# An optimal tableau-based decision algorithm for Propositional Neighborhood Logic

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**Abstract.** In this paper we focus our attention on the decision problem for Propositional Neighborhood Logic (PNL for short). PNL is the proper subset of Halpern and Shoham's modal logic of intervals whose modalities correspond to Allen's relations *meets* and *met by*. We show that the satisfiability problem for PNL over the integers is NEXPTIME-complete. Then, we develop a sound and complete tableau-based decision procedure and we prove its optimality.

## 1 Introduction

Temporal logics play an important role in several areas of computer science, including artificial intelligence, specification and automatic verification of programs, and temporal databases. Even though interval-based temporal logics provide a natural framework for representing and reasoning about time, most work has been devoted to point-based ones, which generally show a better computational behavior. In this paper, we focus our attention on the propositional fragment of the interval logic of temporal neighborhood (PNL for short) [3,4]. We devise a NEXPTIME tableau-based decision procedure for PNL over the integers (or a subset of them) and we prove its optimality.

Various propositional and first-order interval temporal logics have been proposed in the literature (a comprehensive survey can be found in [5]). The most significant propositional ones are Halpern and Shoham's Modal Logic of Time Intervals (HS) [7], Venema's CDT logic [6,12], and Moszkowski's Propositional Interval Temporal Logic (PITL) [11]. Unfortunately, all of them turn out to be undecidable. Halpern and Shoham's logic has been shown to be undecidable for several classes of linear and branching orders [7]. Venema's CDT is powerful enough to embed HS, and thus it is undecidable (at least) over the same classes of orders. Finally, PITL has been shown to be undecidable over discrete linear orders by Moszkowski [11]; its undecidability over dense linear orders easily follows from the undecidability of the *Begin/End* (BE) fragment of HS [5,8].

To get decidability, severe syntactic and/or semantic restrictions have been imposed to interval-based temporal logics to make it possible to reduce them to point-based ones [9]. One can get decidability by making a suitable choice of the interval modalities. This is the case with the $\mathrm{B}\overline{\mathrm{B}}$ (*Begin/Begun by*) and $\mathrm{E}\overline{\mathrm{E}}$ (*End/Ended by*) fragments of HS [5]. As an alternative, decidability can be achieved by constraining the classes of temporal structures over which the interval logic is interpreted. This is the case with the so-called Split Logics (SLs) [10]. Finally, another possibility is to constrain the relation between the truth value of a formula over an interval and its truth value over subintervals of it. As an example, one can constrain a propositional variable to be true over an interval if and only if it is true at its starting point (*locality*) or if and only if it is true over all its subintervals (*homogeneity*) [11]. All these approaches differ in the nature of the restrictions they impose, but they have a common feature: they replace every interval with a point and, accordingly, interval-based temporal operators with point-based ones. Hence, as pointed out in [9], the problem of proving the decidability of interval logics without taking advantage of such a replacement remains largely unexplored.

A first result in this direction has been obtained by Bresolin et al. in [1,2], where the decidability of the future fragment of PNL (RPNL for short) over the natural numbers is established. They basically prove that an RPNL formula is satisfiable if and only if there exist a finite model, or an ultimately periodic (infinite) one, with a finite representation of *bounded size*. In both cases, such a model can be built starting from any model satisfying the formula by progressively removing exceeding points from it until the desired bound is reached. The removal of a point $d$ from a model causes the removal of all intervals either beginning or ending at it. Since RPNL features only future time modalities, the removal of intervals beginning at $d$ is not critical. On the contrary, the removal of intervals ending at $d$ may introduce "defects", that is, there may be existential future temporal formulae that are not satisfied anymore. However, by properly choosing the point $d$ to remove, we can guarantee that there exist sufficiently many points in the future of $d$ which allows us to fix such defects (by possibly changing the truth value of formulas over intervals ending at them) without introducing new defects.

In this paper, we generalize the proof for RPNL to full PNL by showing that a PNL formula is satisfiable if and only if there exist a finite model or an infinite one with a finite representation of *bounded size*. As in the case of RPNL, such a model can be obtained by removing exceeding points from a given model satisfying the formula, but the removal process turns out to be much more involved. In contrast with the case of RPNL, the removal of a point $d$ from a PNL model may affect the satisfiability of formulae over intervals in the past as well as in the future of $d$ . Hence, to fix the defects possibly caused by the removal of $d$, we must guarantee that there exist sufficiently many points with the same characteristics as $d$ both in the future and in the past of $d$. Moreover, we must be sure that changing the valuation of intervals that either end or start at these points does not generate new defects. In the following, we show that this can actually be done.

The paper is organized as follows. In Section 2 we introduce syntax and semantics of PNL. Then, in Section 3 we prove the decidability of PNL over the integers (or a subset of them). In Section 4 we describe an optimal NEXPTIME tableau-based decision procedure, and we prove its soundness and completeness.

Conclusions provide an assessment of the work and outline future research directions.

## 2 Propositional Neighborhood Logic

In this section, we give syntax and semantics of PNL interpreted over the set $\mathbb{Z}$ of the integers or over a subset of it. To this end, we introduce some preliminary notions. Let $\mathbb{D} = \langle D, < \rangle$ be a strict linear order isomorphic to $\mathbb{Z}$ (or to a subset of it). A *strict interval* on $\mathbb{D}$ is an ordered pair $[d_i, d_j]$ such that $d_i, d_j \in D$ and $d_i < d_j$. The set of all strict intervals over $\mathbb{D}$ will be denoted by $\mathbb{I}(\mathbb{D})^-$ (here we conform to the notation proposed in [4], where $^-$ is used to denote the lack of point intervals, that is, intervals of the form $[d_i, d_i]$). The pair $\langle \mathbb{D}, \mathbb{I}(\mathbb{D})^- \rangle$ is called a *strict interval structure*. For every pair of intervals $[d_i, d_j], [d'_i, d'_j] \in \mathbb{I}(\mathbb{D})^-$, we say that $[d'_i, d'_j]$ is a *right* (resp., *left*) *neighbor* of $[d_i, d_j]$ if and only if $d_j = d'_i$ (resp., $d'_j = d_i$).

The language of (Strict) *Propositional Neighborhood Logic* (PNL for short) consists of a set $AP$ of propositional letters, the connectives $\neg$ and $\vee$, and the modal operators $\langle A \rangle$ and $\langle \overline{A} \rangle$. The other connectives, as well as the logical constants $\top$ (true) and $\bot$ (false), can be defined as usual. The *formulae* of PNL, denoted by $\varphi, \psi, \ldots$, are recursively defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle A \rangle \varphi \mid \langle \overline{A} \rangle \varphi.$$

We denote by $|\varphi|$ the length of $\varphi$, that is, the number of symbols in $\varphi$ (in the following, we shall use $|\ |$ to denote the cardinality of a set as well). Whenever there are no ambiguities, we call a PNL formula just a formula. A formula of the forms $\langle A \rangle \psi$, $\neg\langle A \rangle \psi$, $\langle \overline{A} \rangle \psi$, or $\neg\langle \overline{A} \rangle \psi$ is called a *temporal formula* (from now on, we identify $\neg\langle A \rangle \neg\psi$ with $[A]\psi$ and $\neg\langle \overline{A} \rangle \neg\psi$ with $[\overline{A}]\psi$).

A *model* for a PNL formula is a pair $\mathbf{M} = \langle \langle \mathbb{D}, \mathbb{I}(\mathbb{D})^- \rangle, \mathcal{V} \rangle$, where $\langle \mathbb{D}, \mathbb{I}(\mathbb{D})^- \rangle$ is a strict interval structure and $\mathcal{V} : \mathbb{I}(\mathbb{D})^- \longrightarrow 2^{AP}$ is a *valuation function* assigning to every interval the set of propositional letters true over it. Given a model $\mathbf{M} = \langle \langle \mathbb{D}, \mathbb{I}(\mathbb{D})^- \rangle, \mathcal{V} \rangle$ and an interval $[d_i, d_j] \in \mathbb{I}(\mathbb{D})^-$, the semantics of PNL is defined recursively by the *satisfiability relation* $\Vdash$ as follows:

- for every propositional letter $p \in AP$, $\mathbf{M}, [d_i, d_j] \Vdash p$ iff $p \in \mathcal{V}([d_i, d_j])$;
- $\mathbf{M}, [d_i, d_j] \Vdash \neg\psi$ iff $\mathbf{M}, [d_i, d_j] \nVdash \psi$;
- $\mathbf{M}, [d_i, d_j] \Vdash \psi_1 \vee \psi_2$ iff $\mathbf{M}, [d_i, d_j] \Vdash \psi_1$ or $\mathbf{M}, [d_i, d_j] \Vdash \psi_2$;
- $\mathbf{M}, [d_i, d_j] \Vdash \langle A \rangle \psi$ iff $\exists d_k \in D$ such that $d_k > d_j$ and $\mathbf{M}, [d_j, d_k] \Vdash \psi$;
- $\mathbf{M}, [d_i, d_j] \Vdash \langle \overline{A} \rangle \psi$ iff $\exists d_k \in D$ such that $d_k < d_i$ and $\mathbf{M}, [d_k, d_i] \Vdash \psi$.

We place ourselves in the most general (and difficult) setting where there are not constraints on the valuation function. As an example, given an interval $[d_i, d_j]$, it may happen that $p \in \mathcal{V}([d_i, d_j])$ and $p \notin \mathcal{V}([d'_i, d'_j])$ for all intervals $[d'_i, d'_j]$ (strictly) contained in $[d_i, d_j]$.

