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Metric Propositional Neighborhood Interval Logics on Natural Numbers

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Abstract Interval temporal logics formalize reasoning about interval structures over linearly (or partially) ordered domains, where time intervals are the primitive ontological entities and truth of formulae is defined relative to time intervals, rather than time points. In this paper, we introduce and study Metric Propositional Neighborhood Logic (MPNL) over natural numbers. MPNL features two modalities referring, respectively, to an interval that is "met by" the current one and to an interval that "meets" the current one, plus an infinite set of length constraints, regarded as atomic propositions, to constrain the lengths of intervals. We argue that MPNL can be successfully used in different areas of artificial intelligence to combine qualitative and quantitative interval temporal reasoning, thus providing a viable alternative to well-established logical frameworks such as Duration Calculus. We show that MPNL is decidable in double exponential time and expressively complete with respect to a well-defined sub-fragment of the two-variable fragment $FO^2[\mathbb{N},=,<,s]$ of first-order logic for linear orders with successor function, interpreted over natural numbers. Moreover, we show that MPNL can be extended in a natural way to cover full $FO^2[\mathbb{N},=,<,s]$, but, unexpectedly, the latter (and hence the former) turns out to be undecidable.

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1 Introduction

Interval temporal logics provide a natural framework for temporal reasoning about interval structures over linearly (or partially) ordered domains. They take time intervals as the primitive ontological entities and define truth of formulae relative to time intervals, rather than time points. Interval logics feature modal operators that correspond to various relations between pairs of intervals. In particular, the well-known logic HS, introduced by Halpern and Shoham in [21], features a set of modal operators that makes it possible to express all Allen's interval relations [1].

Interval-based formalisms have been extensively used in various areas of AI, such as, for instance, planning and plan validation, theories of action and change, natural language processing, and constraint satisfaction problems. However, most of them are subjected to severe syntactic and semantic restrictions that considerably weaken their expressive power. Interval temporal logics relax these restrictions, thus allowing one to cope with much more complex application domains and scenarios. Unfortunately, many of them, including HS and the majority of its fragments, turn out to be undecidable (a comprehensive survey can be found in [6]). One of the few cases of decidable interval logic with truly interval semantics, i.e., not reducible to point-based semantics, is Propositional Neighborhood Logic (PNL), interpreted over various classes of interval structures (all, dense, and discrete linear orders, integers, natural numbers) [19]. PNL is a fragment of HS with only two modalities, corresponding to Allen's relations *meets* and *met by*. Basic logical properties of PNL (representation theorems, axiomatic systems) have been investigated by Goranko et al. in [19]. The satisfiability problem for PNL has been addressed by Bresolin et al. in [9]. NEXPTIME-completeness with respect to the classes of all linearly ordered domains, well-ordered domains, finite linearly ordered domains, and natural numbers has been proved via a reduction to the satisfiability problem for the two-variable fragment of first-order logic for binary relational structures over ordered domains [28]. Finally, in [13], a tableau system for the right-neighborhood fragment of PNL, interpreted over the natural numbers, has been developed; such a system has been later extended to full PNL over the integers [11].

Various metric extensions to point-based temporal logics have been proposed in the literature. They include Timed Propositional Temporal Logic (TPTL), introduced by Alur and Henzinger in [2], Montanari et al.'s two-sorted metric temporal logics [25,**?**], Hirshfeld and Rabinovich's Quantitative Monadic Logic of Order [23], and Owakine and Worrell's Metric Temporal Logic [29], which refines and extends Koymans' Metric Temporal Logic [**?**]. Little work in that respect has been done in the interval logic setting. Among the few contributions, we mention the extension of Allen's Interval Algebra with a notion of distance developed by Kautz and Ladkin in [24]. The most important quantitative interval temporal logic is Duration Calculus (DC) [22,34], an interval logic for real-time systems originally developed by Zhou Chaochen, C.A.R. Hoare, and A.P. Ravn [15], based on Moszkowski's ITL [27], which is quite expressive, but generally undecidable. A number of variants and fragments of DC have been proposed to model and to reason about real-time processes and systems [4,14,16,34]. Many of them recover decidability by imposing semantic restrictions, such as the *locality* principle, that essentially reduce the interval system to a point-based one.

In this paper, we present a family of non-conservative metric extensions of PNL, which allow one to express *metric properties* of interval structures over natural numbers. We mainly focus our attention on the most expressive language in this class, called *Metric* PNL (MPNL, for short). MPNL features a family of special atomic propositions representing integer constraints (equalities and inequalities) on the length of the intervals over which they are evaluated. MPNL is particularly suitable for quantitative interval reasoning, and thus it emerges as a viable alternative to existing logical systems for quantitative temporal reasoning. The right-neighborhood fragment of MPNL has been introduced and studied in [10]; full MPNL has been considered [7]–the main precursor of this paper, which extends and strengthens it substantially. The main contributions of the paper are:

(i) the proposal of a number of extensions of PNL with metric modalities and with interval length constraints, which turn out to be very expressive and natural to reason about interval structures on natural numbers;
(ii) expressive completeness of MPNL with respect to a proper fragment $FO^2_r[\mathbb{N},=,<,s]$ of the two-variable fragment $FO^2[\mathbb{N},=,<,s]$ of FO with equality, order, successor, and any family of binary relations, interpreted on natural numbers. We also show how to extend MPNL to obtain an interval logic $MPNL^+$ which is expressively complete with respect to full $FO^2[\mathbb{N},=,<,s]$;
(iii) decidability and complexity of the satisfiability problem for MPNL, and undecidability of the satisfiability problem for $MPNL^+$, and thus for $FO^2[\mathbb{N},=,<,s]$;
(iv) analysis and classification of the expressive power of the proposed metric extensions of PNL.

The results in this paper can be compared with analogous results for PNL and $FO^2[=,<]$ (the two-variable fragment of FO with equality on linear orders with a family of uninterpreted binary relations), given in [8,9]. Unlike $FO^2[=,<]$, which was already known to be decidable [28], the decidability of $FO^2_r[\mathbb{N},=,<,s]$ is a consequence of the decidability and expressive completeness results for MPNL. At the best of our knowledge, this result is new and of independent interest.

The paper is organized as follows. In Section 2, we recall some basic features of PNL, and in Section 3 we present the metric language MPNL. In Section 4, we illustrate various possible applications of MPNL. Next, in Section 5, we prove the decidability of the logic. Expressive completeness results are given in Section 6. Finally, in Section 7, we classify various fragments of MPNL with respect to their expressive power. In the conclusions, we provide an assessment of the work and we mention open problems.

2 Propositional Neighborhood Interval Logics: PNL

The language of the propositional neighborhood logics PNL consists of a set $\mathcal{AP}$ of atomic propositions, the propositional connectives $\neg, \vee$, and the modal operators $\Diamond_r$ and $\Diamond_l$, corresponding to the Allen's relation *meets* and its inverse *met-by* [1]. The other propositional connectives, as well as the logical constants $\top$ (*true*) and $\bot$ (*false*), and the dual modal operators $\Box_r$ and $\Box_l$, are defined as usual. Propositional neighborhood logics have been studied both in the so-called *strict semantics*, which excludes point-intervals, and in the *non-strict* one, which includes them. In the latter case, it is natural to include in the language a special atomic proposition (modal constant), usually denoted by π, to identify point-intervals (PNL^π). The differences in the expressive power in the various cases have been studied in [19]. In this paper, we focus on the non-strict semantics. The *formulae*, denoted by $\varphi, \psi, \ldots$, are generated by the following grammar:

$$\varphi ::= \pi \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond_r\varphi \mid \Diamond_l\varphi.$$

Given a linearly ordered domain $\mathbb{D} = \langle D, < \rangle$, a (*non-strict*) *interval* over $\mathbb{D}$ is any ordered pair $[i, j]$ such that $i \leq j$. An *interval structure* is a pair $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$, where $\mathbb{I}(\mathbb{D})$ is the set of all intervals over $\mathbb{D}$. The semantics of PNL is given in terms of *models* of the form $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), V \rangle$, where $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ is an interval structure and $V : \mathcal{AP} \rightarrow 2^{\mathbb{I}(\mathbb{D})}$ is a valuation function assigning to every atomic proposition the set of intervals over which it holds. We recursively define the satisfiability relation $\Vdash$ as follows:

- ($M, [i, j] \Vdash \pi$ iff $i = j$;)
- $M, [i, j] \Vdash p$ iff $p \in V([i, j])$, for any $p \in \mathcal{AP}$;
- $M, [i, j] \Vdash \neg\varphi$ iff it is not the case that $M, [i, j] \Vdash \varphi$;
- $M, [i, j] \Vdash \varphi \vee \psi$ iff $M, [i, j] \Vdash \varphi$ or $M, [i, j] \Vdash \psi$;
- $M, [i, j] \Vdash \Diamond_r \varphi$ iff there exists $h \geq j$ such that $M, [j, h] \Vdash \varphi$;
- $M, [i, j] \Vdash \Diamond_l \varphi$ iff there exists $h \leq i$ such that $M, [h, i] \Vdash \varphi$.

A PNL-formula φ is *satisfiable* if there exists a model M and an interval $[b, e]$ such that $M, [b, e] \Vdash \varphi$.

The logics PNL and PNL$^\pi$ have been studied in [8, 19], where the decidability of their satisfiability problem has been shown. A tableau-based method for deciding satisfiability in the single-modality fragment of PNL, called RPNL, has been presented in [13], and subsequently extended to the full PNL/PNL$^\pi$ in [11]. In this paper, we focus our attention on the class of interval structures over the ordering of the natural numbers $\mathbb{N} = (\omega, <)$.

3 Metric PNL: MPNL

In this sectio, we introduce metric extensions of propositional neighborhood logics interpreted over $\mathbb{N}$. Depending on the choice of the metric operators, a hierarchy of languages can be built. In Section 7, we will study the relative expressive power of these languages. For the moment, we just consider the most expressive language of the hierarchy.

From now on, we denote by δ the *distance* function on $\mathbb{N}$: $\delta : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, defined as $\delta(i, j) = |i - j|$. The results presented here may be suitably rephrased for any function δ satisfying the standard properties of distance over a linear ordering. The most expressive metric extension of PNL is based on *atomic propositions for length constraints*. These are pre-interpreted propositional letters referring to the length of the current interval. Such propositions can be seen as the metric generalizations of the modal constant π. For each $\sim \in \{<, \leq, =, \geq, >\}$, we introduce the length constraint $\mathsf{len}_{\sim \mathsf{k}}$, with the following semantics:

$$M, [i, j] \Vdash \mathsf{len}_{\sim \mathsf{k}} \text{ iff } \delta(i, j) \sim k.$$

Note that the equality and inequality constraints are mutually definable:

$$M, [i, j] \Vdash \mathsf{len}_{=0} \Leftrightarrow M, [i, j] \Vdash \neg\mathsf{len}_{>0}$$
$$M, [i, j] \Vdash \mathsf{len}_{=\mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \mathsf{len}_{>\mathsf{k}-1} \wedge \neg\mathsf{len}_{>\mathsf{k}} \text{ if } k > 0$$
$$M, [i, j] \Vdash \mathsf{len}_{<\mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \mathsf{len}_{=0} \vee \ldots \vee \mathsf{len}_{=\mathsf{k}-1}$$
$$M, [i, j] \Vdash \mathsf{len}_{\leq \mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \mathsf{len}_{=0} \vee \ldots \vee \mathsf{len}_{=\mathsf{k}}$$
$$M, [i, j] \Vdash \mathsf{len}_{>\mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \neg\mathsf{len}_{\leq \mathsf{k}}$$
$$M, [i, j] \Vdash \mathsf{len}_{\geq \mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \neg\mathsf{len}_{<\mathsf{k}}$$

In Section 5 we will limit ourselves to constraints of type $\mathsf{len}_{=\mathsf{k}}$, without taking into account the increase in length of formulae due to the above translation.

4 MPNL at Work

Finding an optimal balance between good expressive power and reasonable computational complexity is a challenge for every knowledge representation and reasoning formalism. Interval temporal logics are not an exception in this respect. We believe that MPNL offers a good compromise between these two requirements. In the following, we show that MPNL makes it possible to encode (*metric versions* of) basic operators of point-based linear temporal logic (LTL) as well as interval modalities corresponding to Allen's relations; in addition, we show that it allows one to express limited forms of fuzziness.

