Right Propositional Neighborhood Logic over Natural Numbers with Integer Constraints for Interval Lengths

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Outline

1. Interval Temporal Logics
2. RPNL + INT
3. Decidability of RPNL+INT
4. Future work

Time and logics

Studying time and its structure is of great importance in **computer science**:

- **Artificial Intelligence.**
  Planning, Natural Language Recognition, ...
- **Databases.**
  Temporal Databases.
- **Formal methods.**
  Specification and Verification of Systems and Protocols, Model Checking, ...

Points vs. Intervals

Usually, time is formalized as a set of **time points** without duration.

But... this concept is extremely abstract:
time is usually viewed as a set of **intervals** (periods) with a duration.

Problem

*It would be nice to have **temporal logics** that take time intervals as primary objects.*

What is an interval?

Definition

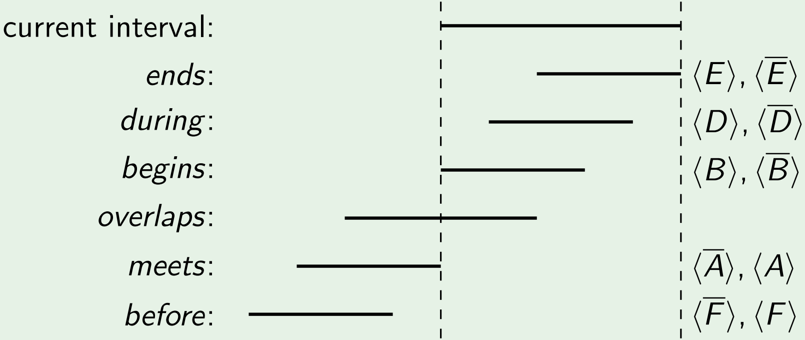
Given a linear order $\mathbb{D} = \langle D, < \rangle$:

- an interval in $\mathbb{D}$ is a pair $[d_0, d_1]$ such that $d_0 < d_1$;
- $\mathbb{I}(\mathbb{D})$ is the set of all intervals on $\mathbb{D}$;
- $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ is an interval structure.

- We consider intervals as pairs of time points.
- A point $d \in D$ belongs to $[d_0, d_1]$ if $d_0 \leq d \leq d_1$.

Allen's binary relations

There are 13 different binary relations between intervals:



together with their inverses.

Between points we have only three binary relations!

Interval temporal logics

- Interval temporal logics, such as HS, CDT, and PITL, are very expressive (compared to point-based temporal logics)
- Most interval temporal logics are (highly) undecidable

Problem

Find expressive, but decidable, interval temporal logics.

A simple path to decidability

Interval logics make it possible to express properties of pairs of time points rather than of single time points.

How has decidability been achieved? By imposing suitable syntactic and/or semantic restrictions that allow one to reduce interval logics to point-based ones:

- **Constraining interval modalities**
  - $\langle B \rangle \langle \overline{B} \rangle$ and $\langle E \rangle \langle \overline{E} \rangle$ fragments of HS.
- **Constraining temporal structures**
  - Split Logics: any interval can be chopped in at most one way (Split Structures).
- **Constraining semantic interpretations**
  - Local QPITL: a propositional variable is true over an interval if and only if it is true over its starting point (Locality Principle).

An alternative path to decidability

A major challenge

Identify expressive enough, yet decidable, logics which are genuinely interval-based.

What is a genuinely interval-based logic?

An logic is genuinely interval-based if it cannot be directly translated into a point-based logic and does not invoke locality, or any other semantic restriction reducing the interval-based semantics to the point-based one.

Known decidability results

The picture of decidable/undecidable non-metric interval logics is almost complete

- Propositional Neighborhood Logic is the first discovered decidable genuine interval logic
- the subinterval logic D
- the logic $AB\overline{B}\overline{A}$ is the most expressive and decidable
- the vast majority of all other fragments is undecidable
- no previous known results for metric extension of any interval logic

We will present a decidability proof for a metric extension of Right PNL over natural numbers

Outline

A simple metric interval logic

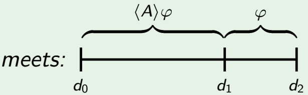
Right Propositional Neighborhood Logic with integer constraints

Syntax of RPNL+INT

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle A\rangle\varphi \mid \ell < k \mid \ell = k \mid \ell > k$$

Semantics

RPNL is based on the right neighborhood operator *meets*:



Metric formulas constraint the length of the current interval:

$$\ell \gtreqless k \text{ holds over } [d_0, d_1] \text{ iff } d_1 - d_0 \gtreqless k$$

The leaking gas burner

- Every time the flame is ignited, a small amount of gas can leak from the burner.
- The propositional letter *Gas* is used to indicate the gas is flowing.
- The propositional letter *Flame* is true when the gas is burning.