## 3 Labeled Interval Structures and Satisfiability

In this section we introduce some preliminary notions and we establish some basic results on which our tableau method for PNL relies (an intuitive account of them can be found in [2]).

Let $\varphi$ be a PNL formula to be checked for satisfiability and let $AP$ be the set of its propositional letters.

**Definition 1.** *The* closure $\mathrm{CL}(\varphi)$ *of* $\varphi$ *is the set of all subformulae of* $\langle A\rangle\varphi$ *and of their negations (we identify* $\neg\neg\psi$ *with* $\psi$*).*

As it will become clear later, we put the formula $\langle A\rangle\varphi$ and its negation in $\mathrm{CL}(\varphi)$ to avoid that the removal process could delete all intervals over which $\varphi$ holds.

**Definition 2.** *The set of* temporal formulae *of* $\varphi$ *is the set* $\mathrm{TF}(\varphi) = \{\zeta \in \mathrm{CL}(\varphi) : \zeta = \langle A\rangle\psi$ *or* $\zeta = [A]\psi$ *or* $\zeta = \langle\overline{A}\rangle\psi$ *or* $\zeta = [\overline{A}]\psi\}$.

By induction on the structure of $\varphi$, we can easily prove that, for every formula $\varphi$, $|\mathrm{CL}(\varphi)|$ is less than or equal to $2 \cdot (|\varphi| + 1)$, while $|\mathrm{TF}(\varphi)|$ is less than or equal to $2 \cdot |\varphi|$. We are now ready to introduce the notion of $\varphi$-atom.

**Definition 3.** *A* $\varphi$-atom *is a set* $A \subseteq \mathrm{CL}(\varphi)$ *such that:*

- *for every* $\psi \in \mathrm{CL}(\varphi)$*,* $\psi \in A$ *iff* $\neg\psi \notin A$*;*
- *for every* $\psi_1 \vee \psi_2 \in \mathrm{CL}(\varphi)$*,* $\psi_1 \vee \psi_2 \in A$ *iff* $\psi_1 \in A$ *or* $\psi_2 \in A$*.*

We denote the set of all $\varphi$-atoms by $\mathcal{A}_\varphi$. We have that $|\mathcal{A}_\varphi| \leq 2^{|\varphi|+1}$. Atoms are connected by the following binary relation.

**Definition 4.** *Let* $LR_\varphi$ *be a relation such that for every pair of atoms* $A_1, A_2 \in \mathcal{A}_\varphi$*,* $A_1\ LR_\varphi\ A_2$ *if and only if (i) for every* $[A]\psi \in \mathrm{CL}(\varphi)$*, if* $[A]\psi \in A_1$ *then* $\psi \in A_2$ *and (ii) for every* $[\overline{A}]\psi \in \mathrm{CL}(\varphi)$*, if* $[\overline{A}]\psi \in A_2$ *then* $\psi \in A_1$*.*

We now introduce a suitable labeling of interval structures based on $\varphi$-atoms.

**Definition 5.** *A* $\varphi$-labeled interval structure *(*LIS *for short) is a pair* $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})^-\rangle, \mathcal{L}\rangle$*, where* $\langle\mathbb{D}, \mathbb{I}(\mathbb{D})^-\rangle$ *is an interval structure and* $\mathcal{L} : \mathbb{I}(\mathbb{D})^- \to \mathcal{A}_\varphi$ *is a* labeling function *such that, for every pair of neighboring intervals* $[d_i, d_j], [d_j, d_k] \in \mathbb{I}(\mathbb{D})^-$*,* $\mathcal{L}([d_i, d_j])\ LR_\varphi\ \mathcal{L}([d_j, d_k])$*.*

If we interpret the labeling function as a valuation function, LISs represent *candidate models* for $\varphi$. The truth of formulae devoid of temporal operators and that of $[A]/[\overline{A}]$ formulae indeed follow from the definition of $\varphi$-atom and $LR_\varphi$, respectively. However, to obtain a model for $\varphi$, we must also guarantee the truth of $\langle A\rangle/\langle\overline{A}\rangle$ formulae. To this end, we introduce the notion of fulfilling LIS.

**Definition 6.** *A $\varphi$-labeled interval structure* $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})^-\rangle, \mathcal{L}\rangle$ *is* fulfilling *if and only if (i) for every temporal formula* $\langle A\rangle\psi \in \mathrm{TF}(\varphi)$ *and every interval* $[d_i, d_j] \in \mathbb{I}(\mathbb{D})^-$*, if* $\langle A\rangle\psi \in \mathcal{L}([d_i, d_j])$*, then there exists* $d_k > d_j$ *such that* $\psi \in \mathcal{L}([d_j, d_k])$ *and (ii) for every temporal formula* $\langle\overline{A}\rangle\psi \in \mathrm{TF}(\varphi)$ *and every interval* $[d_i, d_j] \in \mathbb{I}(\mathbb{D})^-$*, if* $\langle\overline{A}\rangle\psi \in \mathcal{L}([d_i, d_j])$*, then there exists* $d_k < d_i$ *such that* $\psi \in \mathcal{L}([d_k, d_i])$*.*

The next theorem proves that for any given formula $\varphi$, the satisfiability of $\varphi$ is equivalent to the existence of a fulfilling LIS with an interval labeled by $\varphi$.

**Theorem 1.** *A formula $\varphi$ is satisfiable if and only if there exists a fulfilling* LIS $\mathbf{L} = \langle\langle \mathbb{D}, \mathbb{I}(\mathbb{D})^- \rangle, \mathcal{L}\rangle$ *with $\varphi \in \mathcal{L}([d_i, d_j])$ for some $[d_i, d_j] \in \mathbb{I}(\mathbb{D})^-$.*

The implication from left to right is straightforward; the opposite implication is proved by induction on the structure of the formula.

From now on, we say that a fulfilling LIS $\mathbf{L} = \langle\langle \mathbb{D}, \mathbb{I}(\mathbb{D})^- \rangle, \mathcal{L}\rangle$ *satisfies* $\varphi$ if and only if there exists an interval $[d_i, d_j] \in \mathbb{I}(\mathbb{D})^-$ such that $\varphi \in \mathcal{L}([d_i, d_j])$. Since (the domain of) fulfilling LISs satisfying $\varphi$ may be arbitrarily large or even infinite, we must find a way to finitely establish their existence. In the following, we first give a bound on the size of finite fulfilling LISs that must be checked for satisfiability, when searching for finite $\varphi$-models; then, we show that we can restrict ourselves to infinite fulfilling LISs with a finite bounded representation, when searching for infinite $\varphi$-models.

**Definition 7.** *Given a* LIS $\mathbf{L} = \langle\langle \mathbb{D}, \mathbb{I}(\mathbb{D})^- \rangle, \mathcal{L}\rangle$ *and a point $d \in D$, we define the set of* future temporal requests of $d$ *as the set* $\mathrm{REQ}_f^{\mathbf{L}}(d) = \{\langle A\rangle\xi, [A]\xi \in \mathrm{TF}(\varphi) : \exists d' \in D(\langle A\rangle\xi, [A]\xi \in \mathcal{L}([d', d]))\}$ *and the set of* past temporal requests of $d$ *as the set* $\mathrm{REQ}_p^{\mathbf{L}}(d) = \{\langle \overline{A}\rangle\xi, [\overline{A}]\xi \in \mathrm{TF}(\varphi) : \exists d' \in D(\langle \overline{A}\rangle\xi, [\overline{A}]\xi \in \mathcal{L}([d, d']))\}$. *The set of* temporal requests of $d$ *is defined as* $\mathrm{REQ}^{\mathbf{L}}(d) = \mathrm{REQ}_p^{\mathbf{L}}(d) \cup \mathrm{REQ}_f^{\mathbf{L}}(d)$.

We denote by $\mathrm{REQ}_\varphi$ the set of all possible sets of requests. It is not difficult to show that $|\,\mathrm{REQ}_\varphi\,|$ is equal to $2^{\frac{|\,\mathrm{TF}(\varphi)|}{2}}$.

**Definition 8.** *Given a* LIS $\mathbf{L} = \langle\langle \mathbb{D}, \mathbb{I}(\mathbb{D})^- \rangle, \mathcal{L}\rangle$, *$D' \subseteq D$, and $\mathcal{R} \in \mathrm{REQ}_\varphi$, we say that $\mathcal{R}$* occurs $n$ times in $D'$ *if and only if there exist exactly $n$ distinct points $d_{i_1}, \ldots, d_{i_n} \in D'$ such that $\mathrm{REQ}^{\mathbf{L}}(d_{i_j}) = \mathcal{R}$, for all $1 \leq j \leq n$.*

We describe now the process of removing a point from a LIS. Given $\mathbf{L} = \langle\langle \mathbb{D}, \mathbb{I}(\mathbb{D})^- \rangle, \mathcal{L}\rangle$ and $d \in D$, let $\mathbf{L}_{-d}$ be the set of all LIS $\mathbf{L}' = \langle\langle \mathbb{D}', \mathbb{I}(\mathbb{D}')^- \rangle, \mathcal{L}'\rangle$ such that $D' = D \setminus \{d\}$ and $\mathrm{REQ}^{\mathbf{L}'}(\overline{d}) = \mathrm{REQ}^{\mathbf{L}}(\overline{d})$, for all $\overline{d} \in D \setminus \{d\}$. $\mathbf{L}$ and $\mathbf{L}'$ do not necessarily agree on the labeling of intervals, but they agree on the sets of requests of points.