First, MPNL is expressive enough to encode the strict *sometimes in the future* (resp., *sometimes in the past*) operator of LTL:

$$\Diamond_r(\mathsf{len}_{>0} \wedge \Diamond_r(\mathsf{len}_{=0} \wedge p)).$$

Moreover, length constraints allow one to define a metric version of the *until* (resp., *since*) operator. For instance, the condition: '*p is true at a point in the future at distance k from the current interval and, until that point, q is true (pointwise)*' can be expressed as follows:

$$\Diamond_r(\mathsf{len}_{=\mathsf{k}} \wedge \Diamond_r(\mathsf{len}_{=0} \wedge p)) \wedge \Box_r(\mathsf{len}_{<\mathsf{k}} \rightarrow \Diamond_r(\mathsf{len}_{=0} \wedge q)).$$

MPNL can also be used to constrain interval length and to express metric versions of basic interval relations. First, we can constrain the length of the intervals over which a given property holds to be at least (resp., at most, exactly) k. As an example, the following formula constrains p to hold only over intervals of length l, with $k \leq l \leq k'$:

$$[G](p \rightarrow \mathsf{len}_{\geq \mathsf{k}} \wedge \mathsf{len}_{\leq \mathsf{k}'}), \quad \text{(bl)}$$

where the *universal modality* $[G]$ (*for all intervals*) is defined as in [19]. By exploiting such a capability, a metric version of all, but one (the '*subinterval*' relation), Allen's relations can be expressed. As an example, we can state that: '*p holds* *only over intervals of length l, with $k \leq l \leq k'$, and any p-interval begins a q-interval*' as follows:

$$(\mathsf{bl}) \wedge [G] \bigwedge_{i=k}^{k'} (p \wedge \mathsf{len}_{=\mathsf{i}} \rightarrow \Diamond_l \Diamond_r (\mathsf{len}_{>\mathsf{i}} \wedge q)).$$

As another example, a metric version of Allen's relation *during* (the inverse of the 'subinterval' relation) can be expressed by pairing (bl) with:

$$[G] \bigwedge_{i=k}^{k'} (p \wedge \mathsf{len}_{=\mathsf{i}} \rightarrow \bigvee_{j \neq 0, j+j' < i} (\Diamond_l \Diamond_r (\mathsf{len}_{=\mathsf{j}} \wedge \Diamond_r (\mathsf{len}_{=\mathsf{j'}} \wedge q)))).$$

In [12] the problem of the relationship between a metric extension of PNL and the consistency problem for Allen's Interval Networks with quantitative constraints has been considered (and later generalized to its spatial version, Rectangle Algebra, in [30]). Intuitively, one can be interested in exploring the problem of deciding the consistency of a network of constraints using a logical language; although the satisfiability of an interval logic formula is, in general, a much harder problem, this complexity blowup can be paid off by the increase of expressivity, and the possibility of expressing negative constraints and disjunctive information (which is not possible by means of constraint networks). There are at least two different ways of expressing the fact that a constraint network is consistent through a logical formula. The first one consists, as in [30], of using the universal modality to simulate *nominals*, and, then, using them to force two objects (intervals, in this case) to be in a certain relation, even when such a relation is not directly expressible in the language. Notice that this is not in contradiction with the computability properties of the logic (e.g., PNL), since it allow us to express Allen's relation such as *begins* or *ends* only between a bounded number of objects (one for each nominal). If metric constraints are added to the language, one can also formalize a network with quantitative constraints [12]. As we have seen above, the addition of metric constraints allows us also to express a metric version of all, but one, Allen's relations. Such metric interval relations can thus be used as an alternative way to encode the consistency problem for Allen's Interval Networks with quantitative constraints, without introducing nominals. The drawback of this choice is that the constructs only work 'up to a certain length', and, thus, it is not completely general. However, since Allen's relations are expressed without using simulated nominals, we can express properties of an arbitrary (possibly infinite) number of intervals, both existentially and universally. Moreover, the formulae in this second case are much simpler and shorter.

MPNL also allows one to express some form of 'fuzziness'. As an example, the condition: '*p is true over the current interval and q is true over some interval close to it*', where by 'close' we mean that the right endpoint of the p-interval is at distance at most k from the left endpoint of the q-interval, can be expressed as follows:

$$p \wedge (\Diamond_r \Diamond_l (\mathsf{len}_{<\mathsf{k}} \wedge \Diamond_l \Diamond_r q) \vee \Diamond_r (\mathsf{len}_{<\mathsf{k}} \wedge \Diamond_r q)).$$

MPNL capabilities suffice to cope with various application domains. As a source of illustration, we show how to express some basic safety requirements of the classical *gas-burner example* (a formalization of such an example in DC can be found in [34]). Let the propositional letter *Gas* (resp., *Flame*, *Leak*) be used to state that gas is flowing (resp., burning, leaking), e.g., $M, [i, j] \Vdash Gas$ means that gas is flowing over the interval $[i, j]$. The formula

$$[G](Leak \leftrightarrow Gas \wedge \neg Flame)$$

states that *Leak* holds over an interval if and only if gas is flowing and not burning over that interval. The condition: '*it never happens that gas is leaking for more than k time units*' can be expressed as:

$$[G](\neg(\mathsf{len}_{>\mathsf{k}} \wedge Leak)).$$

Similarly, the condition: '*the gas burner will not leak uninterruptedly for k time units after the last leakage*' can be formalized as:

$$[G](Leak \rightarrow \neg \Diamond_l (\mathsf{len}_{<\mathsf{k}} \wedge \Diamond_l Leak)).$$

We conclude the section by mentioning two application domains where MPNL features are well-suited, namely, medical guidelines and ambient intelligence. In the former area (see [31]), events with duration, e.g., '*running a fever*', possibly paired with metric constraints, e.g., '*if a patient is running a fever for more than k time units, then administrate him/her drug D*', are quite common. In general, many relevant phenomena are inherently interval-based as there are no general rules to deduce their occurrence from point-based data. The use of temporal logic in ambient intelligence, specifically in the area of Smart Homes [3, 18], has been advocated by Combi et al. in [17]. MPNL can be successfully used to express safety requirements referring to situations that can be properly modeled only in terms of time intervals, e.g., '*being in the kitchen*'.

5 Decidability of MPNL

In this section, we use a model-theoretic argument to show that the satisfiability problem for MPNL has a bounded-model property with respect to finitely presentable ultimately periodic models, and it is therefore decidable. From now on, let φ be any MPNL-formula and let $\mathcal{AP}$ be the set of proposition letters of the language.

Definition 1 The *closure* of φ is the set $CL(\varphi)$ of all subformulae of $\Diamond_r\varphi$ and their negations (we identify $\neg\neg\psi$ with ψ). Let $\bigodot \in \{\Diamond_r, \Diamond_l, \Box_r, \Box_l\}$. The set of *temporal requests* from $CL(\varphi)$ is the set $TF(\varphi) = \{\bigodot\psi \mid \bigodot\psi \in CL(\varphi)\}$.

Definition 2 A *φ-atom* is a set $A \subseteq CL(\varphi)$ such that for every $\psi \in CL(\varphi)$, $\psi \in A$ iff $\neg\psi \notin A$ and for every $\psi_1 \vee \psi_2 \in CL(\varphi)$, $\psi_1 \vee \psi_2 \in A$ iff $\psi_1 \in A$ or $\psi_2 \in A$.

We denote the set of all φ-atoms by A_φ. One can easily prove that $|CL(\varphi)| \leq 2(|\varphi|+1)$, $|TF(\varphi)| \leq 2|\varphi|$, and $|A_\varphi| \leq 2^{|\varphi|+1}$. We now introduce a suitable labeling of interval structures based on φ-atoms.

Definition 3 A *(φ-)labeled interval structure* (LIS for short) is a structure $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$, where $\langle \mathbb{D}, \mathbb{I}(\mathbb{D})\rangle$ is the interval structure over natural numbers (or over a finite subset of it) and $\mathcal{L} : \mathbb{I}(\mathbb{D}) \to A_\varphi$ is a *labeling function* such that for every pair of neighboring intervals $[i, j], [j, h] \in \mathbb{I}(\mathbb{D})$, if $\Box_r\psi \in \mathcal{L}([i, j])$, then $\psi \in \mathcal{L}([j, h])$, and if $\Box_l\psi \in \mathcal{L}([j, h])$, then $\psi \in \mathcal{L}([i, j])$.

Notice that every interval model M induces a LIS, whose labeling function is the valuation function:

$$\psi \in \mathcal{L}([i, j]) \text{ iff } M, [i, j] \Vdash \psi.$$

Thus, LIS can be thought of as *quasi-models* for φ, in which the truth of formulae containing neither $\Diamond_r$, $\Diamond_l$ nor length constraints is determined by the labeling (due to the definitions of φ-atom and LIS). To obtain a model, we must also guarantee that the truth of the other formulae is in accordance with the labeling. To this end, we introduce the notion of fulfilling LIS.

Definition 4 A LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ is *fulfilling* iff:

- for every length constraint $\mathsf{len}_{=k} \in CL(\varphi)$ and interval $[i, j] \in \mathbb{I}(\mathbb{D})$, $\mathsf{len}_{=k} \in \mathcal{L}([i, j])$ iff $\delta(i, j) = k$;
- for every temporal formula $\Diamond_r\psi$ (resp., $\Diamond_l\psi$) in $TF(\varphi)$ and interval $[i, j] \in \mathbb{I}(\mathbb{D})$, if $\Diamond_r\psi$ (resp., $\Diamond_l\psi$) in $\mathcal{L}([i, j])$, then there exists $h \geq j$ (resp., $h \leq i$) such that $\psi \in \mathcal{L}([j, h])$ (resp., $\mathcal{L}([h, i])$).

Clearly, every interval model is a fulfilling LIS. Conversely, every fulfilling LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ can be transformed into a model $M(\mathbf{L})$ by defining the valuation in accordance with the labeling. Then, one can prove that for every $\psi \in CL(\varphi)$ and interval $[i, j] \in \mathbb{I}(\mathbb{D})$,

$$\psi \in \mathcal{L}([i, j]) \text{ iff } M(\mathbf{L}), [i, j] \models \psi$$

by a routine induction on ψ.

Definition 5 Given a LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ and a point $i \in D$, the set of *left* (resp., *right*) *temporal requests* at i, denoted by $REQ^L(i)$ (resp., $REQ^R(i)$), is the set of temporal formulae of the forms $\Diamond_l\varphi$, $\Box_l\varphi$ (resp., $\Diamond_r\varphi$, $\Box_r\varphi$) in $TF(\varphi)$ belonging to the labeling of any interval beginning in i (resp., ending in i). For any $j \in D$, we write $REQ(j)$ for $REQ^L(j) \cup REQ^R(j)$-

We denote by $REQ(\varphi)$ the set of all possible sets of temporal requests over $CL(\varphi)$. Let m be $\frac{|TF(\varphi)|}{2}$ and k be the maximum among the natural numbers occurring in the length constraints in φ. For example, if $\varphi = \Diamond_r(\mathsf{len}_{>3} \wedge p \to \Diamond_l(\mathsf{len}_{>5} \wedge q))$, then $m = 2$ and $k = 5$. It is easy to show that $|REQ(\varphi)| = 2^m$. Moreover, given any set of temporal requests $REQ^R(j)$ (resp., $REQ^L(i)$), it can be easily proved that all of them can be satisfied using at most m different points greater than j (resp., less than i).

Now, consider any MPNL-formula φ such that φ is satisfiable on a finite model. We have to show that we can restrict our attention to models with a bounded size.

Definition 6 Given any LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$, we say that a *$k$-sequence in* $\mathbf{L}$ is a sequence of k consecutive points in D. Given a k-sequence σ in $\mathbf{L}$, its *sequence of requests* $REQ(\sigma)$ is defined as the k-sequence of temporal requests at the points in σ. We say that $i \in \mathbf{L}$ *starts a k-sequence* σ if the temporal requests at $i, \ldots, i+k-1$ form an occurrence of $REQ(\sigma)$. Moreover, the sequence of requests $REQ(\sigma)$ is said to be *abundant* in $\mathbf{L}$ iff it has at least $2 \cdot (m^2+m) \cdot |REQ(\varphi)| + 1$ disjoint occurrences in D.

Lemma 1 *Let $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ be any LIS such that the sequence $REQ(\sigma)$ is abundant in it. Then, there exists an index q such that for each element $\mathcal{R} \in \{REQ(d) \mid i_q < d < i_{q+1}\}$, where i_q and i_{q+1} begin the q-th and the $q+1$-th occurrence of σ, respectively, $\mathcal{R}$ occurs at least m^2+m times before i_q and at least m^2+m times after $i_{q+1}+k-1$.*

Proof To prove this property, we proceed by contradiction. Suppose that $REQ(\sigma)$ is abundant, that is, it occurs $n > 2 \cdot (m^2+m) \cdot |REQ(\varphi)|$ times in D and, for each q with $1 \leq q \leq n$, there exists a point $d(q)$ with $i_q < d(q) < i_{q+1}$, such that $REQ(d(q))$ occurs less than (m^2+m) times before i_q or less than (m^2+m) times after $j_{q+1}+k-1$. Let $\Delta = \{d(q) | 1 \leq q \leq n\}$ the set of all such points. By hypothesis, there cannot be any $\mathcal{R} \in REQ(\varphi)$ such that $\mathcal{R}$ occurs more than $2 \cdot (m^2+m)$ times in Δ. Then $|\Delta| \leq 2 \cdot (m^2+m) \cdot |REQ(\varphi)|$, which is a contradiction. □

Lemma 2 *Let $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ be a fulfilling LIS that satisfies φ. Suppose that there exists an abundant k-sequence of requests $REQ(\sigma)$ and let q be the index whose existence is guaranteed by Lemma 1. Then, there exists a fulfilling LIS $\overline{\mathbf{L}} = \langle \overline{\mathbb{D}}, \mathbb{I}(\overline{\mathbb{D}}), \overline{\mathcal{L}}\rangle$ that satisfies φ such that $\overline{D} = D \setminus \{i_q, \ldots, i_{q+1}-1\}$.*

Proof Let us fix a fulfilling LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ satisfying φ at some $[i, j]$, an abundant k-sequence $REQ(\sigma)$ in $\mathbf{L}$, and the index q identified by Lemma 1. Now, let $D^- = \{i_q, \ldots, i_{q+1}-1\}$ and $D' = D \setminus D^-$ and, consequently, the set of all intervals $\mathbb{I}(\mathbb{D}')$. For sake of readability, the points in D' will be denoted by the same numbers as in D. Now, we have the problem of suitably re-defining the evaluation of all intervals on D' in a way preserving the temporal requests at all points in D' and still satisfying φ.

