1 \in \mathcal{L}([d,d'])$. By inductive hypothesis we have that $\psi_1 \in \mathcal{L}([d',d''])$ if and only if $\mathbf{M},[d',d''] \Vdash \psi_1$. Summing up we obtain that $\langle A\rangle\psi_1 \in \mathcal{L}([d,d'])$ if and only if $\mathbf{M},[d,d'] \Vdash \langle A\rangle\psi_1$.
- $\psi = \langle\overline{A}\rangle\psi_1$. This case is symmetric to the previous one.

To conclude the proof, it suffices to observe that since L is a fulfilling LIS there exists an interval $[d_i,d_j] \in \mathbb{I}(\mathbb{D})$ for which $\varphi \in \mathcal{L}([d_i,d_j])$ and thus we have that $\mathbf{M},[d_i,d_j] \Vdash \varphi$. □

Lemma 2. *Given a PNL formula φ and a fulfilling* LIS $\mathbf{L} = \langle\langle\mathbb{D},\mathbb{I}(\mathbb{D})\rangle,\mathcal{L}\rangle$ *that satisfies it, there exists a pseudo-model $\mathbf{L}'$ for φ, with $|D'| \leq 2 \cdot |\varphi| \cdot 2^{3\cdot|\varphi|+1}$.*

Proof. Let $IT(\mathbf{L}) = \{\langle R,A,R'\rangle \mid \langle R,A,R'\rangle$ appears in $\mathbf{L}\}$ and let D' be a minimal subset of D such that for every $\langle R,A,R'\rangle \in IT(\mathbf{L})$, there exist two points $d,d' \in D'$ such that $\mathrm{REQ}^{\mathbf{L}}(d) = R$, $\mathrm{REQ}^{\mathbf{L}}(d') = R'$, $\mathcal{L}([d,d']) = A$, and d,d' are fulfilled in D', that is, $ES_f^{d'} \cup ES_f^{d} \cup ES_p^{d'} \cup ES_p^{d'} \subseteq D'$. For every

$d, d' \in D'$, with $d < d'$, we define $\mathcal{L}'([d,d']) = \mathcal{L}([d,d'])$. It is easy to prove that $\mathbf{L}'$ is a pseudo-model for φ. As for the size of D', the number of distinct interval-tuples in $\mathbf{L}'$ is at most $2^{3\cdot|\varphi|+1}$ (the number of atoms is $2^{|\varphi|+1}$ and the number of sets of requests is $2^{|\varphi|}$) and, for every interval-tuple, at most $2 \cdot |\varphi|$ points must be added. Hence, $|D'| \leq 2 \cdot |\varphi| \cdot 2^{3\cdot|\varphi|+1}$. □

Theorem 3. *Let $\mathcal{T}$ be a final tableau for a PNL formula φ and B be a branch of $\mathcal{T}$. We have that $|B| \leq (2 \cdot |\varphi| \cdot 2^{3\cdot|\varphi|+1}) \cdot (2 \cdot |\varphi| \cdot 2^{3\cdot|\varphi|+1} - 1)/2$.*

Proof. By the very same argument of Lemma 2, for any branch B, $|D_B| \leq 2 \cdot |\varphi| \cdot 2^{3\cdot|\varphi|+1}$. As we have exactly one node for any interval over $\mathbb{D}_B$, the length of B is at most $(2 \cdot |\varphi| \cdot 2^{3\cdot|\varphi|+1}) \cdot (2 \cdot |\varphi| \cdot 2^{3\cdot|\varphi|+1} - 1)/2$. □

Theorem 4. *Let $\mathcal{T}$ be a final tableau for a PNL formula φ. If $\mathcal{T}$ features one blocked branch, then φ is satisfiable over all linear orders.*