Given a fulfilling LIS $\mathbf{L}$ and a point $d$, it is not guaranteed that $\mathbf{L}_{-d}$ contains a fulfilling LIS. The removal of $d$ indeed causes the removal of all intervals either beginning or ending at it and thus there can be a point $\overline{d} < d$ (resp., $\overline{d} > d$) such that there exists a formula $\langle A\rangle\psi \in \mathrm{REQ}_f^{\mathbf{L}}(\overline{d})$ (resp., $\langle \overline{A}\rangle\psi \in \mathrm{REQ}_p^{\mathbf{L}}(\overline{d})$) which is fulfilled in $\mathbf{L}$, but not in any $\mathbf{L}' \in \mathbf{L}_{-d}$. The following lemma provides a sufficient condition for preserving the fulfilling property when removing a point from $\mathbf{L}$.

**Lemma 1.** *Let $\mathbf{L} = \langle\langle \mathbb{D}, \mathbb{I}(\mathbb{D})^- \rangle, \mathcal{L}\rangle$ be a fulfilling* LIS, *$f$ be the number of $\langle A\rangle$-formulae in $\mathrm{TF}(\varphi)$, and $p$ be the number of $\langle \overline{A}\rangle$-formulae in $\mathrm{TF}(\varphi)$. If there exists a point $d_e \in D$ such that (i) there exist at least $f \cdot p + p$ distinct points $d < d_e$ such that $\mathrm{REQ}^{\mathbf{L}}(d) = \mathrm{REQ}^{\mathbf{L}}(d_e)$ and (ii) there exist at least $f \cdot p + f$ distinct points $d > d_e$ such that $\mathrm{REQ}^{\mathbf{L}}(d) = \mathrm{REQ}^{\mathbf{L}}(d_e)$, then there is one fulfilling* LIS $\widehat{\mathbf{L}} \in \mathbf{L}_{-d_e}$.

*Proof.* Let $\mathbf{L} = \langle\langle \mathbb{D}, \mathbb{I}(\mathbb{D})^{-}\rangle, \mathcal{L}\rangle$ be a fulfilling LIS and let $d_e \in D$ be a point such that there exist at least $f \cdot p + p$ distinct points $d < d_e$ such that $\mathrm{REQ}^{\mathbf{L}}(d) = \mathrm{REQ}^{\mathbf{L}}(d_e)$ and at least $f \cdot p + f$ distinct points $d > d_e$ such that $\mathrm{REQ}^{\mathbf{L}}(d) = \mathrm{REQ}^{\mathbf{L}}(d_e)$. We define $\mathbb{D}' = \langle D \setminus \{d_e\}, <\rangle$ and $\mathcal{L}' = \mathcal{L}|_{\mathbb{I}(\mathbb{D}')^{-}}$ (the restriction of $\mathcal{L}$ to the intervals on $\mathbb{D}'$). The pair $\mathbf{L}' = \langle\langle \mathbb{D}', \mathbb{I}(\mathbb{D}')^{-}\rangle, \mathcal{L}'\rangle$ is obviously a LIS in $\mathbf{L}_{-d_e}$, but, as already pointed out, it is not necessarily a fulfilling one. We show how the defects possibly caused by the removal of $d_e$ can be fixed one-by-one by properly redefining $\mathcal{L}'$.

Consider the case of a point $d < d_e$ and a formula $\langle A\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}_{f}(d)$ such that $\psi \in \mathcal{L}([d, d_e])$ and there are no $\overline{d} \in D \setminus \{d_e\}$ such that $\psi \in \mathcal{L}'([d, \overline{d}])$ (the symmetric case of $d > d_e$ and $\langle\overline{A}\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}_{p}(d)$ can be dealt with in the same way). Let $R = \{d_r \in D : d_r > d_e \wedge \mathrm{REQ}^{\mathbf{L}}(d_r) = \mathrm{REQ}^{\mathbf{L}}(d_e)\}$. To satisfy the request $\langle A\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}(d)$ we change the labeling of an interval $[d, d_r]$, for a suitable $d_r \in R$. However, to avoid that such a change makes one or more requests in $\mathrm{REQ}^{\mathbf{L}}_{p}(d_r)$ no more satisfied, we preliminarily redefine the labeling $\mathcal{L}'$. First, we take a minimal set of points $P^{d_e} \subseteq D \setminus \{d_e\}$ such that, for every $\langle\overline{A}\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}_{p}(d_e)$ there exists a point $d_i \in P^{d_e}$ such that $\psi \in \mathcal{L}([d_i, d_e])$. We call $P^{d_e}$ the set of *preserved past points for* $d_e$. Then, for every point $d_i \in P^{d_e}$, let $F^{d_i} \subseteq D \setminus \{d_e\}$ be a minimal set of points such that, for every $\langle A\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}_{f}(d_i)$ there is a point $d_f \in F^{d_i}$ such that $\psi \in \mathcal{L}([d_i, d_f])$. We call $F^{d_i}$ the set of *preserved future points for* $d_i$.

Let $G$ be the set of points $R \setminus \bigcup_{d_i \in P^{d_e}} F^{d_i}$. By the minimality requirements, $|P^{d_e}|$ is bounded by $p$ and $|F^{d_i}|$, for each $d_i \in P^{d_e}$, is bounded by $f$. Hence, $|\bigcup_{d_i \in P^{d_e}} F^{d_i}| \leq f \cdot p$ and, by Condition (ii), $|G|$ is greater than or equal to $f$. Now, we can use points in $G$ to fulfill $\langle A\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}_{f}(d)$, without generating new defects, as follows. Since $\mathrm{REQ}^{\mathbf{L}}_{f}(d)$ contains at most $f$ $\langle A\rangle$-formulae, there exists at least one point $d_g \in G$ such that the atom $\mathcal{L}'([d, d_g])$ either fulfills no $\langle A\rangle$-formulae or it fulfills only $\langle A\rangle$-formulae which are also fulfilled by an $\varphi$-atom $\mathcal{L}'([d, d_k])$ for some $d_k$. Let $d_g$ one of such "useless" points. We can redefine $\mathcal{L}'([d, d_g])$ by putting $\mathcal{L}'([d, d_g]) = \mathcal{L}([d, d_e])$, thus fixing the problem for $\langle A\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}_{f}(d)$. Since $\mathrm{REQ}^{\mathbf{L}}(d_g) = \mathrm{REQ}^{\mathbf{L}}(d_e)$, such a change has no impact on the right neighboring intervals of $[d, d_g]$. On the contrary, there may exist one or more $\langle\overline{A}\rangle$-formulae in $\mathrm{REQ}^{\mathbf{L}}_{p}(d_g)$ which, due to the change in the labeling of $[d, d_g]$, are not satisfied anymore. In such a case, however, we can recover satisfiability, without introducing any new defect, by putting $\mathcal{L}'([d_i, d_g]) = \mathcal{L}([d_i, d_e])$ for all $d_i \in P^{d_e}$.

In the same way, we can fix all possible other defects caused by the removal of $d_e$. Let $\widehat{\mathbf{L}} = \langle\langle \mathbb{D}', \mathbb{I}(\mathbb{D}')\rangle, \widehat{\mathcal{L}}\rangle$ be the resulting LIS. It is immediate to show that $\widehat{\mathbf{L}}$ is fulfilling and it belongs to $\mathbf{L}_{-d_e}$. □

By taking advantage of Lemma 1, we can prove the following theorem.

**Theorem 2.** *Let* $\mathbf{L} = \langle\langle \mathbb{D}, \mathbb{I}(\mathbb{D})^{-}\rangle, \mathcal{L}\rangle$ *be a* finite *fulfilling* LIS *that satisfies* $\varphi$, $f$ *be the number of* $\langle A\rangle$*-formulae in* $\mathrm{TF}(\varphi)$*, and* $p$ *be the number of* $\langle\overline{A}\rangle$*-formulae in* $\mathrm{TF}(\varphi)$*. Then, there exists a* finite *fulfilling* LIS $\widehat{\mathbf{L}} = \langle\langle \widehat{\mathbb{D}}, \mathbb{I}(\widehat{\mathbb{D}})^{-}\rangle, \widehat{\mathcal{L}}\rangle$ *that satisfies*

*$\varphi$ such that, for every $\widehat{d}_i \in \widehat{D}$, $\mathrm{REQ}^{\hat{\mathbf{L}}}(\widehat{d}_i)$ occurs at most $m = 2fp + f + p$ times in $\widehat{D}$.*

*Proof.* Let $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})^-\rangle, \mathcal{L}\rangle$ be a finite fulfilling LIS that satisfies $\varphi$. If for every $d_j \in D$, $\mathrm{REQ}^{\mathbf{L}}(d_j)$ occurs at most $m$ times in $D$, we are done. If this is not the case, we show how to build a fulfilling LIS with the requested property by progressively removing exceeding points from $D$.