First, we consider all points $d < i_q$ and for each of them, for all p such that $0 \leq p \leq k-1$, we put $\mathcal{L}'([d, i_{q+1}+p]) = \mathcal{L}([d, i_q + p])$. In such a way, we guarantee that the intervals whose length has been shortened as an effect of the elimination of the points in D^- have a correct labeling in terms of all length constraints of the forms $\mathsf{len}_{=k'}$ and $\neg\mathsf{len}_{=k'}$. Moreover, since the requests (in both directions) in $\mathbf{L}$ at $i_{q+1}+p$ are equal to the requests at $i_q + p$, this operation is safe with respect to universal and existential requirements. Finally, since the lengths of intervals beginning before i_q and ending after $i_{q+1}+k-1$ are greater than k both in $\mathbf{L}$ and in $\mathbf{L}'$, there is no need to change their labeling.

The structure $\mathbf{L}' = \langle \mathbb{D}', \mathbb{I}(\mathbb{D}'), \mathcal{L}' \rangle$ defined so far is obviously a LIS, but it is not necessarily a fulfilling one. The removal of the points in the set D^- and the relabelling needed to guarantee correctness w.r.t. length constraints may generate *defects*, that is, situations in which there exist a point $d < i_q$ (resp., $d \geq i_{q+1}+k$) and a formula of the type $\Diamond_r \psi$ (resp., $\Diamond_l \psi$) belonging to $REQ(d)$, such that ψ was satisfied in $\mathbf{L}$ by some interval $[d, d']$ (resp., $[d', d]$), and it is not satisfied in $\mathbf{L}'$, either because $d' \in D^-$, or because the labelling of $[d, d']$ (resp., $[d', d]$) has changed due to the above relabeling. We have to show how to repair such defects. Suppose that there exists $d < i_q$ (the case when $d \geq i_{q+1}$ is similar) and some formula $\Diamond_r \psi \in REQ(d)$ that it is not satisfied anymore in $\mathbf{L}'$. Since $\mathbf{L}$ is a fulfilling LIS, then there exists an interval $[d, d']$ such that $\psi \in \mathcal{L}([d, d'])$ and either $d' \in D^-$ or $\psi \notin \mathcal{L}'([d, d'])$. Notice that, for this to be the case, $\delta(d', d) > k$ in $\mathbf{L}$. By Lemma 1, there are at least $n = m^2 + m$ points $\{\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_n\}$ after $h_{q+1}+k-1$ such that $REQ(\bar{d}_i) = REQ(d')$ for $i = 1, \ldots, n$. We will chose a point of the type $\bar{d}_i$ to satisfy the request. To prevent such a change making one or more requests in $REQ^L(\bar{d}_i)$ no longer satisfied, we have to preliminarily redefine the labeling $\mathcal{L}'$. First, we take a minimal set of points $P^d \subset D'$ such that, for each $\Diamond_l \tau \in REQ^L(d)$, there exists a point $e \in P^d$ such that $\tau \in \mathcal{L}([e, d])$. Now, for each point $e \in P^d$, let P^d_e be a minimal set of points such that, for every $\Diamond_r \xi \in REQ^R(e)$, there exists a point $f \in P^d_e$ such that $\xi \in \mathcal{L}([e, f])$. Finally, let $Q = \bigcup_{e \in P^d} P^d_e$: by the minimality requirements, we have that $|Q| \leq m^2$, since each set of requests can be satisfied using at most m points. Similarly, requests in $REQ^R(d)$ need at most m points to be satisfied. Consider the set $H = \{\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_n\} \setminus Q$: since, by construction, $|H| \geq m$, there must be some point $\bar{d}_h \in H$ such that in $\mathbf{L}$ the interval $[d, \bar{d}_h]$ satisfies only those $\Diamond_r$-formulae of $REQ(d)$, if any, that are satisfied at other intervals starting at d. Thus we can put $\mathbf{L}'([d, \bar{d}_h]) = \mathbf{L}([d, d'])$, and correct this defect without creating a new one. Since $\delta(\bar{d}_h, d) > k$ in $\mathbf{L}'$, this operation does not introduce inconsistencies with the length constraints in the labeling, either.

Now, if we repeat the above procedure sufficiently many times, we obtain a finite sequence of LISs, the last one of which is the required $\overline{\mathbf{L}}$. To conclude the proof, we have to show that $\overline{\mathbf{L}}$ is still a LIS for φ. Let $[d, d']$ be the interval of $\mathbf{L}$ satisfying the formula φ. Since $\Diamond_r \varphi \in CL(\varphi)$, we have that $\Diamond_r \varphi \in REQ(d)$. If d is still present in $\overline{\mathbf{L}}$, then, since the final LIS is fulfilling, we have that there must exists an interval $[d, d'']$ labelled with φ. If d is not a point of $\overline{\mathbf{L}}$, then Lemma 1 guarantees that there exists another point d'' in $\overline{\mathbf{L}}$ such that $REQ(d'') = REQ(d)$. Again, since $\overline{\mathbf{L}}$ is fulfilling, we have that there must exists an interval $[d'', d''']$ labelled with φ. □

The lemma above guarantees that we can eliminate sequences of requests that occur 'sufficiently many' times in a LIS, without 'spoiling' the LIS. This eventually allows us to prove the following small-model theorem for finite satisfiability of MPNL.

Theorem 1 (Small-Model Theorem) *If φ is any finitely satisfiable formula of MPNL, then there exists a fulfilling, finite LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ that satisfies φ such that* $|D| \leq |REQ(\varphi)|^k \cdot (2 \cdot (m^2+m) \cdot |REQ(\varphi)| + 1) \cdot k + k - 1$.

Proof Let $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ be any finite fulfilling LIS that satisfies φ. If $|D| \leq |REQ(\varphi)|^k (2(m^2+m)|REQ(\varphi)|+1)k + k - 1$, then we are done. Otherwise, by an application of the pigeonhole principle, for at least one sequence $REQ(\sigma)$ of length k, we have that $REQ(\sigma)$ is abundant. In this case, we apply Lemma 2 sufficiently many times to get the requested maximum length. □

To deal with formulae that are satisfiable only over infinite models, we need to provide these models with a finite (periodic) representation, and to bound the lengths of their prefix and period.

Definition 7 A LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ is *ultimately periodic*, with *prefix* L, *period* P, and *threshold* k if, for every interval $[i, j]$,

- if $i \geq L$, then $\mathcal{L}([i, j]) = \mathcal{L}([i+P, j+P])$;
- if $j \geq L$ and $\delta(j, i) > k$, then $\mathcal{L}([i, j]) = \mathcal{L}([i, j+P])$.

It is worth noticing that, in every ultimately periodic LIS, $REQ(i) = REQ(i+P)$, for $i \geq L$, and that every ultimately periodic LIS is finitely presentable: it suffices to define its labeling only on the intervals $[i, j]$ such that $j \leq L + P + max(k, P)$; thereafter, it can be uniquely extended by periodicity. Furthermore, we can identify a finite LIS with an ultimately periodic one with a period $P = 0$.

Lemma 3 *Let $\mathbf{L} = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), \mathcal{L} \rangle$ be an infinite fulfilling LIS over $\mathbb{N}$ that satisfies a formula φ. Then, there exists an infinite ultimately periodic fulfilling LIS $\overline{\mathbf{L}} = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), \overline{\mathcal{L}} \rangle$ over $\mathbb{N}$ that satisfies φ.*

Proof First of all, let $[b,e]$ be the interval satisfying φ in $\mathbf{L}$. We define the set $REQ_{inf}(\varphi)$ as the subset of $REQ(\varphi)$ containing all and only the sets of requests that occurs infinitely often in $\mathbf{L}$. Let $L \in \mathbb{N}$ be the first point in $\mathbf{L}$ such that the following conditions are met:

i) $L \geq e$;

ii) for each point $r \geq L$, $REQ(r) \in REQ_{inf}(\varphi)$;

iii) each set of requests $\mathcal{R} \in REQ_{inf}(\varphi)$, occurs at least m^2+m times before L, and at least m^2+m times between L and M;

iv) for each point $i < L$, and any formula $\Diamond_r \tau \in REQ(i)$, τ is satisfied on some interval $[i,j]$ where $j < M$; and,

v) the k-sequences of requests starting at L and at M are the same.

We put $P = M - L$. We will build an infinite ultimately periodic structure $\overline{\mathbf{L}}$ over the domain $\mathbb{N}$ with prefix L, period P and threshold k. To this end, first, for all points $d < M$ we put $\overline{REQ}(d) = REQ(d)$. Then, for all points $M+n$, where $0 \leq n < P$, we put $\overline{REQ}(M+n) = REQ(L+n)$ (by condition (v), this is already the case with $0 \leq n < k$). Now, we will define the labeling. For all intervals $[i,j]$ such that $j < M$ we put $\overline{\mathcal{L}}([i,j]) = \mathcal{L}([i,j])$. As for any interval $[i,j]$, with $M \leq j < M+P$, (a) if $i \geq M$, we put $\overline{\mathcal{L}}([i,j]) = \mathcal{L}([i-P, j-P])$, (b) if $i < M$, we must distinguish three cases: (b1) if $\delta(i,j) \leq k$, then we put $\overline{\mathcal{L}}([i,j]) = \mathcal{L}([i,j])$ (as $REQ(i)$ has not been modified and $\overline{REQ}(j) = REQ(j)$ by condition (v)); (b2) if $k < \delta(i,j) \leq k+P$, we put $\overline{\mathcal{L}}([i,j]) = \mathcal{L}([i,h])$ for some h such that $\overline{REQ}(j) = REQ(h)$ and $\delta(i,h) > k$, where the existence of such an h is guaranteed by condition (ii) (in fact, if $M \leq j < M+K$, we can take $h = j$); (b3) if $\delta(i,j) > k+P$, we put $\overline{\mathcal{L}}([i,j]) = \mathcal{L}([i,j-P])$. This construction labels all subintervals of $[0, M+P]$ in a way that is consistent with the definition of LIS, but that is not necessarily fulfilling. It could be the case that for some point $L \leq i \leq M$ and some formula $\Diamond_r \psi \in \overline{REQ(i)}$ there are no intervals satisfying ψ, because the only interval(s) satisfying it in $\mathbf{L}$ are of the type $[i,d]$ where $d > M+P$ and $\delta(d,i) > k$. We fix such *defects* as follows. Since $\overline{REQ(i)} = REQ(i)$, there exists a point $j > i$ such that $\psi \in \mathcal{L}([i,j])$ in the original model. By condition iii), there exists at least m^2+m points between M and $M+P$ with the same set of requests of j, and at least m^2+m points between L and M with the same set of requests of j. We proceed exactly as in the proof of Lemma 2, and we fix the defect choosing a point d' between M and $M+P$, putting $\overline{\mathcal{L}}([i,d']) = \mathcal{L}([i,d])$. By repeating such a procedure sufficiently many times going from left to right, we build a LIS where every request of every point $i \leq M$ is fulfilled before $M+P$. To conclude the construction we extend the so defined $\overline{\mathcal{L}}$ over $\mathbb{I}(\mathbb{N})$ in the unique way satisfying the conditions in Definition 7 for an ultimate periodic LIS with prefix L, period P, and threshold k, that is: for every $i > M+P$ we put $\overline{REQ(i)} = \overline{REQ(i-n\cdot P)}$ where n is the least non-negative integer such that $i - n\cdot P \leq M+P$; and, for every interval $[i,j]$ such that $j > M+P$, we put $\overline{\mathcal{L}}([i,j]) = \overline{\mathcal{L}}([i-n\cdot P, j-q\cdot P])$, where n and q are the least non-negative integers such that $i - nP \leq M$ and $j - qP \leq M+P$. It is straightforward to check that the labeling $\overline{\mathcal{L}}$ so defined respects all length constraints $\mathsf{len}_{=k'}$ and their negations for all intervals, and that the resulting structure $\overline{\mathbf{L}} = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), \overline{\mathcal{L}} \rangle$ is an ultimately periodic fulfilling LIS satisfying φ on $[b,e]$. □

Theorem 2 (Small Periodic Model Theorem) *If φ is any satisfiable formula of MPNL, then there exists a fulfilling, ultimately periodic LIS satisfying φ such that both the length L of the prefix and the length P of the period are less or equal to* $|REQ(\varphi)|^k \cdot (2\cdot(m^2+m)\cdot|REQ(\varphi)|+1)\cdot k + k - 1$.

Proof Existence of an ultimately periodic fulfilling LIS is guaranteed by Lemma 3. The bound on the prefix and of the period can be proved by exploiting Lemma 2. □

Corollary 1 *The satisfiability problem for MPNL, interpreted over $\mathbb{N}$, is decidable.*

The results of this section immediately give a double exponential time nondeterministic procedure for checking the satisfiability of any MPNL-formula φ. Such a procedure nondeterministically checks models whose size is in general $O(2^{|\varphi|^k})$, where $|\varphi|$ is the length of the formula to be checked for satisfiability. It has been shown in [10] that, in the case in which k is represented in binary, the right-neighborhood fragment of MPNL is complete for the class EXPSPACE. This means that, in the general case, the complexity for MPNL is located somewhere in between EXPSPACE and 2NEXPTIME (the exact complexity is still an open problem). It is worth noticing that, whenever k is a constant, it does not influence the complexity class and thus, since we have a NTIME($2^{|\varphi|}$) procedure for satisfiability and a NEXPTIME-hardness result [13], we can conclude that MPNL is NEXPTIME-complete. Similarly, when k is expressed in unary, the value of k increases linearly with the length of the formula and thus NTIME($2^{k|\varphi|}$)=NTIME($2^{|\varphi|^2}$); therefore, as in the previous case, MPNL is NEXPTIME-complete.