Safety of the gas burner:

1. It is never the case that the gas is leaking for more than 2 seconds.
2. The gas burner will not leak for 30 seconds after the last leakage.

Safety of the gas burner in RPNL+INT

Universal modality: φ holds everywhere in the future

$$[G]\varphi ::= \varphi \wedge [A]\varphi \wedge [A][A]\varphi$$

Leaking = gas flowing but not burning

$$[G](\mathit{Leak} \leftrightarrow \mathit{Gas} \wedge \neg \mathit{Flame})$$

Safety properties:

1. $[G](\mathit{Leak} \rightarrow \ell \leq 2)$
2. $[G](\mathit{Leak} \rightarrow \neg\langle A\rangle(\ell < 30 \wedge \langle A\rangle \mathit{Leak}))$

RPNL+INT is simple but powerful

RPNL is expressive enough to encode some metric form of Until:

"q holds exactly k time units in the future, while p holds at every shorter-than-k interval"

$$\langle A\rangle(\ell = k \wedge \langle A\rangle q) \wedge [A](\ell < k \rightarrow p)$$

Unbounded until is not expressible in RPNL+INT.

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Definition

An atom is a maximal, locally consistent set of subformulae of φ.

A relation connecting atoms

Connect every pair of atoms that can be associated with neighbor intervals:

$$A \, \mathrm{R}_\varphi \, B \quad \text{iff} \quad [A]\psi \in A \Rightarrow \psi \in B$$

Labelled Interval Structures

Definition

A (fulfilling) Labelled Interval Structure (LIS) is a pair $\langle \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ where:

- $\mathbb{I}(\mathbb{D})$ is the set of intervals over $\mathbb{D}$;
- the labelling function $\mathcal{L}$ assigns an atom to every interval $[d_i, d_j]$;
- metric formulae in $\mathcal{L}([d_i, d_j])$ are consistent with respect to the interval length;
- atoms assigned to neighbor intervals are related by R_φ;
- for every $[d_i, d_j]$ and $\langle A \rangle \psi \in \mathcal{L}([d_i, d_j])$ there exists $d_k > d_j$ such that $\psi \in \mathcal{L}([d_j, d_k])$.

Theorem

A formula φ is satisfiable if and only if there exists a (fulfilling) LIS $\langle \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ and an interval $[d_i, d_j]$ such that $\varphi \in \mathcal{L}([d_i, d_j])$.

A small-model theorem for LIS

- We have reduced the satisfiability problem for PNL to the problem of finding a (fulfilling) LIS for φ.
- LIS can be of arbitrary size and even infinite!

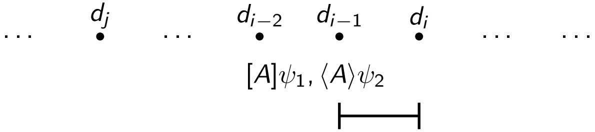
Problems

- How to bound the size of finite LIS?
- How to finitely represent infinite LIS?

Solution

Any large (resp., infinite) model can be turned into a bounded (resp., bounded periodic) one by progressively removing exceeding points

The set of requests of a point



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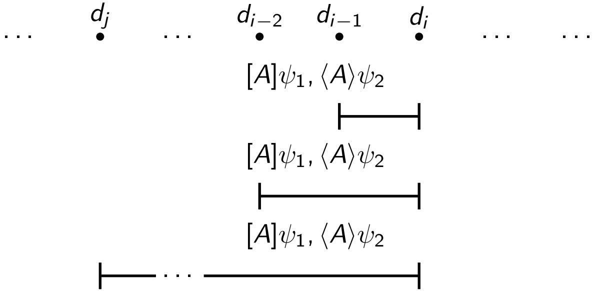

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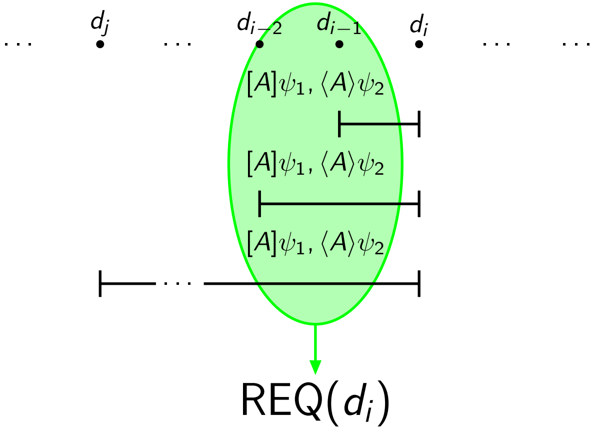
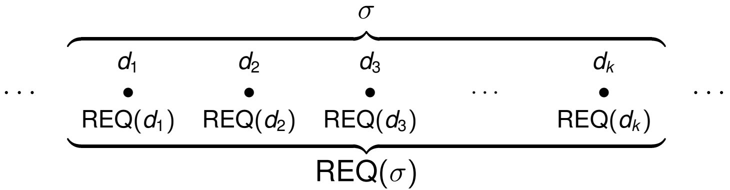


k-sequences of requests

Given a formula φ, let k be the greatest constant that appears in φ.