Let $\mathbf{L}_0 = \mathbf{L}$ and let $\mathcal{R}_0 = \{\mathrm{REQ}_1, \mathrm{REQ}_2, \ldots, \mathrm{REQ}_k\}$ be the (arbitrarily ordered) finite set of all and only the sets of requests that occur more than $m$ times in $D$. $\mathbf{L}_0$ can be turned into a fulfilling LIS $\mathbf{L}_1 = \langle\langle\mathbb{D}_1, \mathbb{I}(\mathbb{D}_1)^-\rangle, \mathcal{L}_1\rangle$ satisfying $\varphi$, which contains exactly $m$ points $d \in D_1$ such that $\mathrm{REQ}^{\mathbf{L}_1}(d) = \mathrm{REQ}_1$ as follows. Since $\mathrm{REQ}_1$ occurs more that $m$ times in $D$, there exists a point $d_e \in D$ such that $\mathrm{REQ}^{\mathbf{L}_0}(d_e) = \mathrm{REQ}_1$ and there exist at least $fp + p$ distinct points $d < d_e$ such that $\mathrm{REQ}^{\mathbf{L}_0}(d) = \mathrm{REQ}^{\mathbf{L}_0}(d_e)$ and at least $fp + f$ distinct points $d > d_e$ such that $\mathrm{REQ}^{\mathbf{L}_0}(d) = \mathrm{REQ}^{\mathbf{L}_0}(d_e)$. Hence, by Lemma 1, there exists a fulfilling LIS $\mathbf{L}' \in \mathbf{L}_{-d_e}$. We repeat the application of Lemma 1 until we get a fulfilling LIS $\mathbf{L}_1$ such that $\mathrm{REQ}_1$ occurs exactly $m$ times in $D_1$. It remains to show that $\mathbf{L}_1$ satisfies $\varphi$. Since $\mathbf{L}_0$ satisfies $\varphi$, we have that there exists an interval $[d_i, d_j]$ such that $\varphi \in \mathcal{L}_0([d_i, d_j])$. By definition of $\mathrm{CL}(\varphi)$, $\langle A\rangle\varphi \in \mathrm{CL}(\varphi)$, hence $\langle A\rangle\varphi \in \mathrm{REQ}^{\mathbf{L}_0}(d_i)$. In $\mathbf{L}_1$ two cases are possible: either $d_i \in D_1$ or it does not. If $d_i \in D_1$, then $\langle A\rangle\varphi \in \mathrm{REQ}^{\mathbf{L}_1}(d_i)$ and, being $\mathbf{L}_1$ fulfilling, there exists an interval $[d_i, d_k]$ such that $\varphi \in \mathcal{L}_1([d_i, d_k])$. If $d_i \notin D_1$, then it has been deleted at some stage of the construction of $\mathbf{L}_1$. This implies that $\mathrm{REQ}^{\mathbf{L}_1}(d_i) = \mathrm{REQ}_1$ and thus there exist $m$ points $d$ in $\mathbf{L}_1$ such that $\mathrm{REQ}^{\mathbf{L}_1}(d) = \mathrm{REQ}^{\mathbf{L}_1}(d_i)$. Since $\mathbf{L}_1$ is fulfilling, there exists an interval $[d, d']$ such that $\varphi \in \mathcal{L}_1([d, d'])$. In both cases $\mathbf{L}_1$ satisfies $\varphi$.

By iterating such a transformation $k-1$ times, we can turn $\mathbf{L}_1$ into a fulfilling LIS devoid of exceeding points that satisfies $\varphi$. □

Let us consider now the case of infinite (fulfilling) LISs. We start with a classification of points belonging the domain of the structure.

**Definition 9.** *Given an infinite* LIS $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})^-\rangle, \mathcal{L}\rangle$, *we partition the points in $D$ into the following sets:*

- *$Fin(\mathbf{L})$ is the set of all points $d \in D$ such that $\mathrm{REQ}^{\mathbf{L}}(d)$ occurs finitely many times in $D$;*
- *$Inf_l(\mathbf{L})$ is the set of all points $d \in D$ such that $\mathrm{REQ}^{\mathbf{L}}(d)$ occurs infinitely many times in $D$, but there exists a point $d_{max}$ such that, for all $d' > d_{max}$, $\mathrm{REQ}^{\mathbf{L}}(d') \neq \mathrm{REQ}^{\mathbf{L}}(d)$;*
- *$Inf_r(\mathbf{L})$ is the set of all points $d \in D$ such that $\mathrm{REQ}^{\mathbf{L}}(d)$ occurs infinitely many times in $D$, but there exists a point $d_{min}$ such that, for all $d' < d_{min}$, $\mathrm{REQ}^{\mathbf{L}}(d') \neq \mathrm{REQ}^{\mathbf{L}}(d)$;*
- *$Inf(\mathbf{L})$ is the set of all points $d \in D$ such that $\mathrm{REQ}^{\mathbf{L}}(d)$ occurs infinitely many times in $D$ and, for every point $d'$, there exists $d'' < d'$ such that $\mathrm{REQ}^{\mathbf{L}}(d'') = \mathrm{REQ}^{\mathbf{L}}(d)$ and there exists $d''' > d'$ such that $\mathrm{REQ}^{\mathbf{L}}(d''') = \mathrm{REQ}^{\mathbf{L}}(d)$.*

The following definition captures a particular subclass of infinite LISs that enjoy a finite representation.

**Definition 10.** *An infinite* LIS $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})^-\rangle, \mathcal{L}\rangle$ *is* ultimately periodic*, with left period* $l$*, infix* $i$ *and right period* $r$*, if and only if there exists* $d_0 \in D$ *such that for all* $k < 0$, $\mathrm{REQ}^{\mathbf{L}}(d_k) = \mathrm{REQ}^{\mathbf{L}}(d_{k-l})$ *and for all* $k \geq 0$, $\mathrm{REQ}^{\mathbf{L}}(d_{i+k}) = \mathrm{REQ}^{\mathbf{L}}(d_{i+k+r})$.

The following theorem proves that if there exists an infinite fulfilling LIS that satisfies $\varphi$, then there exists also an ultimately periodic fulfilling LIS that satisfies it. Furthermore, it provides a bound to the left period, infix, and right period of such a fulfilling LIS which closely resembles the one that we established for finite ones.

**Theorem 3.** *Let* $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})^-\rangle, \mathcal{L}\rangle$ *be an* infinite *fulfilling* LIS *that satisfies* $\varphi$*,* $f$ *be the number of* $\langle A\rangle$*-formulae in* $\mathrm{TF}(\varphi)$*, and* $p$ *be the number of* $\langle\overline{A}\rangle$*-formulae in* $\mathrm{TF}(\varphi)$*. Then, there exists an ultimately periodic fulfilling* LIS $\widehat{\mathbf{L}} = \langle\langle\widehat{\mathbb{D}}, \mathbb{I}(\widehat{\mathbb{D}})^-\rangle, \widehat{\mathcal{L}}\rangle$*, with left period* $l$*, infix* $i$ *and right period* $r$*, such that*

1. *for every* $d_j \in Fin(\widehat{\mathbf{L}})$*,* $\mathrm{REQ}^{\hat{\mathbf{L}}}(d_j)$ *occurs at most* $m = 2fp + f + p$ *times in* $D$*;*
2. *for every* $d_j \in Inf_r(\widehat{\mathbf{L}})$*,* $\mathrm{REQ}^{\hat{\mathbf{L}}}(d_j)$ *occurs exactly* $fp + p$ *times in* $I$*, where* $I$ *is the set of points in the infix part of* $\widehat{\mathbf{L}}$*;*
3. *for every* $d_j \in Inf_l(\widehat{\mathbf{L}})$*,* $\mathrm{REQ}^{\hat{\mathbf{L}}}(d_j)$ *occurs exactly* $fp + f$ *times in* $I$*;*
4. *for all points* $d_j \in Inf(\widehat{\mathbf{L}})$*,* $d_j \notin I$*;*
5. $r \leq |\,\mathrm{REQ}_\varphi\,|$ *and* $l \leq |\,\mathrm{REQ}_\varphi\,|$*;*
6. *for every* $d_j \in Fin(\mathbf{L})$ *and every formula* $\langle A\rangle\psi \in \mathrm{REQ}^{\hat{\mathbf{L}}}_f(d_j)$*, there exists a point* $d_h \leq d_{i+(f\cdot p+f)\cdot r}$ *such that* $\psi \in \widehat{\mathcal{L}}([d_j, d_h])$*;*
7. *for every* $d_j \in Fin(\mathbf{L})$ *and every formula* $\langle\overline{A}\rangle\psi \in \mathrm{REQ}^{\hat{\mathbf{L}}}(d_j)$*, there exists a point* $d_h \geq d_{-(f\cdot p+p)\cdot l}$ *such that* $\psi \in \widehat{\mathcal{L}}([d_h, d_j])$

*that satisfies* $\varphi$*.*

*Proof.* Let $\varphi$ be a satisfiable formula and let $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})^-\rangle, \mathcal{L}\rangle$ be an infinite fulfilling LIS that satisfies $\varphi$. By exploiting Lemma 1, we briefly show how to build a fulfilling LIS $\widehat{\mathbf{L}}$ which respects Conditions 1–7.

1. Let $d_0$ be the smallest point in $Fin(\mathbf{L}) \cup Inf_r(\mathbf{L})$ and $d_{i-1}$ be the greatest point in $Fin(\mathbf{L}) \cup Inf_l(\mathbf{L})$. The set $I = \{d_0, \ldots, d_{i-1}\}$ will be the infix of $\widehat{\mathbf{L}}$. By repeatedly applying Lemma 1 we can remove from the infix all points $d \in Fin(\mathbf{L})$ such that $\mathrm{REQ}^{\mathbf{L}}(d)$ occurs more that $m$ times in $D$.
2. Suppose that there exists a point $d_j \in Inf_r(\mathbf{L})$ such that $\mathrm{REQ}^{\mathbf{L}}(d_j)$ does not occurs $fp + p$ times in $I$: two cases may arise. If $\mathrm{REQ}^{\mathbf{L}}(d_j)$ occurs more than $fp + p$ times in $I$, we can exploit Lemma 1 to remove the exceeding occurrences of $\mathrm{REQ}^{\mathbf{L}}(d_j)$. If $\mathrm{REQ}^{\mathbf{L}}(d_j)$ occurs less than $fp+p$ times in $I$, let

$d_k > d_{i-1}$ be the point such that $\text{REQ}^{\mathbf{L}}(d_k) = \text{REQ}^{\mathbf{L}}(d_j)$ and $\text{REQ}^{\mathbf{L}}(d_k)$ occurs *exactly* $fp+p$ times in $\{d_0, \ldots, d_{i-1}, \ldots, d_k\}$. $I = \{d_0, \ldots, d_{i-1}, \ldots, d_k\}$ becomes the new infix of the ultimately periodic LIS. We repeat such a procedure until $\text{REQ}^{\mathbf{L}}(d)$ occurs exactly $fp + p$ times in $I$, for all $d \in Inf_r(\mathbf{L})$.