6 MPNL and Two-Variable Fragments of First Order Logic for $(\mathbb{N}, <, s)$

6.1 PNL and Two-Variable Fragments of First Order Logic

Here we will recall some results from [9] which will then be extended to MPNL. Let us denote by $\mathrm{FO}^2[=]$ the fragment of first-order logic with equality whose language contains only two distinct variables. We denote its formulae by $\alpha, \beta, \ldots$. For example, the formula $\forall x(P(x) \rightarrow \forall y \exists x Q(x,y))$ belongs to FO^2, and the formula $\forall x(P(x) \rightarrow \forall y \exists z(Q(z,y) \wedge$

$Q(z,x)))$ does not. We first focus our attention on the extension $FO^2[=,<]$ of $FO^2[=]$ over a purely relational vocabulary $\{=,<,P,Q,\ldots\}$ including equality and a distinguished binary relation $<$ interpreted as a linear ordering. Since atoms in the two-variable fragment can involve at most two distinct variables, we may further assume without loss of generality that the arity of every relation in the considered vocabulary is exactly 2. Let x and y be the two variables of the language. The formulae of $FO^2[=,<]$ can be defined recursively as follows:

$$\begin{aligned} \alpha &::= A_0 \mid A_1 \mid \neg\alpha \mid \alpha \vee \beta \mid \exists x\alpha \mid \exists y\alpha \\ A_0 &::= x = x \mid x = y \mid y = x \mid y = y \mid x < y \mid y < x \\ A_1 &::= P(x,x) \mid P(x,y) \mid P(y,x) \mid P(y,y), \end{aligned}$$

where A_1 deals with (uninterpreted) binary predicates. For technical convenience, we assume that both variables x and y occur as (possibly vacuous) free variables in every formula $\alpha \in FO^2[<]$, that is, $\alpha = \alpha(x,y)$. Formulas of $FO^2[=,<]$ are interpreted over *relational models* of the form $\mathbf{M} = \langle \mathbb{D}, V \rangle$, where $\mathbb{D} = \langle D, < \rangle$ is a linearly ordered set, and V is a *valuation function* that assigns to every binary relation P a subset of $D \times D$. When we evaluate a formula $\alpha(x,y)$ on a pair of elements a,b, we write $\alpha(a,b)$ for $\alpha[x := a, y := b]$.

The satisfiability problem for FO^2 without equality was proved decidable by Scott [32] by a satisfiability preserving reduction of any FO^2-formula to a formula of the form $\forall x \forall y \psi_0 \wedge \bigwedge_{i=1}^{m} \forall x \exists y \psi_i$, which belongs to the Gödel's prefix-defined decidable class of first-order formulae, shown by Gödel to have decidable satisfiability problem [5]. Later on, Mortimer extended this result by including equality in the language [26]. More recently, Grädel, Kolaitis, and Vardi improved Mortimer's result by lowering the complexity [20]. Finally, by building on techniques from [20] and taking advantage of an in-depth analysis of the basic 1-types and 2-types in $FO^2[=,<]$-models, Otto proved the decidability of $FO^2[=,<]$ over various classes of orderings, and in particular over the natural numbers. It has been shown in [8] that $FO^2[=,<]$ is expressively complete with respect to PNL^{π}. For the comparison of these logics suitable truth-preserving model transformations between interval models and relational models have been defined. We will sketch this transformations here, since they will be used to extend the result to expressive completeness of MPNL with respect to a suitable extension of $FO^2[=,<]$.

In order to define the mapping from interval models to relational models, we associate a binary relation P with every propositional variable $p \in \mathcal{AP}$ of the considered interval logic [33].

Definition 8 ([8]) Given an interval model $M = \langle \mathbb{I}(\mathbb{D}), V_M \rangle$, the corresponding relational model $\eta(M)$ is a pair of the type $\langle \mathbb{D}, V_{\eta(M)} \rangle$, where for all $p \in \mathcal{AP}$, $V_{\eta(M)}(P) = \{(i,j) \in D \times D : [i,j] \in V_M(p)\}$.

To define the mapping from relational models to interval ones, we have to solve a technical problem: the truth of formulae in interval models is evaluated only on ordered pairs $[i,j]$, with $i \leq j$, while in relational models there is no such constraint. To deal with this problem, we associate two propositional letters $p^{\leq}$ and $p^{\geq}$ of the interval logic with every binary relation P.

Definition 9 ([8]) Given a relational model $\mathbf{M} = \langle \mathbb{D}, V_{\mathbf{M}} \rangle$, the corresponding interval model $\zeta(\mathbf{M})$ is a pair $\langle \mathbb{I}(\mathbb{D}), V_{\zeta(\mathbf{M})} \rangle$ such that for any binary relation P and any interval $[i,j]$,we have that $[i,j] \in V_{\zeta(\mathbf{M}}(p^{\leq})$ iff $(i,j) \in V_M(P)$ and that $[i,j] \in V_{\zeta(\mathbf{M})}(p^{\geq})$ iff $(j,i) \in V_{\mathbf{M}}(P)$.

Definition 10 Given an interval logic L_I and a first-order logic L_{FO}, we say that L_{FO} is *at least as expressive as* L_I, denoted by $L_I \preceq L_{FO}$, if there exists an effective translation τ from L_I to L_{FO} such that for any interval model M, any interval $[a,b]$, and any formula φ of L_I, $M,[a,b] \Vdash \varphi$ iff $\eta(M) \models \tau(\varphi)(a,b)$. Conversely, we say that L_I is *at least as expressive as* L_{FO}, denote by $L_{FO} \preceq L_I$, if there exists an effective translation τ' from L_{FO} to L_I such that for any relational model M, any pair (i,j) of elements, and any formula φ of L_{FO}, $M \models \varphi(i,j)$ iff $\zeta(M),[i,j] \Vdash \tau'(\varphi)$ if $i \leq j$ or $\zeta(M),[j,i] \Vdash \tau'(\varphi)$ otherwise. We say that L_I is *as expressive as* L_{FO}, denoted by $L_I \equiv L_{FO}$, if $L_I \preceq L_{FO}$ and $L_{FO} \preceq L_I$. Then, $L_I \prec L_{FO}$ and $L_{FO} \prec L_I$ are defined as expected.

Theorem 3 ([8]) $PNL^{\pi} \equiv FO^2[=,<]$*, when interpreted over any class of linearly ordered sets.*

6.2 The Logic $FO^2[\mathbb{N},=,<,s]$

Here we consider the extension of $FO^2[=,<]$ interpreted over $\mathbb{N}$ with the successor function s, denoted by $FO^2[\mathbb{N},=,<,s]$. The terms of the language $FO^2[\mathbb{N},=,<,s]$ are of the type $s^k(z)$, where $z \in \{x,y\}$ and $s^k(z)$ denotes z when $k = 0$ and $\underbrace{s(s(\ldots s}_{k}(z)\ldots))$ when $k > 0$. Then, the formulae of $FO^2[\mathbb{N},=,<,s]$ can be defined as in the case of the logic $FO^2[=,<]$, mutatis mutandis. Using 2-pebble games and a standard model-theoretic argument, it is possible to prove that $FO^2[\mathbb{N},=,<,s]$ is strictly more expressive than $FO^2[=,<]$. That result, however, is also a direct consequence of the expressive completeness results established in [8] and in this paper.

Theorem 4 *The satisfiability problem for* $FO^2[\mathbb{N},=,<,s]$*, interpreted over any class of linearly ordered sets with at least one infinite ascending or descending sequence, is undecidable.*

Proof For the sake of simplicity, we assume that $FO^2[\mathbb{N},=,<,s]$ is interpreted over $\mathbb{N}$; nevertheless, the proof can be adapted to any class of linearly ordered sets with at least one infinite ascending or descending sequence. Let $\mathcal{O}=\{(i,j): i,j\in\mathbb{N}\wedge 0\leq i\leq j\}$ be the second octant of the integer plane $\mathbb{Z}\times\mathbb{Z}$. The *tiling problem for* $\mathcal{O}$ is the problem of establishing whether a given finite set of tile types $\mathcal{T}=\{t_1,\ldots,t_k\}$ can tile $\mathcal{O}$. For every tile type $t_i\in\mathcal{T}$, let $right(t_i)$, $left(t_i)$, $up(t_i)$, and $down(t_i)$ be the colors of the corresponding sides of t_i. To solve the problem, one must find a function $f:\mathcal{O}\rightarrow\mathcal{T}$ such that

$$right(f(n,m)) = left(f(n+1,m)), \text{ with } n<m, \text{ and}$$
$$up(f(n,m)) = down(f(n,m+1)).$$

Using König's lemma one can prove that a tiling system tiles $\mathcal{O}$ if and only if it tiles arbitrarily large squares if and only if it tiles $\mathbb{N}\times\mathbb{N}$ if and only if it tiles $\mathbb{Z}\times\mathbb{Z}$. The undecidability of the first of these tiling problems immediately follows from that of the last one [5]. The reduction from the tiling problem for $\mathcal{O}$ to the satisfiability problem for $FO^2[\mathbb{N},=,<,s]$ takes advantage of some special relational symbols, namely those in the set $Let=\{*,Tile,Id,\overline{Id_e},\overline{Id_b},\overline{Id_d},Corr,T_1,T_2,\ldots,T_k\}$. The reduction consists of three main steps: (i) the encoding of an infinite chain that will be used to represent the tiles, (ii) the encoding of the above-neighbor relation by means of the relation $Corr$, and (iii) the encoding of the right-neighbor relation, which will make use of the successor function. Pairs of successive points are used as cells to arrange the tiling. Next, we use the relation Id to represent a row of the octant. Any Id consists of a sequence of intervals, each one of which is used either to represent a part of the plane or to separate two Ids. In the former case, it is labeled with the relation $Tile$, while, in the latter case, it is labeled with the relation $*$. Consider now the following formulae:

$$\forall x,y \bigwedge_{P\in Let}(P(x,y)\leftrightarrow x<y) \quad (1)$$
$$\forall x,y(y=s(x)\leftrightarrow *(x,y)\vee Tile(x,y)) \quad (2)$$
$$\forall x,y(*(x,y)\rightarrow\neg Tile(x,y)) \quad (3)$$
$$y=s(x)\wedge *(x,y)\wedge\forall x\exists y(y=s(x)) \quad (4)$$
$$\exists x(x=s(y)\wedge Tile(y,x)\wedge *(s(y),s(x))) \quad (5)$$

The conjunction α_1 of the above formulae, guarantees that there exists an infinite sequence $x_0,x_1,\ldots,x_\omega$ of points. Moreover, α_1 guarantees that each pair x_i,x_{i+1} is labelled either by $*$ or by $Tile$, but not both. Finally, we have that $*(x_0,x_1)$, $Tile(x_1,x_2)$, and $*(x_2,x_3)$. Now, consider the conjunction α_2 of α_1 and the following formulae:

$$\exists y(y=s^2(x)\wedge Id(x,y)) \quad (6)$$
$$\forall x,y(Id(x,y)\rightarrow *(y,s(y))) \quad (7)$$
$$\forall x,y(Id(x,y)\rightarrow *(x,s(x))) \quad (8)$$
$$\forall x,y(*(x,y)\rightarrow\exists y(s(x)<y\wedge Id(x,y))) \quad (9)$$
$$\forall x,y(Id(x,y)\rightarrow\overline{Id_e}(s(x),y)) \quad (10)$$
$$\forall x,y(\overline{Id_e}(x,y)\wedge s(x)<y\rightarrow\overline{Id_e}(s(x),y)) \quad (11)$$
$$\forall x,y(Id(x,s(y))\rightarrow\overline{Id_b}(x,y)) \quad (12)$$
$$\forall x,y(\overline{Id_b}(x,s(y))\wedge x<y\rightarrow\overline{Id_b}(x,y)) \quad (13)$$
$$\forall x,y((\overline{Id_e}(x,s(y))\vee\overline{Id_d}(x,s(y)))\wedge x<y\rightarrow\overline{Id_d}(x,y)) \quad (14)$$

$$\forall x,y((\overline{Id_b}(x,y)\vee\overline{Id_e}(x,y)\vee\overline{Id_d}(x,y))\rightarrow\neg Id(x,y)) \quad (15)$$
$$\forall x,y \bigwedge_{\nu,\mu\in\{b,d,e\},\nu\neq\mu}(\overline{Id_\nu}(x,y)\rightarrow\neg\overline{Id_\mu}(x,y)). \quad (16)$$