Proof. Let B be a blocked branch in $\mathcal{T}$. From the completeness of B, it follows that for every pair $d, d' \in D_B$, with $d < d'$, there exists one node n in B labeled with $\langle [d,d'], A \rangle$. Moreover, such a node is unique by construction. Let $\mathbf{L} = \langle (\mathbb{D}_B, \mathbb{I}(\mathbb{D}_B)), \mathcal{L}_B \rangle$, where, for every pair $d, d' \in D_B$, we put $\mathcal{L}_B([d,d']) = A$. By construction (expansion rules), it immediately follows that $\mathbf{L}$ is a finite LIS. Since B is blocked, all its active points are fulfilled in B and thus in $\mathbf{L}$. Hence, every interval-tuple $\langle S, A, S' \rangle$ in $\mathbf{L}$ is fulfilled. This allows us to conclude that $\mathbf{L}$ is a pseudo-model for φ. From Lemma 1 and Theorem 1, it immediately follows that φ is satisfiable. □

Theorem 5. *Let φ be a PNL formula which is satisfiable over the class of all linear orders. Then, there exists a final tableau for φ with at least one blocked branch.*

Proof. First, by Theorem 1, we have that if φ is satisfiable, then there exists a fulfilling LIS $\mathbf{L} = \langle (\mathbb{D}, \mathbb{I}(\mathbb{D})), \mathcal{L} \rangle$ for it. Next, we show that such a fulfilling LIS can be exploited to construct a final tableau $\mathcal{T}$ for φ, that features a blocked branch B. Formally, by an induction on the number i of expansion steps, we prove that there exists a non-closed branch B such that (i) for all $d \in D_B$, $d \in D$ and $REQ_B(d) = REQ^{\mathbf{L}}(d)$, (ii) for all $d, d' \in D_B$, if $d < d'$ in $\mathbb{D}$, then $d < d'$ in $\mathbb{D}_B$, and (iii) if there exists a node n in B labeled with $\langle [d,d'], A \rangle$, then $A = \mathcal{L}([d,d'])$. The base case is straightforward. Since $\mathbf{L}$ is a LIS for φ, there exist two points $d, d' \in D$ such that $\varphi \in \mathcal{L}([d,d'])$. We start with an initial tableau $\mathcal{T}_0$, consisting of a single branch B_0, whose unique node n_0 is labeled with $\langle [d,d'], A \rangle$, with $\mathrm{REQ}_{B_0}(d) = \mathrm{REQ}^{\mathbf{L}}(d)$, $\mathrm{REQ}_{B_0}(d') = \mathrm{REQ}^{\mathbf{L}}(d')$, and $A = \mathcal{L}([d,d'])$. Let $\mathcal{T}_i$ be the tableau generated at the i-th step of the expansion process and let B_i be a branch of $\mathcal{T}_i$ that satisfies the inductive hypothesis. We expand it as follows:

- If the *Fill-in rule* is applicable, then there exists a pair of points $d, d' \in D_{B_i}$ such that there exists no node in B_i labeled with the interval $[d,d']$. By the inductive hypothesis, $d, d' \in D$. We expand B_i with a new node n labeled with $\langle [d,d'], \mathcal{L}([d,d']) \rangle$.

- If the $\langle A \rangle$-rule is applicable, then there exist an active point $d \in D_{B_i}$ and a formula $\langle A \rangle \psi \in \mathrm{REQ}_{B_i}(d)$, which is not fulfilled in B_i for d. By the inductive hypothesis, $d \in D$. Since $\mathcal{L}$ is fulfilling, there exists $d' \in D$, with $d' > d$, such that $\psi \in \mathcal{L}([d, d'])$. Since $\langle A \rangle \psi$ is not fulfilled in B_i for d and B_i is complete (the *Fill-in rule* is not applicable), by condition (iii), we can conclude that $d' \notin D_{B_i}$. We add a new point d' to D_{B_i} in such a way that for all $d'' \in D_{B_i}$, if $d'' < d'$ in $\mathbb{D}$, then $d'' < d'$ in $\mathbb{D}_{B_i}$, $d' < d''$ in $\mathbb{D}_{B_i}$ otherwise, and we apply the $\langle A \rangle$-rule to expand B_i with a new node $n = \langle [d, d'], \mathcal{L}[d, d'] \rangle$.
- The case of the $\langle \overline{A} \rangle$-rule is completely symmetric, and thus its description is omitted.

By Theorem 3, the expansion of the tableau terminates in a finite (bounded) number of steps. Since no contradiction is introduced by any of the above steps, the final branch B is necessarily blocked. $\square$