3. Suppose that there exists a point $d_j \in Inf_l(\mathbf{L})$ such that $\text{REQ}^{\mathbf{L}}(d_j)$ does not occurs $fp + f$ times in $I$. We proceed as in the previous case, either by removing the exceeding occurrences of $\text{REQ}^{\mathbf{L}}(d_j)$ or by extending the infix to the left if $\text{REQ}^{\mathbf{L}}(d_j)$ occurs less than $fp + f$ times in $I$.

4. Suppose now that there exists a point $d_j \in Inf(\mathbf{L})$ such that $d_j \in I$. By the definition of $Inf(\mathbf{L})$, there are infinitely many points $d < d_j$ such that $\text{REQ}^{\mathbf{L}}(d) = \text{REQ}^{\mathbf{L}}(d_j)$ and infinitely many points $d > d_j$ such that $\text{REQ}^{\mathbf{L}}(d) = \text{REQ}^{\mathbf{L}}(d_j)$. Hence, by exploiting Lemma 1, we can obtain a fulfilling LIS satisfying $\varphi$ where $d_j$ is removed.

5. Let $I = \{d_0, \ldots, d_{i-1}\}$ be the infix of $\mathbf{L}$ and suppose that it respects Conditions 1–4. To turn $\mathbf{L}$ into an ultimately periodic LIS respecting Condition 5, we must show how to define the right and left period. Consider the set $\mathcal{R} = \{\text{REQ}^{\mathbf{L}}(d) : d \in Inf(\mathbf{L}) \cup Inf_r(\mathbf{L})\}$ and let $\mathcal{R} = \{\text{REQ}_0, \ldots, \text{REQ}_{r-1}\}$ be an arbitrary enumeration of it. The cardinality $r$ of $\mathcal{R}$ will be the right period of $\widehat{\mathbf{L}}$. We inductively define $\widehat{\mathbf{L}}$ in such a way that, for all $k \geq 0$, $\text{REQ}^{\widehat{\mathbf{L}}}(\widehat{d}_{i+k}) = \text{REQ}_{k \,\texttt{mod}\, r}$. Let $k = 0$, and consider $\text{REQ}^{\widehat{\mathbf{L}}}(d_i)$: if $\text{REQ}^{\mathbf{L}}(d_i) = \text{REQ}_0$, we are done. Otherwise, let $d_h > d_i$ be the first occurrence of $\text{REQ}_0$ after $d_i$. Since $\mathbf{L}$ respects Conditions 1–4, we have that, for every point $d_i \leq d' < d_h$, there exist sufficiently many points $d'' < d_i$ such that $\text{REQ}^{\mathbf{L}}(d'') = \text{REQ}^{\mathbf{L}}(d')$. Hence, by Lemma 1, there exists a LIS $\mathbf{L}_0$ where all points $d_i \leq d' < d_h$ have been removed. Thus, $\mathbf{L}_0$ is such that $\text{REQ}^{\mathbf{L}_0}(d_i) = \text{REQ}^{\mathbf{L}}(d_h) = \text{REQ}_0$. Now, let $k > 0$ and suppose that $\mathbf{L}_{k-1} = \langle\langle \mathbb{D}_{k-1}, \mathbb{I}(\mathbb{D}_{k-1})\rangle, \mathcal{L}_{k-1}\rangle$ respect the condition for all $h < k$. We can proceed as in the case of $k = 0$ and define a LIS $\mathbf{L}_k = \langle\langle \mathbb{D}_k, \mathbb{I}(\mathbb{D}_k)\rangle, \mathcal{L}_k\rangle$ such that $\text{REQ}^{\mathbf{L}_k}(d_{i+k}) = \text{REQ}_{k \,\texttt{mod}\, r}$.
   The left period of $\widehat{\mathbf{L}}$ can be defined in an analogous way starting from an arbitrary enumeration of the set $\mathcal{L} = \{\text{REQ}^{\mathbf{L}}(d) : d \in Inf(\mathbf{L}) \cup Inf_l(\mathbf{L})\}$.

6. Suppose that $\mathbf{L}$ respects Conditions 1-5. Let $d_j \in Fin(\mathbf{L})$ and $\langle A\rangle\psi \in \text{REQ}^{\mathbf{L}}(d_j)$ be a formula that is fulfilled only by intervals $[d_j, d_h]$ such that $d_h > d_{i+f(p+1)r}$. Since $\mathbf{L}$ respects Condition 2, for every point $d'$ such that $d_i \leq d' < d_h$ we have that there exist at least $fp + p$ points $d'' < d_i$ with $\text{REQ}^{\mathbf{L}}(d'') = \text{REQ}^{\mathbf{L}}(d')$. Hence, we can exploit Lemma 1 to remove points between $d_i$ and $d_h$, thus building a fulfilling LIS $\widehat{\mathbf{L}}$ that satisfies $\varphi$ and such that $\langle A\rangle\psi \in \text{REQ}^{\mathbf{L}}(d_j)$ is fulfilled by an interval $[d_j, d_h]$ with $d_h \leq d_{i+(fp+f)r}$.

7. To build a fulfilling LIS $\widehat{\mathbf{L}}$ that respects Condition 7, we suppose that $\mathbf{L}$ respects Conditions 1-5, and we proceed with a removal procedure analogous to the one for the previous case. □

## 4 A tableau-based decision procedure for PNL

In this section we define a tableau method for PNL over the integers (or a subset of them). We begin with some basic definitions.

Given a formula $\varphi$, let $m = 2fp + f + p$, where $f$ (resp. $p$) is the number of $\langle A\rangle$-formulae (resp. $\langle\overline{A}\rangle$-formulae) in $\mathrm{CL}(\varphi)$. A tableau for PNL is a special *decorated tree* $\mathcal{T}$. For each node $n$ in a branch $B$, the *decoration* $\nu(n)$ is a tuple $\langle[d_i, d_j], A_n, \mathrm{REQ}_n, \mathbb{D}_n, x\rangle$, where:

- $[d_i, d_j] \in \mathbb{I}(\mathbb{D}_n)^-$;
- $\mathrm{REQ}_n : D_n \mapsto \mathrm{REQ}_\varphi$ is a *request function*;
- $\mathbb{D}_n = \langle D_n, <\rangle$ is a finite linear order;
- $A_n \in \mathcal{A}_\varphi$ is such that: *(i)* for all $[A]\psi \in \mathrm{REQ}_n(d_i)$, $\psi \in A_n$, *(ii)* for all $[\overline{A}]\psi \in \mathrm{REQ}_n(d_j)$, $\psi \in A_n$, *(iii)* for all $\psi \in A_n$, if $\psi = \langle\overline{A}\rangle\xi$ or $\psi = [\overline{A}]\xi$, then $\psi \in \mathrm{REQ}_n(d_i)$, and *(iv)* for all $\psi \in A_n$, if $\psi = \langle A\rangle\xi$ or $\psi = [A]\xi$, then $\psi \in \mathrm{REQ}_n(d_j)$;
- $x \in \{R, L, F\}$, where $R$, $L$, and $F$ respectively stand for right blocked, left blocked, and free.

The root $r$ of the tree is decorated by the *empty decoration* $\langle\emptyset, \emptyset, \emptyset, \emptyset, F\rangle$.

Given a node $n \in B$, decorated with $\langle[d_i, d_j], A_n, \mathrm{REQ}_n, \mathbb{D}_n, x\rangle$, and a future existential formula $\langle A\rangle\psi \in A_n$, we say that $\langle A\rangle\psi \in A_n$ is *fulfilled on* $B$ if and only if there exists a node $n' \in B$ such that $\nu(n') = \langle[d_j, d_k], A_{n'}, \mathrm{REQ}_{n'}, \mathbb{D}_{n'}, x\rangle$ and $\psi \in A_{n'}$. Conversely, we say that a past existential formula $\langle\overline{A}\rangle\psi \in A_n$ is *fulfilled on* $B$ if and only if there exists a node $n' \in B$ such that $\nu(n') = \langle[d_k, d_i], A_{n'}, \mathrm{REQ}_{n'}, \mathbb{D}_{n'}, x\rangle$ and $\psi \in A_{n'}$. A node $n$ is said to be *active on* $B$ if and only if $A_n$ contains at least one (future or past) existential formula which is not fulfilled on $B$.