The formula α_2 builds a chain of Id, in such a way that it holds $Id(x_0,x_3)$, each Id is followed by another Id, for each pair $x<y$ such that $Id(x,y)$ then $*(x,x+1)$, and if $Id(x,y)$ then $\neg Id(z,t)$, for all $x\leq z\leq t\leq y$ $((x,y)\neq(z,t))$. The relations of the type $\overline{Id_\nu}$ are used to ensure the last condition. For example, if $Id(x,y)$, then, for all $x<z<y$ we put $\overline{Id_b}(x,z)$, and similarly for $\overline{Id_e}$ and $\overline{Id_d}$; then, we impose that no pair of points is labeled by $\overline{Id_\nu}$ and $\overline{Id_\mu}$ at the same time, thus preventing two Id to be one inside, overlapping, starting, or ending the other. As a third step, let α_3 be the conjunction of α_2 with the following formulae:

$$\forall x,y(Id(x,y)\rightarrow Corr(s(x),s(y))) \quad (17)$$
$$\forall x,y(Corr(x,y)\rightarrow Tile(x,s(x))\wedge Tile(y,s(y))) \quad (18)$$
$$\forall x,y(Corr(x,y)\wedge *(s(x),s^2(x))\rightarrow$$
$$Tile(y,s(y))\wedge Tile(s(y),s^2(y))\wedge *(s^2(x),s^3(x))) \quad (19)$$
$$\forall x,y(Corr(x,y)\wedge\neg *(s(x),s^2(x))\rightarrow Corr(s(x),s(y))) \quad (20)$$
$$\forall x,y(Id(x,y)\rightarrow\neg Corr(x,y)). \quad (21)$$

If $Tile(x,y)$ and $Tile(z,t)$, we say that the two tiles are *above connected* if and only if $Corr(x,z)$. If α_3 holds, then, as a first consequence, we have that the first tile of each Id is above connected with the first tile of the successive Id. Then, by taking advantage of the successor function, from this initial connection we make sure that each i-th $Tile$ of any Id is above connected with the i-th $Tile$ of the successive Id, and, finally, the second formula of the above set enures that each Id has exactly one tile less than the successive one. This means that, if α_3 holds, the j-th Id codifies exactly the j-th layer of the octant. Finally, let $\alpha_{\mathcal{T}}$ be the conjunction of

α_3 and the following formulae:

$$\forall x,y(Tile(x,y) \rightarrow \bigvee_{T\in\mathcal{T}} T(x,y) \wedge \bigwedge_{T,T'\in\mathcal{T},T\neq T'} \neg(T(x,y)\wedge T'(x,y)) \quad (22)$$

$$\forall x,y(T(x,y)\wedge Tile(s(x),s(y)) \rightarrow \bigvee_{T'\in\mathcal{T},right(T)=left(T')} T'(s(x),s(y))) \quad (23)$$

$$\forall x,y(Corr(x,y)\wedge T(x,s(x))) \rightarrow \bigvee_{T'\in\mathcal{T},up(T)=down(T')} T'(y,s(y))). \quad (24)$$

Given any set of tiles $\mathcal{T}$ the formula $\alpha_{\mathcal{T}}$ is satisfiable if and only if $\mathcal{T}$ can tile $\mathcal{O}$, as claimed. Thus, the satisfiability problem of $\mathrm{FO}^2[\mathbb{N},=,<,s]$ is undecidable. □

6.3 Expressive completeness of MPNL for a fragment of $\mathrm{FO}^2[\mathbb{N},=,<,s]$

Let $\mathrm{FO}^2_r[\mathbb{N},=,<,s]$ be the fragment of $\mathrm{FO}^2[\mathbb{N},=,<,s]$ with the following restriction: if both variables x and y occur in the scope of an occurrence of a binary relation, other than $=$ and $<$, then the successor function s cannot occur in the scope of that occurrence. As an example, each of the formulae $s^k(x)=s^m(y), s^k(x)<s^m(y), P(s^k(x),s^m(x)), P(x,y)$ belongs to $\mathrm{FO}^2_r[\mathbb{N},=,<,s]$, but none of $P(x,s(y))$ and $P(s(x),y)$ does. By using 2-pebble games and a standard model-theoretic argument, one can show that:

$$\mathrm{FO}^2[=,<] \prec \mathrm{FO}^2_r[\mathbb{N},=,<,s] \prec \mathrm{FO}^2[\mathbb{N},=,<,s].$$

First, we define the standard translation $ST_{x,y}$ of MPNL$_l$-formulae into $\mathrm{FO}^2_r[\mathbb{N},=,<,s]$, as follows:

$$ST_{x,y}(\varphi) = x\leq y \wedge ST'_{x,y}(\varphi),$$

where x,y are the two first-order variables in $\mathrm{FO}^2_r[\mathbb{N},=,<,s]$, and

$$\begin{array}{ll} ST'_{x,y}(p) & = P(x,y)\\ ST'_{x,y}(\mathsf{len}_{=\mathsf{k}}) & = s^k(x)=y\\ ST'_{x,y}(\varphi\vee\psi) & = ST'_{x,y}(\varphi)\vee ST'_{x,y}(\psi)\\ ST'_{x,y}(\neg\varphi) & = \neg ST'_{x,y}(\varphi)\\ ST'_{x,y}(\Diamond_l\varphi) & = \exists y(y\leq x\wedge ST'_{y,x}(\varphi))\\ ST'_{x,y}(\Diamond_r\varphi) & = \exists x(y\leq x\wedge ST'_{y,x}(\varphi)).\end{array}$$

Lemma 4 *For any MPNL$_l$-formula φ, any interval model $M=\langle\mathbb{N},\mathbb{I}(\mathbb{N}),V\rangle$, and any interval $[a,b]$ in M:*

$M,[a,b]\Vdash\varphi$ iff $\eta(M)\models ST_{x,y}(\varphi)[x:=a,y:=b]$.

Proof Routine structural induction on φ.

Now, the inverse translation τ from $\mathrm{FO}^2[\mathbb{N},=,<,s]$ to MPNL$_l$ is given in Table 1, and we have the following lemma.

Lemma 5 *For every $\mathrm{FO}^2_r[\mathbb{N},=,<,s]$-formula $\alpha(x,y)$, every $\mathrm{FO}^2[\mathbb{N},=,<,s]$-model $\mathbf{M}=\langle\mathbb{N},V_M\rangle$ and every pair $i,j\in\mathbb{N}$, with $i\leq j$:*

(i) $\mathbf{M}\models\alpha(i,j)$ if and only if $\zeta(\mathbf{M}),[i,j]\Vdash\tau[x,y](\alpha)$, and

(ii) $\mathbf{M}\models\alpha(j,i)$ if and only if $\zeta(\mathbf{M}),[i,j]\Vdash\tau[y,x](\alpha)$.

Proof The proof is by structural induction on the complexity of α (for the sake of simplicity, we only prove claim (i); the other one can be proved similarly):

- $\alpha=(s^k(x)=s^m(x))$. If $k=m$, then both α and its translation $\tau[x,y](\alpha)=\top$ are true, while if $k\neq m$, then α and $\tau[x,y](\alpha)=\bot$ are both false; the same applies when x is used instead of y;
- $\alpha=(s^k(x)<s^m(x))$. If $k=m$, then both α and its translation $\tau[x,y](\alpha)=\bot$ are false, while if $k\neq m$, then α and $\tau[x,y](\alpha)=\top$ are both true; the same applies when x is used instead of y;
- $\alpha=(s^k(x)=s^m(y))$. Assuming $i<j$, if $k<m$ then $s^k(i)<s^m(j)$, and, since $\mathbf{M}\models\alpha(i,j)$ iff $s^k(i)<s^m(j)$, we have that $\mathbf{M}\not\models\alpha(i,j)$; on the other hand $\tau[x,y](\alpha)=\bot$. If $m\leq k$, $s^k(i)=s^m(j)$ iff $j-i=k-m$, that is $\mathbf{M}\models\alpha(i,j)$ iff $\zeta(\mathbf{M}),[i,j]\models\mathsf{len}_{=\mathsf{k-m}}$. Likewise for the cases $\alpha=(s^m(y)=s^k(x))$, $\alpha=(s^k(x)=s^m(y))$, $\alpha=(s^m(y)<s^k(x))$;
- $\alpha=(P(s^k(x),s^m(x)))$. Assuming $i<j$, if $k<m$ then we have that $s^m(x)-s^k(x)=m-k$, and that $s^k(x)-x=k$. Thus, $\mathbf{M}\models\alpha(i,j)$ iff P is true over the pair $(s^k(i), s^{m-k}(s^k(i)))$, that is, $\mathbf{M}\models\alpha(i,j)$ iff $\zeta(\mathbf{M}),[i,j]\Vdash\Diamond_l\Diamond_r(\mathsf{len}_{=\mathsf{k}}\wedge\Diamond_r(\mathsf{len}_{=\mathsf{m-k}}\wedge p^{\leq}))$. A similar reasoning can be followed for the case of $m<k$. If $k=m$, then $s^k(x)=s^m(x)$, so P must be true over a point-interval, specifically, identified by the pair $(s^k(i),s^k(i))$. Thus, we have that $\mathbf{M}\models\alpha(i,j)$ iff $\zeta(\mathbf{M}),[i,j]\Vdash\Diamond_l\Diamond_r(\mathsf{len}_{=\mathsf{k}}\wedge\Diamond_r(\mathsf{len}_{=0}\wedge p^{\leq}\wedge p^{\geq}))$. Likewise, when y substitutes x;
- $\alpha=P(x,y)$ or $\alpha=P(y,x)$. The claim follows from the valuation of $p^{\leq}$ and $p^{\geq}$;
- The Boolean cases are straightforward;
- $\alpha=\exists x\beta$. Suppose that $\mathbf{M}\models\alpha(i,j)$. Then, there is $l\in\mathbf{M}$ such that $\mathbf{M}\models\beta(l,j)$. There are two (non-exclusive) cases: $j\leq l$ and $l\leq j$. If $b\leq c$, by the inductive hypothesis, we have that $\zeta(\mathbf{M}),[j,l]\Vdash\tau[y,x](\beta)$ and thus $\zeta(\mathbf{M}),[i,j]\Vdash\Diamond_r(\tau[y,x](\beta))$. Likewise, if $l\leq j$, by the inductive hypothesis, we have that $\zeta(\mathbf{M}),[l,j]\Vdash\tau[x,y](\beta)$ and thus for every r such that $j\leq r$, $\zeta(\mathbf{M}),[j,r]\Vdash\Diamond_l(\tau[x,y](\beta))$, that is, $\zeta(M),[a,b]\Vdash\Box_r\Diamond_l(\tau[x,y](\beta))$. Hence $\zeta(\mathbf{M}),[i,j]\Vdash\Diamond_r(\tau[y,x](\beta))\vee\Box_r\Diamond_l(\tau[x,y](\beta))$, that is, $\zeta(\mathbf{M}),[i,j]\Vdash\tau[x,y](\alpha)$. For the converse direction, it suffices to note that the interval $[i,j]$ has at least one right neighbor, viz. $[j,j]$, and thus the above argument can be reversed;
- $\alpha=\exists y\beta$. Analogous to the previous case.

$\tau[x,y](s^k(z)=s^m(z)) = \top \quad (z \in \{x,y\})$,	if $k=m$	$\tau[x,y](\alpha \vee \beta) = \tau[x,y](\alpha) \vee \tau[x,y](\beta)$	
$= \bot \quad (z \in \{x,y\})$,	if $k \neq m$	$\tau[x,y](\exists x\beta) = \Diamond_r(\tau[y,x](\beta)) \vee \Box_r\Diamond_l(\tau[x,y](\beta))$	
$\tau[x,y](s^k(z)<s^m(z)) = \bot \quad (z \in \{x,y\})$,	if $k \geq m$	$\tau[x,y](\exists y\beta) = \Diamond_l(\tau[y,x](\beta)) \vee \Box_l\Diamond_r(\tau[x,y](\beta))$	
$= \top \quad (z \in \{x,y\})$,	if $k<m$	$\tau[x,y](P(s^k(x),s^m(x))) = \Diamond_l\Diamond_r(\mathsf{len}_{=\mathsf{k}} \wedge \Diamond_r(\mathsf{len}_{=\mathsf{m-k}} \wedge p^{\leq}))$,	if $k<m$
$\tau[x,y](s^k(x)=s^m(y)) = \bot$,	if $k<m$	$= \Diamond_l\Diamond_r(\mathsf{len}_{=\mathsf{k}} \wedge \Diamond_r(\mathsf{len}_{=0} \wedge p^{\leq} \wedge p^{\geq}))$,	if $k=m$
$= \mathsf{len}_{=\mathsf{k-m}}$,	if $k \geq m$	$= \Diamond_l\Diamond_r(\mathsf{len}_{=\mathsf{m}} \wedge \Diamond_r(\mathsf{len}_{=\mathsf{k-m}} \wedge p^{\geq}))$,	if $k>m$
$\tau[x,y](s^k(x)<s^m(y)) = \top$,	if $k<m$	$\tau[x,y](P(s^k(y),s^m(y))) = \Diamond_r(\mathsf{len}_{=\mathsf{k}} \wedge \Diamond_r(\mathsf{len}_{=\mathsf{m-k}} \wedge p^{\leq}))$,	if $k<m$
$= \mathsf{len}_{>\mathsf{k-m}}$,	if $k \geq m$	$= \Diamond_r(\mathsf{len}_{=\mathsf{k}} \wedge \Diamond_r(\mathsf{len}_{=0} \wedge p^{\leq} \wedge p^{\geq}))$,	if $k=m$
$\tau[x,y](s^m(y)<s^k(x)) = \bot$,	if $k<m$	$= \Diamond_r(\mathsf{len}_{=\mathsf{m}} \wedge \Diamond_r(\mathsf{len}_{=\mathsf{k-m}} \wedge p^{\geq}))$,	if $k>m$
$= \mathsf{len}_{<\mathsf{k-m}}$,	if $k \geq m$	$\tau[x,y](P(x,y)) = p^{\leq}$	
$\tau[x,y](\neg\alpha) = \neg\tau[x,y](\alpha)$		$\tau[x,y](P(y,x)) = p^{\geq}$	

Table 1 Translation clauses from $\mathrm{FO}^2_r[\mathbb{N},=,<,s]$ to MPNL.