Definition

Given a LIS, a (k-)sequence is a sequence of (k) consecutive points. Given a sequence σ, its sequence of requests $REQ(\sigma)$ is defined as the sequence of temporal requests at the points in σ.



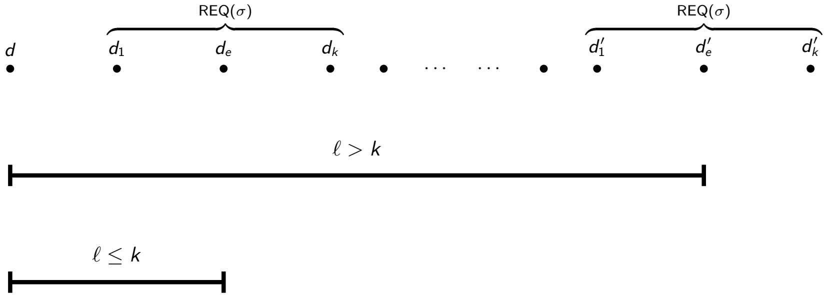
Removing k-sequences from a LIS

Lemma

- *Let* m *be the number of* $\langle A \rangle$*-subformulae of* φ *and* r *the number of possible sets of requests* REQ.
- *Let* $\langle \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ *be a (fulfilling) LIS for* φ *and* $\text{REQ}(\sigma)$ *be a* k*-sequence of request that occurs more than* $m \cdot r + 1$ *times.*

$\Rightarrow$ *We can remove one occurrence of* $\text{REQ}(\sigma)$ *from the LIS in such a way that the resulting LIS is still fulfilling.*

The removal process: fixing the length of intervals



- Remove all points up to the next occurrence of REQ(σ)
- Some intervals became shorter, and do not respect metric formulas anymore
- Since REQ(d_e) = REQ(d'_e), we can relabel problematic intervals

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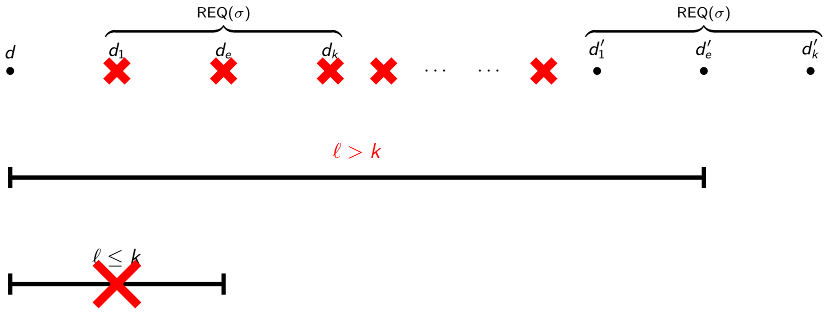

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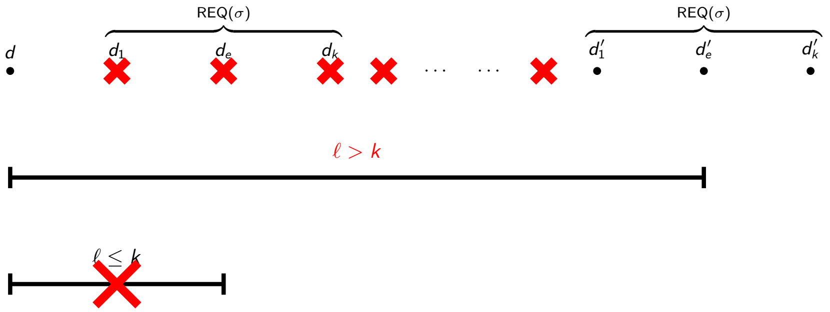