**Expansion rules.** Let $B$ a branch of a decorated tree $\mathcal{T}$. We denote by $\mathbb{D}_B$ and $\mathrm{REQ}_B$ the linear order and the request function of the decoration of the last node in $B$, respectively. Moreover, let $d_l$ and $d_r$ be the minimum and maximum element of $\mathbb{D}_B$, respectively. The *expansion rules* for $B$ are:

1. *Right step rule*: if there exists an active node $n \in B$, with $\nu(n) = \langle[d_i, d_j], A_n, \mathrm{REQ}_n, \mathbb{D}_n, x\rangle$ and a non-fulfilled future existential formula in $A_n$, then extend $D_B$ to $D' = D_B \cup \{d_{r+1}\}$, with $d_{r+1} > d_r$. Then, take an atom $A'$ such that $A_n\ LR_\varphi\ A'$ and extend $\mathrm{REQ}_B$ to $\mathrm{REQ}' : D' \mapsto \mathrm{REQ}_\varphi$ in such a way that for all $[\overline{A}]\psi \in \mathrm{REQ}'(d_{r+1})$, $\psi \in A'$ and for all $\psi \in A'$, if $\psi = \langle A\rangle\xi$ or $\psi = [A]\xi$, then $\psi \in \mathrm{REQ}'(d_{r+1})$. Finally, add an immediate successor $n'$ to the last node in $B$ decorated as follows:
   - if the number $p$ of points $d \in D'$ with $\mathrm{REQ}'(d) = \mathrm{REQ}'(d_{r+1})$ is less than or equal to $m$, then $\nu(n') = \langle[d_j, d_{r+1}], A', \mathrm{REQ}', \mathbb{D}', F\rangle$;
   - otherwise ($p = m + 1$), $\nu(n') = \langle[d_j, d_{r+1}], A', \mathrm{REQ}', \mathbb{D}', R\rangle$.
2. *Left step rule*: if there exists an active node $n \in B$, with $\nu(n) = \langle[d_i, d_j], A_n, \mathrm{REQ}_n, \mathbb{D}_n, x\rangle$ and a non-fulfilled past existential formula in $A_n$, then extend

$D_B$ to $D' = D_B \cup \{d_{l-1}\}$, with $d_{l-1} < d_l$. Then, take an atom $A'$ such that $A'\ LR_\varphi\ A_n$ and extend $\mathrm{REQ}_B$ to $\mathrm{REQ}' : D' \mapsto \mathrm{REQ}_\varphi$ in such a way that for all $[A]\psi \in \mathrm{REQ}'(d_{l-1})$, $\psi \in A'$ and for all $\psi \in A'$, if $\psi = \langle \overline{A} \rangle \xi$ or $\psi = [\overline{A}]\xi$, then $\psi \in \mathrm{REQ}'(d_{l-1})$. Finally, add an immediate successor $n'$ to the last node in $B$ decorated as follows:

  - if the number $p$ of points $d \in D'$ with $\mathrm{REQ}'(d) = \mathrm{REQ}'(d_{l-1})$ is less than or equal to $m$, then $\nu(n') = \langle [d_{l-1}, d_i], A', \mathrm{REQ}', \mathbb{D}', F \rangle$;
  - otherwise ($p = m + 1$), $\nu(n') = \langle [d_{l-1}, d_i], A', \mathrm{REQ}', \mathbb{D}', L \rangle$.

3. *Fill-in rule*: if there exist two points $d_i < d_j$ such that there are no nodes in $B$ decorated with the interval $[d_i, d_j]$ and there exists a decoration $\langle [d_i, d_j], A', \mathrm{REQ}_B, \mathbb{D}_B, F \rangle$, then expand $B$ by adding an immediate successor $n'$, with such a decoration, to the last node in $B$.

All rules expand the branch $B$ with a new node. However, while the left and right step rules add a new point $d$ to $D_B$ and decorate the new node with a new interval beginning or ending at $d$, the fill-in rule decorates it with a new interval whose endpoints already belong to $D_B$.

**Expansion strategy.** Given a decorated tree $\mathcal{T}$ and a branch $B$, we say that $B$ is *right-blocked* if there exists a node $n$ decorated with $\langle [d_i, d_j], A_n, \mathrm{REQ}_n, \mathbb{D}_n, R \rangle$, while it is *left-blocked* if there exists a node $n$ decorated with $\langle [d_i, d_j], A_n, \mathrm{REQ}_n, \mathbb{D}_n, L \rangle$. A branch is *blocked* if it is both left and right blocked.

An expansion rule is *applicable on B* if $B$ is non-blocked and the application of the rule generates a new node. The *branch expansion strategy* for a branch $B$ is the following one:

1. if the fill-in rule is applicable, apply the fill-in rule to $B$ and, for every possible choice for the decoration, add an immediate successor to the last node in $B$;
2. if the fill-in rule is not applicable and there exist two points $d_i < d_j \in D_B$ such that there are no nodes in $B$ decorated with $[d_i, d_j]$, *close* the branch;
3. if $B$ is not right-blocked and the right-step rule is applicable, then apply it to $B$ and, for every possible choice for the decoration, add an immediate successor to the last node in $B$;
4. if $B$ is not left-blocked and the left-step rule is applicable, then apply it to $B$ and, for every possible choice for the decoration, add an immediate successor to the last node in $B$.

**Tableau.** Let $\varphi$ be the formula to be checked for satisfiability and let $\langle [d_0, d_1], A_1, \mathrm{REQ}_1, \{d_0, d_1\}, F \rangle, \ldots, \langle [d_0, d_1], A_k, \mathrm{REQ}_k, \{d_0, d_1\}, F \rangle$ be the set of decorations with $\langle A \rangle \varphi \in \mathrm{REQ}_i(d_0)$. The *initial tableau* for $\varphi$ consists of the root, with the empty decoration, and $k$ immediate successors $n_1, \ldots n_k$. For each $1 \leq i \leq k$, $n_i$ is decorated by $\langle [d_0, d_1], A_i, \mathrm{REQ}_i, \{d_0, d_1\}, F \rangle$. A *tableau* for $\varphi$ is any decorated tree $\mathcal{T}$ obtained by expanding the initial tableau for $\varphi$ through successive applications of the branch-expansion strategy to existing branches, until the branch-expansion strategy cannot be applied anymore.

**Fulfilling branches.** Given a branch $B$ of a tableau $\mathcal{T}$ for $\varphi$, we say that $B$ is a *fulfilling branch* if and only if $B$ is not closed and one of the following conditions holds:

1. $B$ does not contain active nodes (finite model case);
2. $B$ is right blocked and there exists at least one formula $\langle A\rangle\psi$ not fulfilled in $B$ (right unbounded model case). Moreover, let $d_r$ be the greatest point in $D_B$. By the blocking condition, $\mathrm{REQ}_B(d_r)$ is repeated $m+1$ times in $D_B$. Let $d_k$ be the greatest point in $D_B$, with $d_k < d_r$, such that $\mathrm{REQ}_B(d_k) = \mathrm{REQ}_B(d_r)$. The set $\{d_{k+1}, \ldots, d_r\}$, called *fulfilling right period*, satisfies the following conditions:
   (a) for all $d_i, d_j \in \{d_{k+1}, \ldots, d_r\}$, there exists an atom $A_{ij}$ such that *(i)* for all $[A]\psi \in \mathrm{REQ}_B(d_i)$, $\psi \in A_{ij}$, and *(ii)* for all $[\overline{A}]\psi \in \mathrm{REQ}_B(d_j)$, $\psi \in A_{ij}$;
   (b) for all $d_i \in \{d_{k+1}, \ldots, d_r\}$ and $\langle A\rangle\psi \in \mathrm{REQ}_B(d_i)$, there exist a point $d_j \in \{d_{k+1}, \ldots, d_r\}$ and an atom $A_{ij}$ such that *(i)* $\psi \in A_{ij}$, *(ii)* for all $[A]\xi \in \mathrm{REQ}_B(d_i)$, $\xi \in A_{ij}$, and *(iii)* for all $[\overline{A}]\xi \in \mathrm{REQ}_B(d_j)$, $\xi \in A_{ij}$;
   (c) for all $d_i \leq d_k$ such that $\mathrm{REQ}_B(d_i)$ does not occur in the right period, all $\langle A\rangle$-formulae in $\mathrm{REQ}_B(d_i)$ are fulfilled in $B$.
3. $B$ is left blocked and there exists at least one formula $\langle\overline{A}\rangle\psi$ not fulfilled in $B$ (left unbounded model case). Moreover, let $d_l$ be the smallest point in $D_B$. By the blocking condition, $\mathrm{REQ}_B(d_l)$ is repeated $m+1$ times in $D_B$. Let $d_k$ be the smallest point in $D_B$, with $d_k > d_l$, such that $\mathrm{REQ}_B(d_k) = \mathrm{REQ}_B(d_l)$. The set $\{d_l, \ldots, d_{k-1}\}$, called *fulfilling left period*, satisfies the following conditions:
   (a) for all $d_i, d_j \in \{d_l, \ldots, d_{k-1}\}$, there exists an atom $A_{ij}$ such that *(i)* for all $[A]\psi \in \mathrm{REQ}_B(d_i)$, $\psi \in A_{ij}$, and *(ii)* for all $[\overline{A}]\psi \in \mathrm{REQ}_B(d_j)$, $\psi \in A_{ij}$;
   (b) for all $d_i \in \{d_l, \ldots, d_{k-1}\}$ and $\langle\overline{A}\rangle\psi \in \mathrm{REQ}_B(d_i)$, there exists a point $d_j \in \{d_l, \ldots, d_{k-1}\}$ and an atom $A_{ji}$ such that *(i)* $\psi \in A_{ji}$, *(ii)* for all $[A]\xi \in \mathrm{REQ}_B(d_j)$, $\xi \in A_{ji}$, and *(iii)* for all $[\overline{A}]\xi \in \mathrm{REQ}_B(d_i)$, $\xi \in A_{ji}$;
   (c) for all $d_i \geq d_k$ such that $\mathrm{REQ}_B(d_i)$ does not occur in the left period, all $\langle\overline{A}\rangle$-formulae in $\mathrm{REQ}_B(d_i)$ are fulfilled in $B$.
4. if $B$ is both right and left blocked, Conditions 2. and 3. must hold.

The decision procedure works as follows: given a formula $\varphi$, it constructs a tableau $\mathcal{T}$ for $\varphi$ and it returns "satisfiable" if and only if there exists at least one fulfilling branch in $\mathcal{T}$.