$\Diamond_{be}^{+k}\Diamond_e^{+(m-k)}p^{\leq}$,	if $k<m$
$(\mathsf{len}_{>0} \wedge \Diamond_{be}^{+k}p^{\leq}) \vee (\mathsf{len}_{=0} \wedge \Diamond_{be}^{+k}(p^{\leq} \wedge p^{\geq})$,	if $k=m$
$(\mathsf{len}_{>\mathsf{k-m}} \wedge \Diamond_{be}^{+m}\Diamond_b^{+(k-m)}p^{\leq}) \vee$	
$(\mathsf{len}_{=\mathsf{k-m}} \wedge \Diamond_{be}^{+m}\Diamond_b^{+(k-m)}(p^{\leq} \wedge p^{\geq}) \vee$	
$(\mathsf{len}_{<\mathsf{k-m}} \wedge \Diamond_{be}^{+k}\Diamond_b^{+(k-m)}p^{\geq}$,	if $k>m$

Table 2 Extending the translation from $\mathrm{FO}^2[\mathbb{N},=,<,s]$ to MPNL: the clause for $\tau[x,y](P(s^k(x),s^m(y)))$

Corollary 2 *For every* $\mathrm{FO}^2_r[\mathbb{N},=,<,s]$*-formula* $\alpha(x,y)$ *and every* $\mathrm{FO}^2[\mathbb{N},=,<,s]$*-model* $M=\langle\mathbb{N},V_M\rangle$, $M \models \forall x\forall y\alpha(x,y)$ *if and only if* $\zeta(M) \Vdash \tau[x,y](\alpha) \wedge \tau[y,x](\alpha)$.

Theorem 5 $\mathrm{FO}^2_r[\mathbb{N},=,<,s] \equiv$ MPNL.

6.4 Extension of MPNL expressively complete for $\mathrm{FO}^2[\mathbb{N},=,<,s]$

A natural way to extend MPNL to cover the entire $\mathrm{FO}^2[\mathbb{N},=,<,s]$ would be to add diamond modalities that shift respectively the beginning, the end, or both endpoints of the current interval to the right by a prescribed distance, viz:

- $M,[i,j] \Vdash \Diamond_e^{+k}\psi$ iff $M,[i,j+k] \Vdash \psi$
- $M,[i,j] \Vdash \Diamond_b^{+k}\psi$ iff ($i+k \leq j$ and $M,[i+k,j] \Vdash \psi$) or ($i+k>j$ and $M,[j,i+k] \Vdash \psi$).
- $M,[i,j] \Vdash \Diamond_{be}^{+k}\psi$ iff $M,[i+k,j+k] \Vdash \psi$

We denote the resulting language as MPNL$^+$. The standard translation $ST'_{x,y}$ of MPNL-formulae into $\mathrm{FO}^2[\mathbb{N},=,<,s]$ extends to MPNL$^+$ as follows, where $\alpha[t/z]$ is the result of simultaneous substitution of the term t for all free occurrences of z in α.

$$ST'_{x,y}(\Diamond_e^{+k}\psi) = ST'_{x,y}(\psi)[s^k(y)/y].$$
$$ST'_{x,y}(\Diamond_b^{+k}\psi) = ST'_{x,y}(\psi)[s^k(x)/x].$$
$$ST'_{x,y}(\Diamond_{be}^{+k}\psi) = ST'_{x,y}(\psi)[s^k(x)/x, s^k(y)/y].$$

Note that if $ST'_{x,y}(\psi) \in \mathrm{FO}^2[\mathbb{N},=,<,s]$ then $ST'_{x,y}(\psi)[s^k(x)/x, s^m(y)/y] \in \mathrm{FO}^2[\mathbb{N},=,<,s]$ for any $k,m \in \mathbb{N}$, so the translation of all formulae of MPNL$^+$ will remain within $\mathrm{FO}^2[\mathbb{N},=,<,s]$. Conversely, we can now extend the translation τ of $\mathrm{FO}^2_r[\mathbb{N},=,<,s]$ into MPNL to a translation of $\mathrm{FO}^2[\mathbb{N},=,<,s]$ into MPNL$^+$ by adding the clauses for the atomic formulae in Table 2. The extensions of the expressive completeness results are routine.

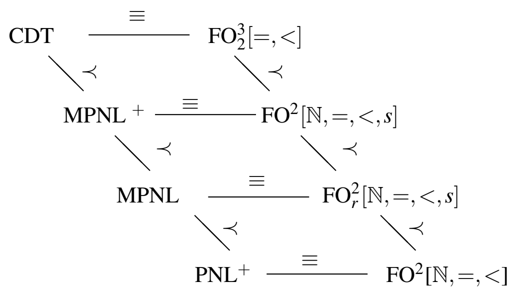

Fig. 1 Expressive completeness results for interval logics.

To conclude this subsection, we recall that Venema [33] has shown in a similar way that the interval temporal logic CDT, involving binary modalities based on the ternary interval relation ’chop’ and its residuals (denoted respectively C, D and T) is expressively complete for the fragment of first-order logic with equality with three variables of which at most two are free, denoted by $\mathrm{FO}^3_2[=,<]$. Note that, when interpreted in $\mathbb{N}$ the successor function is definable in this fragment, which therefore strictly extends $\mathrm{FO}^2[\mathbb{N},=,<,s]$. Thus, a hierarchy of expressive completeness results arises, depicted in Fig. 1. Note also that the proposed translations from the first order languages towards the interval ones are exponential in the size of the input formula in all three cases, due to the clauses for the existential quantifier[1].

7 Classifying the Expressive Power of Metric Propositional Neighborhood Interval Logics

In the previous sections, we discussed the expressive power and the computational properties of MPNL. A natural question is whether there exist other interesting variants of PNL

[1] At present we do not know whether a polynomial translation for any of these cases exists.

$\Diamond_o^{<k}\psi \Leftrightarrow$	$\bot\ (k=0)$	$\Diamond_o \psi \Leftrightarrow$	$\Diamond_o^{\geq 0}\psi$
	$\Diamond_o^{\leq k-1}\psi\ (k>0)$		$\Diamond_o^{[0,\infty]}\psi$
$\Diamond_o^{[k,k']}\psi \Leftrightarrow$	$\Diamond_o^{[k,k')}\psi\ (k'=\infty)$	$\Diamond_o^{<k}\psi \Leftrightarrow$	$\Diamond_o^{=0}\psi \quad \vee \quad \ldots \quad \vee$ $\Diamond_o^{=k-1}\psi$
	$\Diamond_o^{[k,k'+1)}\psi\ (k'\neq\infty)$	$\Diamond_o^{=k}\psi \Leftrightarrow$	$\Diamond_o^{[k,k]}\psi$
$\Diamond_o^{[k,k')}\psi \Leftrightarrow$	$\bot\ (k'=0)$	$\Diamond_o^{>k}\psi \Leftrightarrow$	$\Diamond_o^{\geq k+1}\psi$
	$\Diamond_o^{[k,k'-1]}\ (k'>0)$		$\Diamond_o^{(k,\infty)}\psi$
	$\Diamond_o^{[k,k']}\psi\ (k'=\infty)$	$\Diamond_o^{\geq k}\psi \Leftrightarrow$	$\Diamond_o^{[k,\infty]}\psi$
$\Diamond_o^{(k,k']}\psi \Leftrightarrow$	$\Diamond_o^{(k,k'+1)}\psi\ (k'\neq\infty)$	$\Diamond_o^{(k,k')}\psi \Leftrightarrow$	$\bot\ (k'=0)$
	$\Diamond_o^{(k,k')}\psi\ (k'=\infty)$		$\Diamond_o^{[k+1,k'-1]}\psi\ (k'>0)$
$\Diamond_o^{(k,k')}\psi \Leftrightarrow$	$\bot\ (k'=0)$		$\Diamond_o^{[k+1,k']}\psi\ (k'=\infty)$
	$\Diamond_o^{(k,k'-1]}\ (k'>0)$		
	$\Diamond_o^{(k,k']}\psi\ (k'=\infty)$		

Table 3 Equivalences between metric operators, $o \in \{r,l\}$.

that can be further analyzed. In this section we define a family of metric languages, and we compare their expressive power. As it will be proved in the following, MPNL is able to encode all the languages in the family, thus being the most expressive metric extension of PNL.

Let $\sim \in \{<,\leq,=,\geq,>\}$, $k\in\mathbb{N}$, and $k'\in\mathbb{N}\cup\{\infty\}$. We consider a set of *metric modalities* of the type $\Diamond_r^{\sim k}$, $\Diamond_r^{[k,k']}$, $\Diamond_r^{(k,k')}$, $\Diamond_r^{[k,k')}$, $\Diamond_r^{(k,k']}$, as well as their inverses $\Diamond_l^{\sim k}$, $\Diamond_l^{[k,k']}$, $\Diamond_l^{(k,k')}$, $\Diamond_l^{[k,k')}$, $\Diamond_l^{(k,k']}$, with the following semantics:

- $M,[i,j]\Vdash\Diamond_r^{\sim k}\psi$ iff there exists $m\geq j$ such that $\delta(j,m)\sim k$ and $M,[j,m]\Vdash\psi$;
- $M,[i,j]\Vdash\Diamond_r^{[k,k']}\psi$ iff there exists $m\geq j$ such that $k\leq\delta(j,m)\leq k'$ and $M,[j,m]\Vdash\psi$;
- $M,[i,j]\Vdash\Diamond_r^{(k,k')}\psi$ iff there exists $m\geq j$ such that $k<\delta(j,m)<k'$ and $M,[j,m]\Vdash\psi$;

The truth clauses for $\Diamond_r^{[k,k')}$ and $\Diamond_r^{[k,k')}$, as well as those for the inverse modalities, are defined likewise. It is easy to show that all metric modalities are definable by exploiting the length constraints, e.g.:

$$\Diamond_r^{\sim k}\psi := \Diamond_r(\psi\wedge \mathsf{len}_{\sim k}),$$

and thus that all languages of the family are fragments of MPNL. Let $\kappa\in\{<k,\leq k,=k,\geq k,>k,[k,k'],(k,k'),[k,k'),(k,k']\}$, and let $\Diamond_o^{\kappa}$ be any of the two operators $\Diamond_l^{\kappa}$ or $\Diamond_r^{\kappa}$. The dual operators are defined as usual, that is, $\Box_o^{\kappa}\psi=\neg\Diamond_o^{\kappa}\neg\psi$. Let ε be a special symbol such that $\Diamond_r^{\varepsilon k}=\Diamond_r$ and $\Diamond_l^{\varepsilon k}=\Diamond_l$ for any k and let $S\subseteq\{\varepsilon,<,\leq,=,\geq,>,[],(),[),(]\}$. We will denote by MPNL^S the language that features:

(i) the modal operators $\Diamond_l^{\sim k}$ and $\Diamond_r^{\sim k}$ for each $k\in\mathbb{N}$ and $\sim\in S\cap\{\varepsilon,<,\leq,=,\geq,>\}$;
(ii) the modal operators $\Diamond_l^{[k,k']}$ and $\Diamond_r^{[k,k']}$ (resp., $\Diamond_l^{(k,k')}$ and $\Diamond_r^{(k,k')}$, $\Diamond_l^{[k,k')}$ and $\Diamond_r^{[k,k')}$, $\Diamond_l^{(k,k']}$ and $\Diamond_r^{(k,k']}$), for each $k\in\mathbb{N}$, $k'\in\mathbb{N}\cup\{\infty\}$, if $[]\in S$ (resp., $()\in S$, $[)\in S$, $(]\in S$).

We will denote by MPNL_l^S the extension of MPNL^S with the length constraints (this means that $\mathrm{MPNL}_l^{\emptyset}$ is exactly the language MPNL of the previous sections). For the sake of simplicity, we will omit the curly brackets in the superscript; for example, if $S=\{<,>\}$, we will write simply $\mathrm{MPNL}^{<,>}$ instead of $\mathrm{MPNL}^{\{<,>\}}$. Thus, we have that $\mathrm{MPNL}^{\varepsilon}\equiv$PNL and $\mathrm{MPNL}_l^{\varepsilon}\equiv\mathrm{MPNL}_l$. Moreover, by the following lemma, we can reduce the number of interesting fragments:

Lemma 6 *If $o\in\{r,l\}$, whenever $\Diamond_o^{<k}$ (resp., $\Diamond_o^{[k,k']}$, $\Diamond_o^{(k,k']}$) is included in the language, then $\Diamond_o^{\leq k}$ (resp., $\Diamond_o^{[k,k')}$, $\Diamond_o^{(k,k')}$) can be defined, and the other way around.*

Proof See Table 3, left column. □

Thus, without loss of generality, from now on we can focus our attention on languages characterized by subsets of $\{\varepsilon,<,=,>,\geq,[],()\}$. As we will see, some languages will be expressive enough to embed non-metric PNL, and some others will not. We will use the term *Weak Metric Propositional Neighborhood Logics* (wMPNL) to denote the latter.