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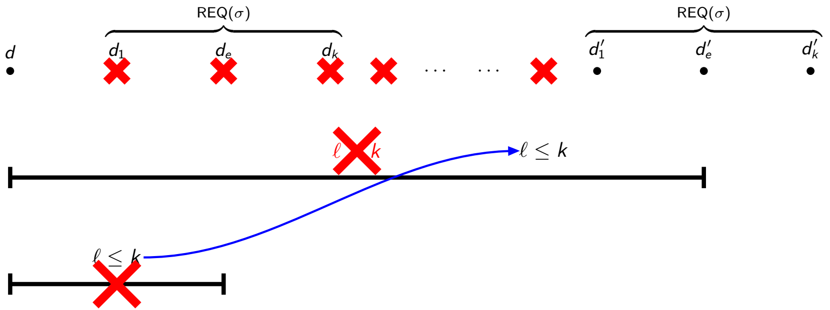

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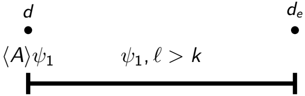
The removal process: fixing defects on $\langle A \rangle$-formulae



m points on the right of d_e
with the same set of requests of d_e

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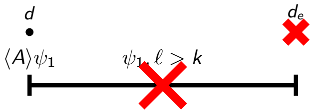

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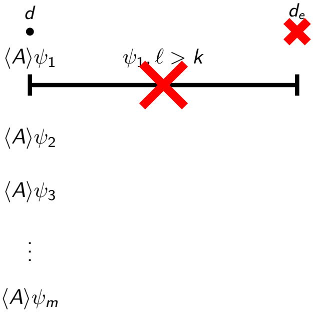

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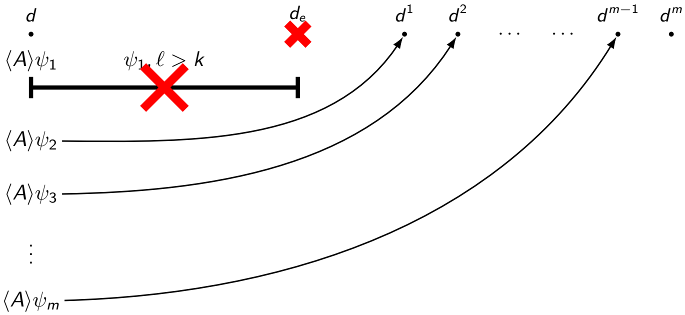

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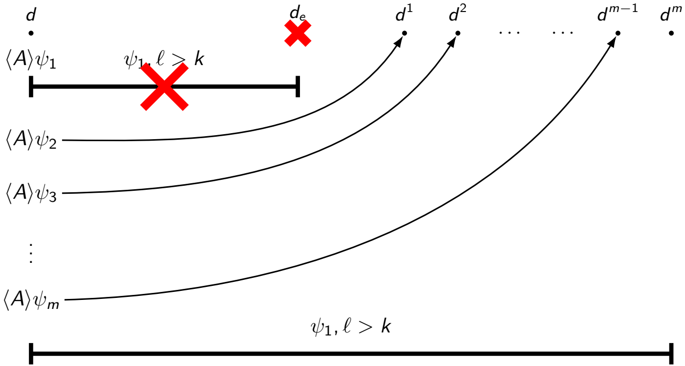

m points on the right of d_e

with the same set of requests of d_e

The small model theorem for RPNL+INT

By taking advantage of such a removal process, we can prove the following theorem:

Theorem

A formula φ is satisfiable if and only if there exists a LIS $\langle \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ such that:

- *if $\mathbb{D}$ is finite, then every k-sequence of requests occurs at most $m \cdot r + 1$ times in $\mathbb{D}$;*
- *if $\mathbb{D}$ is infinite, then the LIS is ultimately periodic with prefix and period bounded by $r^k \cdot m \cdot r \cdot k + k + 1$.*

Decidability and complexity

- "Plain" RPNL is known to be NEXPTIME-complete.
- A model for an RPNL+INT formula φ can be obtained by a non-deterministic decision procedure that uses space $O(k \cdot 2^n$ and time $O(2^{k \cdot n})$.
- The k-corridor tiling problem can be encoded by a formula that is polynomial in k and n.

The complexity depends on k!

The exact complexity class depens on how k is encoded:

- k is a constant: $k = O(1)$
  RPNL+INT is NEXPTIME-complete
- k is encoded in unary: $k = O(n)$
  RPNL+INT is NEXPTIME-complete
- k is encoded in binary: $k = O(2^n)$
  RPNL+INT is EXPSPACE-complete

Outline

Future work

- **Conclude the study of PNL with metric operators:**
  - adapt the small model theorem for full PNL
  - decidability over other classes of linear orderings (e.g. $\mathbb{R}$)
- **Decidability/undecidability of other Metric Interval Logics:**
  - the sub-interval logic $\langle D \rangle$
  - the logic $A\,B\,\overline{B}\,\overline{A}$
  - other combinations of Allen's relations
- **Model Checking of Metric Interval logics:**
  - no known results