### 4.1 Soundness and completeness

Soundness and completeness of the proposed method can be proved as follows. Soundness is proved by showing how to construct a fulfilling LIS satisfying $\varphi$ from a fulfilling branch $B$ in a tableau $\mathcal{T}$ for $\varphi$ (by Theorem 1, it follows that $\varphi$

has a model). The proof must encompass both the case of non-blocked branches (finite case) and of blocked ones (infinite case). Proving completeness consists in showing that for any satisfiable formula $\varphi$, there exists a fulfilling branch $B$ in any tableau $\mathcal{T}$ for $\varphi$. Given a model for $\varphi$ and the corresponding fulfilling LIS $\mathbf{L}$, we prove the existence of a fulfilling branch in $\mathcal{T}$ by exploiting Theorems 2 and 3.

**Theorem 4.** *Given a formula $\varphi$ and a tableau $\mathcal{T}$ for $\varphi$, if there exists a fulfilling branch in $\mathcal{T}$, then $\varphi$ is satisfiable.*

*Proof.* Let $\mathcal{T}$ be a tableau for $\varphi$ and $B$ a fulfilling branch in $\mathcal{T}$. We show that, starting from $B$, we can build up a fulfilling LIS $\mathbf{L}$ satisfying $\varphi$. We first consider the LIS $\mathbf{L}_B = \langle\langle\mathbb{D}_B, \mathbb{I}(\mathbb{D}_B)^-\rangle, \mathcal{L}_B\rangle$, where $\mathcal{L}_B$ is such that, for every $[d_i, d_j] \in \mathbb{I}(\mathbb{D}_B)^-$, $\mathcal{L}_B([d_i, d_j]) = A_n$, with $n$ being the unique node in $B$ decorated with $\langle[d_i, d_j], A_n, \mathrm{REQ}_n, \mathbb{D}_n, x\rangle$. Given the expansion rules of the tableau, we have that $\mathbf{L}_B$ is a LIS, but it is not necessarily fulfilling. Four cases may arise.

$B$ does not contains active nodes (*finite model case*). In this case, all $\langle A\rangle$ and $\langle\overline{A}\rangle$-formulae that occur in $B$ are fulfilled in $B$ and thus in $\mathbf{L}_B$. By the definition of initial tableau we have that $\langle A\rangle\varphi \in \mathrm{REQ}_B(d_0)$. Hence, $\varphi$ is satisfied in $\mathbf{L}_B$.

$B$ is right blocked and it contains at least one non-fulfilled $\langle A\rangle$ formula, while all $\langle\overline{A}\rangle$-formulae are fulfilled in $B$ (*right-unbounded model case*). In this case, we extend $\mathbf{L}_B$ to a right unbounded LIS $\mathbf{L}'$ where all $\langle A\rangle$-formulae are fulfilled. Let $d_r$ be the greatest point of $D_B$ and $d_k$ be the greatest point in $D_B$ such that $d_k < d_r$ and $\mathrm{REQ}_B(d_k) = \mathrm{REQ}_B(d_r)$. We extend $D_B$ to $D'$ by putting an infinite sequence of point $d_{r+1}, d_{r+2}, \ldots$ after $d_r$ and we build the right-periodic LIS $\mathbf{L}' = \langle\langle\mathbb{D}', \mathbb{I}(\mathbb{D}')^-\rangle, \mathcal{L}'\rangle$ as follows:

- for all intervals $[d, d'] \in \mathbb{I}(\mathbb{D}_B)^-$, $\mathcal{L}'([d, d']) = \mathcal{L}_B([d, d'])$;
- for all points $d_{r+h} > d_r$, we put $\mathrm{REQ}^{\mathbf{L}'}(d_{r+h}) = \mathrm{REQ}^{\mathbf{L}_B}(d_{k+(h \,\mathtt{mod}(r-k))})$;
- for every point $d_{r+h} > d_r$, we fulfill the $\langle\overline{A}\rangle$-formulae in $\mathrm{REQ}^{\mathbf{L}'}(d_{r+h})$ as follows. First, for every $d_i < d_k$ such that $\mathrm{REQ}^{\mathbf{L}'}(d_i)$ does not occur in the period, we put $\mathcal{L}'([d_i, d_{r+h}]) = \mathcal{L}_B([d_i, d_{k+(h \,\mathtt{mod}(r-k))}])$. Then, for every formula $\langle\overline{A}\rangle\psi \in \mathrm{REQ}^{\mathbf{L}'}(d_{r+h})$ which has not been fulfilled yet, we consider the point $d_{k+(h \,\mathtt{mod}(r-k))}$. Since in $B$ all $\langle\overline{A}\rangle$-formulae are fulfilled, there exists an interval $[d, d_{k+(h \,\mathtt{mod}(r-k))}]$ such that $\psi \in \mathcal{L}_B([d, d_{k+(h \,\mathtt{mod}(r-k))}])$. Two may cases arise. Either $\mathrm{REQ}^{\mathbf{L}'}(d)$ occurs in the period or it does not. If $\mathrm{REQ}^{\mathbf{L}'}(d)$ does not occur in the period, then $\mathcal{L}'([d, d_{r+h}]) = \mathcal{L}_B([d, d_{k+(h \,\mathtt{mod}(r-k))}])$ and $\langle\overline{A}\rangle\psi \in \mathrm{REQ}^{\mathbf{L}'}(d_{r+h})$ is already fulfilled. If $\mathrm{REQ}^{\mathbf{L}'}(d)$ occurs in the period, we take the greatest point $d' < d_{r+h}$ such that $\mathrm{REQ}^{\mathbf{L}'}(d') = \mathrm{REQ}^{\mathbf{L}'}(d)$ and the labeling of the interval $[d', d_{r+h}]$ has not been defined yet, and we put $\mathcal{L}'([d', d_{r+h}]) = \mathcal{L}_B([d, d_{k+(h \,\mathtt{mod}(r-k))}])$. By making such a choice for $d'$, we guarantee that there always exist infinitely many points $d'' > d$ with the same set of requests of $d_{r+h}$ such that the labeling of $[d, d'']$ is still undefined;

- for every point $d \in D'$ and every $\langle A\rangle\psi \in \mathrm{REQ}^{\mathbf{L}'}(d)$ which has not been fulfilled yet, proceed as follows. By Condition 2.(c) of the definition of fulfilling branch, there exists a point $d_{k+1} \leq d_i \leq d_r$ such that $\mathrm{REQ}_B(d) = \mathrm{REQ}_B(d_i)$. Hence, by Condition 2.(b), there exist a point $d_j \in \{d_{k+1}, \ldots, d_r\}$ and an atom $A_{ij}$ such that $\psi \in A_{ij}$, for all $[A]\xi \in \mathrm{REQ}_B(d_i)$, $\xi \in A_{ij}$, and for all $[\overline{A}]\xi \in \mathrm{REQ}_B(d_j)$, $\xi \in A_{ij}$. By the definition of $\mathbf{L}'$, we have that there exist infinitely many points $d_n \geq d_r$ in $D'$ such that $\mathrm{REQ}^{\mathbf{L}'}(d_n) = \mathrm{REQ}_B(d_j)$. We can take one of such points $d_n$ such that $\mathcal{L}([d, d_n])$ has not been defined yet and put $\mathcal{L}([d, d_n]) = A_{ij}$;
- once we have fulfilled all diamond formulae in $\mathrm{REQ}^{\mathbf{L}'}(d)$, for all $d \in D'$, we define the labeling of the remaining intervals $[d, d']$, where $d' > d_r$. Since $B$ is fulfilling, we can always define $\mathcal{L}'([d, d'])$ by exploiting Condition 2.(b) for fulfilling branches.

$B$ is left blocked and it contains at least one non-fulfilled $\langle\overline{A}\rangle$ formula, while all $\langle A\rangle$-formulae are fulfilled in $B$ (*left-unbounded model case*). In this case we proceed as in the right-unbounded model case to extend $\mathbf{L}_B$ to a left unbounded LIS where all $\langle A\rangle$-formulae are fulfilled.

$B$ is left and right blocked and it contains both $\langle A\rangle$ and $\langle\overline{A}\rangle$ non-fulfilled formulae (*unbounded model case*). We apply the construction for the right unbounded model case and that for the left unbounded model case to build an unbounded LIS where all diamond formulae are fulfilled. □

**Theorem 5.** *Given a satisfiable formula $\varphi$, there exists a fulfilling branch in every tableau $\mathcal{T}$ for $\varphi$.*

*Proof.* Let $\varphi$ be a satisfiable formula and let $\mathbf{L} = \langle\langle\mathbb{D}, \mathbb{I}(\mathbb{D})^-\rangle, \mathcal{L}\rangle$ be a fulfilling LIS satisfying $\varphi$, whose existence is guaranteed by Theorem 1. Without loss of generality, we may assume that $\mathbf{L}$ respects the constraints of Theorem 2, if it is finite, and of Theorem 3, if it is infinite. Furthermore, we assume that $\langle A\rangle\varphi \in \mathrm{REQ}^{\mathbf{L}}(d_0)$. Given a linear order $\mathbb{D}' \subseteq \mathbb{D}$, we denote with $\mathrm{REQ}^{\mathbf{L}}\mid_{\mathbb{D}'}$ the restriction of $\mathrm{REQ}^{\mathbf{L}}$ to the intervals in $\mathbb{I}(\mathbb{D}')^-$. We prove that there exists a fulfilling branch $B$ in $\mathcal{T}$ which corresponds to $\mathbf{L}$. To this end, we prove the following property: *there exists a non-closed branch $B$ such that, for every node $n \in B$, if $n$ is decorated with $\langle[d_j, d_k], A_n, \mathrm{REQ}_n, \mathbb{D}_n, x\rangle$, then $A_n = \mathcal{L}([d_j, d_k])$ and* $\mathrm{REQ}_n = \mathrm{REQ}^{\mathbf{L}}\mid_{\mathbb{D}_n}$. The proof is by induction on the height $h(\mathcal{T})$ of $\mathcal{T}$.