Definition 11 Let L and L' be two languages for interval logic. We say that L' is *at least as expressive as* L denoted by L $\preceq$ L', if there exists an effective translation τ from L to L' (usually, defined inductively on the structure of formulae) such that for every formula φ of L, $M,[i,j]\Vdash\varphi$ if and only if $M,[i,j]\Vdash\tau(\varphi)$, and we say that L is *as expressive as* L', denoted by L $\equiv$ L', if both L $\preceq$ L' and L' $\preceq$ L, while we say that L' is *strictly more expressive than* L, denoted by L $\prec$ L', if L $\preceq$ L' and L' $\not\preceq$ L.

In order to compare the expressive power of interval languages, we use bisimulation games [?] and bisimulation [?]; since the former can be considered a generalization of the latter, we give here a quick remind of bisimulation games (defined here for interval logics).

We define the notion of a *N-moves bisimulation game* (for the interval logic L) to be played by two players, Player I and Player II, on a pair of L-models M, M', with $M=\langle\mathbb{D},\mathbb{I}(D),V\rangle$ and $M'=\langle\mathbb{D}',\mathbb{I}(D'),V'\rangle$. The game starts from a given *initial configuration*, where a *configuration* is a pair of intervals $([a,b],[a',b'])$, with $[a,b]\in\mathbb{I}(D)$ and $[a',b']\in\mathbb{I}(D')$. A configuration $([a,b],[a',b'])$ is *matching* if $[a,b]$ and $[a',b']$ satisfy the same atomic propositions in their respective models. The *moves* of the game depend on the modal operators of L: for each $\Diamond$ in the language of L, where $R_{\Diamond}$ is the (interval) relation on which $\Diamond$ is based, Player I can play the corresponding move: choose M (resp., M'), and an interval $[c,d]$ (resp., $[c',d']$) such that $[a,b]\ R_{\Diamond}\ [c,d]$ (resp., $[a',b']\ R_{\Diamond}\ [c',d']$). Player II must reply by choosing an interval $[c',d']$ (resp., $[c,d]$) in M' (resp., M), which leads to the new configuration $([c,d],[c',d'])$. If after any given round the current configuration is not matching, Player I wins the game; otherwise, after N rounds, Player II wins the game. Intuitively, Player II has a *winning strategy* in the N-moves bisimulation game on the models M and M' with a given initial configuration if she can win regardless of the moves played by Player I; otherwise, Player I has a winning strategy. A formal definition of winning strategy can be found in [**?**]. The following key property of the N-move games can be proved routinely, in analogy with similar results about bisimulation games in modal logic [**?**][2].

Proposition 1 *Let L be a finite interval language. For all $N \geq 0$, Player II has a winning strategy in the N-move L-bisimulation game on M and M' with initial configuration $([a,b],[a',b'])$ iff $[a,b]$ and $[a',b']$ satisfy the same N-formulas over L with modal depth at most N.*

7.1 The class of Weak Metric Propositional Neighborhood Logics

Here we analyze the set of languages in wMPNL. Formally, wMPNL is the subset of MPNL defined as follows:

$$\text{wMPNL} = \{\text{L} \mid \text{L} \in \text{MPNL and PNL} \not\preceq \text{L}\}.$$

The following lemma states some basic results which we will use to classify languages in wMPNL.

Lemma 7 *If $o \in \{r,l\}$, whenever any of the modalities in $\{\Diamond_o^{\geq k}, \Diamond_o^{[k,k']}\}$ (resp., $\{\Diamond_o^{=k}, \Diamond_o^{[k,k']}\}$, $\{\Diamond_o^{\geq k}, \Diamond_o^{(k,k')}, \Diamond_o^{[k,k']}\}$), are included in the language, then $\Diamond_o$ (resp., $\Diamond_o^{<k}$, $\Diamond_o^{>k}$) can be defined. Similarly, whenever $\Diamond_o^{[k,k']}$ is included, then $\Diamond_o^{=k}$, $\Diamond_o^{\geq k}$, and $\Diamond_o^{(k,k')}$ can be defined.*

Proof See Table 3, right column. □

Theorem 6 *Let $\mathcal{S}_w = \{\{<\},\{>\},\{=\},\{()\}\}$. We have that wMPNL $= \{\text{MPNL}^S, \text{MPNL}_l^S \mid S \in \mathcal{S}_w\}$.*

Proof First, we show that MPNL^S and MPNL_l^S belong to *w*MPNL for each $S \in \mathcal{S}_w$. We have to show that PNL $\not\preceq$ MPNL_l^S for each $S \in \mathcal{S}_w$. As a consequence, we also have that PNL $\not\preceq$ MPNL^S for each $S \in \mathcal{S}_w$. By Lemma 7, we have that $\text{MPNL}_l^< \preceq \text{MPNL}_l^=$ and $\text{MPNL}_l^> \preceq \text{MPNL}_l^{()}$. Thus, it suffices to show that PNL $\not\preceq$ $\text{MPNL}_l^=$ and PNL $\not\preceq$ $\text{MPNL}_l^{()}$, as follows.

PNL $\not\preceq$ $\text{MPNL}_l^=$. It is easy to show that classical, non-metric modal operators of PNL can be expressed using formulae of $\text{MPNL}_l^=$ of infinite length. For example, it is possible to express the formula $\Diamond_r p$ of PNL by means the infinite formulae $\Diamond_r^{=0} p \vee \Diamond_r^{=1} p \vee \ldots \Diamond_r^{=i} p \vee \ldots$. Nevertheless, suppose, by contradiction, that there exists a finite formula $\varphi \in \text{MPNL}_l^=$ such that $\varphi \equiv \Diamond_r p$. This means that φ contains a finite number of modal operators. Let $t \in \mathbb{N}$ be the largest number such that $\Diamond_r^{=t}$ or $\Diamond_l^{=t}$ occurs in φ, and, for any $t \in \mathbb{N}$, define ${}^t\text{MPNL}_l^=$ as the restriction of $\text{MPNL}_l^=$ to the set of modalities $\{\Diamond_r^{=k}, \Diamond_l^{=k} \mid 0 \leq k \leq t\}$. Now, let $M = \langle \mathbb{D} = \mathbb{N}, \mathbb{I}(\mathbb{D}), V\rangle$ and $M' = \langle \mathbb{D}' = \mathbb{N}, \mathbb{I}(\mathbb{D}'), V'\rangle$, $\mathcal{AP} = \{p\}$, $V(p) = \{[1,t+2]\}$, $V'(p) = \emptyset$, and $Z \subset \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ defined as $Z = \{([i,j],[i',j']) \mid \delta(i,j) \leq t\}$. It is possible to show that Z is a bisimulation for ${}^t\text{MPNL}_l^=$. Since $M,[1,1] \Vdash \Diamond_r p$, $M',[1',1'] \not\Vdash \Diamond_r p$, and $[1,1]$ is Z-related with $[1',1']$, we have a contradiction.

PNL $\not\preceq$ $\text{MPNL}_l^{()}$. Again, suppose that for some $\varphi \in \text{MPNL}_l^{()}$ it is the case that $\varphi \equiv \Diamond_r p$. Consider $M = \langle \mathbb{D} = \mathbb{N}, \mathbb{I}(\mathbb{D}), V\rangle$, $M' = \langle \mathbb{D}' = \mathbb{N}, \mathbb{I}(\mathbb{D}'), V'\rangle$, $\mathcal{AP} = \{p\}$, $V(p) = \{[1,1]\}$, and $V'(p) = \emptyset$, while $Z \subset \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ is defined as $Z = \{([i,j],[i',j']) \mid i \neq j\}$. As before, Z is a bisimulation for $\text{MPNL}_l^{()}$. Since $M,[0,1] \Vdash \Diamond_r p$, $M',[0',1'] \not\Vdash \Diamond_r p$, and $[0,1]$ is Z-related with $[0',1']$, we have a contradiction.

Now,we show that no other language belongs to wMPNL, that is, neither MPNL^S nor MPNL_l^S belongs to wMPNL for any $S \notin \mathcal{S}_w$. Let $S \subseteq \{\varepsilon,<,=,>,\geq,[],()\}$ such that $S \notin \mathcal{S}_w$. We must show that PNL $\preceq$ MPNL^S and PNL $\preceq$ MPNL_l^S. Since $\text{MPNL}^S \preceq \text{MPNL}_l^S$, it suffices to show that PNL $\preceq$ MPNL^S. If $\varepsilon \in S$, then clearly PNL $\preceq$ MPNL^S, since PNL $\equiv$ MPNL^ε. If $\geq \in S$ or $[] \in S$, then the result immediately follows from Lemma 7. If $\{<,>\} \subseteq S$, then the thesis immediately follows by the fact that $\Diamond_o \psi$ is defined by $\Diamond_o^{<1} \psi \vee \Diamond_o^{>0} \psi$ for each $o \in \{r,l\}$. The rest of the cases are consequences of the others and of previous lemmas. □

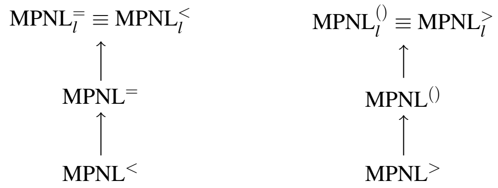

Fig. 2 Relative expressive power of the metric languages belonging to wMPNL. An arrow going from L to L′ denotes that L′ is strictly more expressive than L. Languages that are no connected through any path are incomparable.

We now establish how the various languages of wMPNL relate to each other in terms of expressive power.

Theorem 7 *The relative expressive power of the languages of the class wMPNL is as depicted in Fig. 2, where each arrow means that the language at the top is strictly more expressive than the one at the bottom.*

[2] We refer to the notion of modal depth of a L-formula φ, which is defined in the usual way. Let us denote by $mdepth(\varphi)$ the modal depth of φ. It can be inductively defined as follows: (i) mdepth(p) = 0, for each $p \in \mathcal{AP}$; (ii) $mdepth(\neg\varphi) = mdepth(\varphi), mdepth(\varphi \vee \psi) = max\{mdepth(\varphi), mdepth(\psi)\}, mdepth(\Diamond\varphi) = mdepth(\varphi)+1$, for each $\Diamond$ of the language

Proof By Lemma 7, we already know that $\mathrm{MPNL}^{<} \preceq \mathrm{MPNL}^{=}$, $\mathrm{MPNL}_l^{<} \preceq \mathrm{MPNL}_l^{=}$, $\mathrm{MPNL}^{>} \preceq \mathrm{MPNL}^{()}$, and that $\mathrm{MPNL}_l^{>} \preceq \mathrm{MPNL}_l^{()}$. To complete the proof, it remains to show that $\mathrm{MPNL}^{=} \not\preceq \mathrm{MPNL}^{<}$, $\mathrm{MPNL}_l^{=} \preceq \mathrm{MPNL}_l^{<}$, $\mathrm{MPNL}^{()} \not\preceq \mathrm{MPNL}^{>}$, and $\mathrm{MPNL}_l^{()} \preceq \mathrm{MPNL}_l^{>}$.

$\mathrm{MPNL}^{=} \not\preceq \mathrm{MPNL}^{<}$. It suffices to show that $\Diamond_r^{=k}$ cannot be defined in $\mathrm{MPNL}^{<}$. Suppose the contrary, and let $M = \langle \mathbb{D} = \mathbb{N}, \mathbb{I}(\mathbb{D}), V\rangle$, $M' = \langle \mathbb{D}' = \{0'\}, \mathbb{I}(\mathbb{D}'), V'\rangle$, $\mathcal{AP} = \{p\}$, $V(p) = \mathbb{I}(\mathbb{D})$, $V'(p) = \mathbb{I}(\mathbb{D}') = \{[0',0']\}$, and $Z = \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$. It is possible to show that Z is a bisimulation for $\mathrm{MPNL}^{<}$. Since it holds that $M,[0,0] \Vdash \Diamond_r^{=1} p$, $M',[0',0'] \not\Vdash \Diamond_r^{=1} p$, and $[0,0]$ is Z-related to $[0',0']$, we have a contradiction.

$\mathrm{MPNL}^{()} \not\preceq \mathrm{MPNL}^{>}$. Consider, for any $t \in \mathbb{N}$, the language ${}^t\mathrm{MPNL}^{>}$, that is, as before, the restriction of $\mathrm{MPNL}^{>}$ to the set of modalities $\{\Diamond_r^{>k}, \Diamond_l^{>k} \mid 0 \le k \le t\}$. Let $M = \langle \mathbb{D} = \mathbb{N}, \mathbb{I}(\mathbb{D}), V\rangle$, $M' = \langle \mathbb{D}' = \mathbb{N}, \mathbb{I}(\mathbb{D}'), V'\rangle$, $\mathcal{AP} = \{p\}$, $V(p) = \{[i,j] \mid \delta(i,j)$ is odd and $\delta(i,j) \le t+N+1\}$, $V'(p) = \{[i',j'] \mid \delta(i,j)$ is odd, $\delta(i,j) \le t+N+1$, and $[i,j] \neq [a-1,a]\}$, where $a = N(t+N+1)$, and consider the relation $Z = \{([i,j],[k',l']) \mid \delta(i,j) = \delta(k,l) \le t+N+1$ and $[k,l] \neq [a-1,a]\} \cup \{([i,j],[i',k']) \mid \delta(i,j) > t+N+1$ and $\delta(i,k) > t+N+1\} \cup \{([a-1,a],[(a-3)',a']), ([a-1,a],[(a-1)',(a+2)'])\} \cup \{([i,j],[(a-1)',a']) \mid \delta(i,j) = 2\}$. It is possible to show that Z represents a winning strategy for Player II at the initial configurarion $([a,b],[a',b'])$ (for any b) in the N-moves bisimulation game for ${}^t\mathrm{MPNL}^{>}$. But $M,[a,b] \Vdash \Diamond_l^{(0,2)} p$ and $M',[a',b'] \not\Vdash \Diamond_l^{(0,2)} p$, which means that the formula $\Diamond_l^{(0,2)} p$ cannot be expressed in ${}^t\mathrm{MPNL}^{>}$ for any $t,N \in \mathbb{N}$. Thus, we have the result.