If $h(\mathcal{T}) = 1$, then $\mathcal{T}$ is the initial tableau for $\varphi$ and, by construction, it contains a branch $B_0 = \langle\emptyset, \emptyset\rangle \cdot \langle[d_0, d_1], A_n, \mathrm{REQ}_n, \mathbb{D}_n, F\rangle$, with $A_n = \mathcal{L}([d_0, d_1])$ and $\mathrm{REQ}_n = \mathrm{REQ}^{\mathbf{L}}\mid_{\{0,1\}}$.

Let $h(\mathcal{T}) = i + 1$. By the inductive hypothesis, there exists a branch $B_i$ of length $i$ that satisfies the property. Let $D_{B_i} = \{d_{-h}, \ldots, d_0, d_1, \ldots, d_k\}$. We distinguish three cases, depending on the expansion rule that has been applied to $B_i$ in the construction of $\mathcal{T}$.

- **The right-step rule has been applied.**
  Let $n$ be the active node, decorated with $\langle[d_j, d_l], A_n, \mathrm{REQ}_n, \mathbb{D}_n, x\rangle$, which

the right-step rule has been applied to. By the inductive hypothesis, $A_n = \mathcal{L}([d_j, d_l])$ and $\mathrm{REQ}_n = \mathrm{REQ}^{\mathbf{L}}|_{\mathbb{D}_n}$. Let $D' = \{d_{-h}, \ldots, d_{k+1}\}$. Since $\mathbf{L}$ is a LIS, $\mathcal{L}([d_j, d_l])\ LR_\varphi\ \mathcal{L}([d_l, d_{k+1}])$ and $\mathrm{REQ}^{\mathbf{L}}|_{\mathbb{D}'}$ is a possible extension of $\mathrm{REQ}_n$. Hence, there must exist in $\mathcal{T}$ a successor $n'$ of the last node of $B_i$ decorated with $\langle [d_l, d_{k+1}], \mathcal{L}([d_l, d_{k+1}]), \mathrm{REQ}^{\mathbf{L}}|_{\mathbb{D}'}, \mathbb{D}', x\rangle$. Let $B_{i+1} = B_i \cdot n'$. Since the step rule can be been applied only to non-closed branches (and it does not close any branch), $B_{i+1}$ is non-closed.

- **The left-step rule has been applied.**
  Let $n$ be the active node, decorated with $\langle [d_j, d_l], A_n, \mathrm{REQ}_n, \mathbb{D}_n, x\rangle$, which the left-step rule has been applied to. By proceeding as in the case of the right-step rule, we can extend $B_i$ to a non closed branch $B_{i+1}$ that respects the property.
- **The fill-in rule has been applied.**
  Let $d_j < d_l$ be the points in $D_{B_i}$ such that there are no nodes in $B_i$ decorated with $[d_j, d_l]$. By the inductive hypothesis (and by the definition of LIS), we have that $\langle [d_i, d_j], \mathcal{L}([d_j, d_l]), \mathrm{REQ}_{B_i}, \mathbb{D}_{b_i}, F\rangle$ is a possible decoration. Hence, there must exist in $\mathcal{T}$ a successor $n'$ of the last node of $B_i$ decorated with $\langle [d_i, d_j], \mathcal{L}([d_j, d_l]), \mathrm{REQ}_{B_i}, \mathbb{D}_{b_i}, F\rangle$. Let $B_{i+1} = B_i \cdot n'$. As before, since the fill-in rule can be applied only to non-closed branches (and it does not close any branch), $B_{i+1}$ is not closed.

Now we show that $B$ is the fulfilling branch we are searching for. Since $B$ is not closed, one of the following cases may arise.

- *$B$ is non-blocked and the expansion strategy cannot be applied anymore.* Since $B$ is not closed, this means that there exist no active nodes in $B$, that is, for every node $n \in B$ and every formula $\langle A\rangle\psi \in A_n$ (resp. $\langle \overline{A}\rangle\psi \in A_n$), there exists a node $n'$ fulfilling it. Hence, $B$ is a fulfilling branch.
- *$B$ is right-blocked.* This implies that $\mathrm{REQ}_B(d_k)$ is repeated $m+1$ times in $B$. Since $B$ is decorated coherently to $\mathbf{L}$, by Theorem 2, we can assume $\mathbf{L}$ to be infinite. Let $d_j < d_k$ be the greatest point in $D_B$ such that $\mathrm{REQ}_B(d_j) = \mathrm{REQ}_B(d_k)$. We have that $\mathbf{L}$ is ultimately periodic, with right prefix $r = k-j$, since (by Theorem 3) the only set of requests which has been repeated $m+1$ times in $B$ is the one associated with the first point in the right period. Furthermore, we have that there are exactly $fp + f$ repetitions of the right period in $B$. This allows us to exploit the structural properties of $\mathbf{L}$ to prove that $B$ is fulfilling.
  For every pair of points $d, d' \in \{d_j, \ldots, d_k\}$, we have that $d, d' \in Inf(\mathbf{L})$. Hence, there exist infinitely many points $d''$ in $\mathbf{L}$ such that $\mathrm{REQ}_B(d'') = \mathrm{REQ}_B(d')$ and $d < d''$. Let $d''$ be one of such points. We can choose the atom $A'' = \mathcal{L}([d, d''])$ to satisfy Condition 2.(a) of the definition of fulfilling branch.
  For every point $d \in \{d_j, \ldots, d_k\}$ and for every formula $\langle A\rangle\psi \in \mathrm{REQ}_B(d)$, since $\mathbf{L}$ is fulfilling, there exists a point $d''$ in $D$ such that $\psi \in \mathcal{L}([d', d''])$. If $d'' \leq d_k$, then $\langle A\rangle\psi$ is fulfilled in $B$. Otherwise, there exists a point $d_m$, with $d_i \leq d_m \leq d_k$, such that $\mathrm{REQ}_B(d'') = \mathrm{REQ}_B(d_m)$. Hence, the atom

$A' = \mathcal{L}([d', d''])$ can be chosen in order to satisfy Condition 2.(b) of the definition of fulfilling branch.
For every point $d \in D$ such that $\mathrm{REQ}_B(d_i)$ does not occur in the period, we have that $d \in Fin(\mathbf{L})$. Hence, by Theorem 3, we have that every formula $\langle A\rangle\psi \in \mathrm{REQ}_B(d)$ is fulfilled by an interval $[d, d']$ such that $d' \leq d_{i+(fp+f)r}$. Since $d_k$ corresponds to the first point of the $(fp + f + 1)$-th occurrence of the right period, we have that $d' < d_k$ and hence $\langle A\rangle\psi$ is fulfilled in $B$. This shows that $B$ respects Condition 2.(c) of the definition of fulfilling branch.

– The cases when $B$ is left-blocked, or both right and left-blocked can be proved as the case when $B$ is a right-blocked branch. □

### 4.2 Optimality of the proposed method

In this section we provide a precise characterization of the computational complexity of the satisfiability problem for PNL.

As for the computational complexity of the proposed decision procedure, observe that, by the blocking condition, after at most $|\mathrm{REQ}_\varphi| \cdot m+1$ applications of the step rules, the expansion strategy cannot be applied anymore to a branch. Moreover, given a branch $B$, between two successive applications of the step rules, the fill-in rule can be applied at most $k$ times, being $k$ the number of points in $D_B$ (as a matter of fact, $k$ is exactly the number of applications of the step rules up to that point). Since $m = 2fp + p \leq 2 \cdot |\mathrm{TF}(\varphi)|^2 + |\mathrm{TF}(\varphi)|$, we have that $m$ is polynomial in the length of $\varphi$, while $|\mathrm{REQ}_\varphi|$ is exponential in it. If $|\varphi| = n$, the length of any branch $B$ of a tableau $\mathcal{T}$ for $\varphi$ is bounded by $\left(|\mathrm{REQ}_\varphi| \cdot (2 \cdot |\mathrm{TF}(\varphi)|^2 + |\mathrm{TF}(\varphi)|)\right)^2 = 2^{O(n)}$, that is, the length of a branch is exponential in $|\varphi|$. This implies that the satisfiability problem for PNL can be solved by a (nondeterministic) algorithm that guesses a fulfilling branch $B$ for the formula $\varphi$ in nondeterministic exponential time.

To give a NEXPTIME lower bound to the complexity of the satisfiability problem for PNL we can exploit the computational complexity results for the future-only fragment of PNL [2]. NEXPTIME-hardness of RPNL is proved by reducing the exponential tiling problem to the satisfiability problem for RPNL. Since RPNL is a fragment of PNL, the reduction presented in [2] proves NEXPTIME-hardness of PNL as well.

**Theorem 6.** *The satisfiability problem for* RPNL *is NEXPTIME-complete.*

## 5 Conclusions

In this paper, we focussed our attention on interval logics of temporal neighborhood. We addressed the satisfiability problem for Propositional Neighborhood Logic (PNL), interpreted over the integers (or a subset of them), and we showed that it is NEXPTIME-complete. Moreover, we developed a sound and complete tableau-based decision procedure for PNL and we proved its optimality. As for possible extensions of the method, we are working on its generalization to the whole class of linear orders as well as to other specific classes of temporal structures, such as dense ones.

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