$\mathrm{MPNL}_l^{=} \preceq \mathrm{MPNL}_l^{<}$, $\mathrm{MPNL}_l^{()} \preceq \mathrm{MPNL}_l^{>}$. This is immediate by observing that, for each $o \in \{r,l\}$, $\Diamond_o^{=k}\psi$ is defined by $\Diamond_o^{<k+1}(\mathsf{len}_{=k} \wedge \psi)$, and that $\Diamond_o^{(k,k')}\psi$ is defined by $\Diamond_o^{>k}(\mathsf{len}_{<k'} \wedge \psi)$ (if $k' \neq \infty$) or by $\Diamond_o^{>k}\psi$ (if $k' = \infty$).

We have $\mathrm{MPNL}^{<} \prec \mathrm{MPNL}^{=}$, $\mathrm{MPNL}_l^{<} \equiv \mathrm{MPNL}_l^{=}$, $\mathrm{MPNL}^{>} \prec \mathrm{MPNL}^{()}$, and $\mathrm{MPNL}_l^{>} \equiv \mathrm{MPNL}_l^{()}$ as a consequence of the above results. Now, we want to show that each language in the set $\{\mathrm{MPNL}^{<}, \mathrm{MPNL}^{=}, \mathrm{MPNL}_l^{=}\}$ is incomparable with any of the languages of the set $\{\mathrm{MPNL}^{>}, \mathrm{MPNL}^{()}, \mathrm{MPNL}_l^{()}\}$. To this end it suffices to show that $\mathrm{MPNL}^{<} \not\preceq \mathrm{MPNL}_l^{()}$ and $\mathrm{MPNL}^{>} \not\preceq \mathrm{MPNL}_l^{=}$, which can be done as in Theorem 6. Finally, we must show that $\mathrm{MPNL}^{=} \prec \mathrm{MPNL}_l^{=}$ and $\mathrm{MPNL}^{()} \prec \mathrm{MPNL}_l^{()}$. It is easy to see that $\mathrm{MPNL}^{=} \preceq \mathrm{MPNL}_l^{=}$ and $\mathrm{MPNL}^{()} \preceq \mathrm{MPNL}_l^{()}$. To show that $\mathrm{MPNL}_l^{=} \not\preceq \mathrm{MPNL}^{=}$, consider, for any $t \in \mathbb{N}$, the language ${}^t\mathrm{MPNL}^{=}$, defined as usual. Let $M = \langle \mathbb{D} = \mathbb{N}, \mathbb{I}(\mathbb{D}), V\rangle$, $M' = \langle \mathbb{D}' = \mathbb{N}, \mathbb{I}(\mathbb{D}'), V'\rangle$, $\mathcal{AP} = \emptyset$, $V(p) = V'(p) = \emptyset$, and consider the relation $Z = \{([i,j],[i',j']) \mid i,j \in \mathbb{N}\} \cup \{([a,a+1],[a',(a+2)'])\} \cup \{([i,j],[(i+1)',(j+1)']) \mid i,j \in \mathbb{N}\}$, where $a = Nt$. It is possible to show that Z represents a winning strategy for Player II at the initial configurarion $([a,a+1],[a',(a+2)'])$ in the N-moves bisimulation game for ${}^t\mathrm{MPNL}^{=}$. But $M,[a,a+1] \Vdash \mathsf{len}_{=1}$ and $M',[a',(a+2)'] \not\Vdash \mathsf{len}_{=1}$, which means that the formula $\mathsf{len}_{=1}$ cannot be expressed in the language ${}^t\mathrm{MPNL}^{=}$ for any $t,N \in \mathbb{N}$. Thus, we have the result. By exploiting a very similar argument, it is possible to show that $\mathrm{MPNL}_l^{()} \not\preceq \mathrm{MPNL}^{()}$. $\square$

7.2 Expressive Power of Languages of the Class MPNL

In this section we deal with the problem of classifying all the fragments of the class MPNL with respect to their relative expressive power. Fig. 3 shows how the various languages are related to each other.

Lemma 8 *The following equivalences hold:*

1. $\mathrm{MPNL}^{<,>} \equiv \mathrm{MPNL}^{<,\geq}$;
2. $\mathrm{MPNL}^{<,()} \equiv \mathrm{MPNL}^{=,()} \equiv \mathrm{MPNL}^{=,>} \equiv \mathrm{MPNL}^{=,\geq} \equiv \mathrm{MPNL}^{[]}$;
3. $\mathrm{MPNL}^{>,\varepsilon} \equiv \mathrm{MPNL}^{\geq}$;
4. $\mathrm{MPNL}^{\geq,()} \equiv \mathrm{MPNL}^{(),\varepsilon}$.

Proof It suffices to use Lemma 7 and the equivalences in Table 4 (left column). $\square$

Corollary 3 *If $S = \{\varepsilon, <, =, >, \geq, (), []\}$, then we have that $MPNL^S \equiv MPNL^{[]}$ and $MPNL_l^S \equiv MPNL_l^{[]}$.*

Theorem 8 *The relative expressive power of the languages of the class MPNL is as depicted in Fig. 3, where each arrow means that the language at the top is strictly more expressive than the one at the bottom.*

Proof To prove this result, one can use very similar arguments based on bisimulations (and bisimulation games) as in the previous theorems, plus the equivalences in Table 4, right column, and all the above results. As an example, we present here only the proof of one case, namely $\mathrm{MPNL}^{<} \not\preceq \mathrm{MPNL}^{(),\varepsilon}$. To this end, consider, for any $t \in \mathbb{N}$, the language ${}^t\mathrm{MPNL}^{(),\varepsilon}$, defined as usual. Let $M = \langle \mathbb{D} = \mathbb{N}, \mathbb{I}(\mathbb{D}), V\rangle$, $M' = \langle \mathbb{D}' = \mathbb{N}, \mathbb{I}(\mathbb{D}'), V'\rangle$, $\mathcal{AP} = p$, $V(p) = \{[i,i],[i,i+1] \mid i \in \mathbb{N}\}$, $V'(p) = \{[i',i'],[i',(i+1)'] \mid i \in \mathbb{N}\} \setminus \{[a',a']\}$, where $a = N(t+2N)$, and consider the relation $Z = \{([i,j],[k',l']) \mid \delta(i,j) = \delta(k,l)$ and $[k,l] \neq [a,a]\} \cup \{([a,a],[a',(a+1)']), ([a,a],[(a-1)',a'])\} \cup \{([i,i+2],[a',a']) \mid i \in \mathbb{N}\}$. It is possible to show that Z represents a winning strategy for Player II at the initial configuration $([a,b],[a',b'])$ (for any b) in the N-moves bisimulation game for ${}^t\mathrm{MPNL}^{(),\varepsilon}$. But $M,[a,b] \Vdash \Diamond_o^{<1} p$ and $M',[a',b'] \not\Vdash \Diamond_o^{<1} p$, which means that the formula $\Diamond_o^{<1} p$ cannot be expressed in ${}^t\mathrm{MPNL}^{(),\varepsilon}$ for any $t,N \in \mathbb{N}$. Thus, we have the result. $\square$

$\Diamond_o^{\geq k}\psi \Leftrightarrow$	$\Diamond_o^{<1}\psi \vee \Diamond_o^{>0}\psi$	$k=0$	$\Diamond_o^{<k}\psi \Leftrightarrow$	$\Diamond_o^{[0,k-1]}\psi$	$k>0$
	$\Diamond_o^{>k-1}\psi$	$k>0$		$\bot$	$k=0$
$\Diamond_o^{(k,k')}\psi \Leftrightarrow$	$\Diamond_o^{=k+1}\psi \vee \ldots \vee \Diamond_o^{=k'-1}\psi \vee \bot$	$k \neq \infty$	$\Diamond_o^{>k}\psi \Leftrightarrow$	$\Diamond_o^{[k+1,\infty]}\psi$	
	$\Diamond_o^{>k}\psi$	$k=\infty$	$\Diamond_o^{[k,k']}\psi \Leftrightarrow$	$\Diamond_o(\mathsf{len}_{\geq k} \wedge \mathsf{len}_{\leq k'} \wedge \psi)$	$k' \neq \infty$
$\Diamond_o^{[k,k']}\psi \Leftrightarrow$	$\Diamond_o^{(k-1,k'+1)}\psi$	$k>0, k' \neq \infty$		$\Diamond_o(\mathsf{len}_{\geq k} \wedge \psi)$	$k'=\infty$
	$\Diamond_o^{<k'+1}\psi$	$k=0, k' \neq \infty$	$\Diamond_o^{=k}\psi \Leftrightarrow$	$\Diamond_o(\mathsf{len}_{=k} \wedge \psi)$	
	$\Diamond_o^{(k-1,k')}\psi$	$k>0, k'=\infty$	$\Diamond_o^{(k,k')}\psi \Leftrightarrow$	$\Diamond_o(\mathsf{len}_{>k} \wedge \mathsf{len}_{<k'} \wedge \psi)$	$k' \neq \infty$
	$\Diamond_o^{(k,k')}\psi \vee \Diamond_o^{<1}\psi$	$k=0, k'=\infty$		$\Diamond_o(\mathsf{len}_{>k} \wedge \psi)$	$k'=\infty$
$\Diamond_o^{\geq k}\psi \Leftrightarrow$	$\Diamond_o\psi$	$k=0$			
	$\Diamond_o^{>k-1}\psi$	$k>0$			

Table 4 More equivalences between metric operators, $o \in \{r, l\}$.

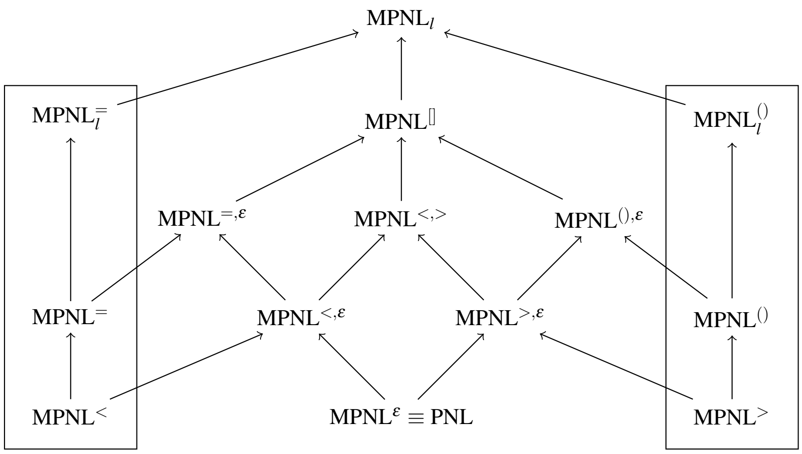

Fig. 3 Relative expressive power of the metric languages belonging to MPNL. Fragments inside the boxes belong to wMPNL (see Fig. 2).

PNL$^\pi$	NEXPTIME complete	FO$^2[=,<]$ [8]	NEXPTIME complete [28]
MPNL	2NEXPTIME, NEXPTIME hard	FO$^2_r[\mathbb{N},=,<,s]$	3NEXPTIME, NEXPTIME hard
MPNL$^+$	undecidable	FO$^2[\mathbb{N},=,<,s]$	undecidable

Table 5 Complexity and expressive completeness results.

8 Concluding remarks

In this paper we have presented and studied metric extensions of Propositional Neighborhood Logic over the interval structure of natural numbers $\mathbb{N}$. We have demonstrated that these are expressive and natural languages to reason about that structure by proving the complexity and expressive completeness results summarized in Table 5. First, we have considered the most expressive language of this class, MPNL, and shown the decidability of its satisfiability problem. Then, we have considered an appropriate fragment, called FO$^2_r[\mathbb{N},=,<,s]$, of FO$^2[\mathbb{N},=,<,s]$, that is, the two-variable fragment of first order logic with equality, order, successor, and any family of binary relations, interpreted on the structure of natural numbers, and have proved that MPNL is expressively complete for it. As a consequence, we have obtained a decidability result for FO$^2_r[\mathbb{N},=,<,s]$. We have then showed how to extend MPNL to obtain an interval logic expressively complete for the entire FO$^2[\mathbb{N},=,<,s]$, which we have proved to be undecidable. Finally, we have discussed the variety of metric logics and their expressiveness. The results obtained here are amenable to some fairly straightforward generalizations, e.g., from $\mathbb{N}$ to $\mathbb{Z}$.

One important open problem directly is the exact complexity of MPNL, when constraints are represented in binary, and the identification of the fragment(s) of MPNL where the complexity jumps occur. Another one is to identify more precisely the (un)decidability border amongst the family of MPNL logics.

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