What's decidable about Halpern and Shoham's interval logic? The maximal fragment $AB\bar{B}\bar{L}$

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***Abstract*—The introduction of Halpern and Shoham's modal logic of intervals (later on called HS) dates back to 1986. Despite its natural semantics, this logic is undecidable over all interesting classes of temporal structures. This discouraged research in this area until recently, when a number of non-trivial decidable fragments have been found. This paper is a contribution toward the complete classification of HS fragments. Different combinations of Allen's interval relations *begins* (B), *meets* (A), and *later* (L), and their inverses $\bar{A}$, $\bar{B}$, and $\bar{L}$, have been considered in the literature. We know from previous work that the combination $AB\bar{B}\bar{A}$ is decidable over finite linear orders and undecidable everywhere else. We extend these results by showing that $AB\bar{B}\bar{L}$ is decidable over the class of all (resp., dense, discrete) linear orders, and that it is maximal with respect to decidability over these classes: adding any other interval modality immediately leads to undecidability.**

I. Introduction

Interval temporal logics are quite expressive modal logics for temporal representation and reasoning based on time intervals instead of time points. Their introduction dates back to 1986, when Halpern and Shoham's modal logic of intervals (HS) was proposed [1]. HS allows one to express all possible ordering relations between any pair of intervals (the so-called Allen's interval relations [2]), and it features one modal operator for each of them (obviously, no modal operator is needed for the equality relation), that is, $\langle A\rangle$ for *meets*, $\langle B\rangle$ for *begins*, $\langle E\rangle$ for *finishes*, $\langle O\rangle$ for *overlaps*, $\langle D\rangle$ for *during*, and $\langle L\rangle$ for *later*, plus the operators $\langle \overline{A}\rangle$, $\langle \overline{B}\rangle$, $\langle \overline{E}\rangle$, $\langle \overline{O}\rangle$, $\langle \overline{D}\rangle$, and $\langle \overline{L}\rangle$ for the inverse relations (in fact, some operators are definable in terms of the others). Unfortunately, as already pointed out by Halpern and Shoham in their original contribution [1], the satisfiability problem for HS turns out to be undecidable over all interesting classes of temporal structures. Fifteen years later [3], Lodaya proved that a suitable sharpening of the reduction technique from [1] can be exploited to prove the undecidability of BE, that is, the fragment of HS only featuring the pair of operators $\langle B\rangle$ and $\langle E\rangle$, over dense linear orders (from now on, we will always denote by $X_1 \dots X_n$ the fragment of HS featuring the modalities $\langle X_1\rangle \dots \langle X_n\rangle$). Since density is expressible in BE by a constant formula, it immediately follows that BE is undecidable over the class of all linear orders as well [4]. Since then, a number of undecidability results for simple fragments of HS, many of them featuring only two, or even one, operators has been obtained, e.g., [5], [6], [7]. All together, these results disclose a landscape of interval temporal logics where undecidability is the the rule and decidability the exception. As an example, decidability of $B\bar{B}$ and $E\bar{E}$ over all classes of interval structures can be easily proved, as shown in [4]; however, any other combination of these four operators turns out to be undecidable [5], [3].

Such a situation discouraged research in the area until recently, when some meaningful decidable fragments of HS have been identified. Among them, we mention the fragments $D\bar{D}$ and $A\bar{A}$. In [8], Montanari et al. introduce a spatial modal logic based on cone-shaped cardinal directions over the rational plane (*cone logic* for short) and they prove PSPACE-completeness of its satisfiability problem. Moreover, they show that the decidability of $D\bar{D}$, interpreted over the rational line, can be easily derived from that of cone logic. Decidability of $A\bar{A}$ over a number of interesting classes has been proved by its reduction to the satisfiability problem for the two-variable fragment of first-order logic over ordered domains [9]. Its single-modality fragment A [10] has been later extended to the fragment $AB\bar{B}$ (and, by symmetry, its single-modality fragment $\bar{A}$ to $\bar{A}E\bar{E}$), which has been proved to be decidable when interpreted over natural numbers [11]. The problem of precisely defining the boundary between decidability and undecidability, that is, to identify maximal decidable fragments of HS, is definitely not trivial. Not surprisingly, a few such results can be found in the literature. In [12], Montanari et al. improve the result given in [8] by proving PSPACE-completeness and maximality with respect to decidability of the fragment $B\bar{B}D\bar{D}L\bar{L}$ over the rational line. In [13], the satisfiability problem for $AB\bar{B}\bar{A}$ (resp., $AE\bar{E}\bar{A}$), interpreted over finite linear orders, has been shown to be decidable, but not primitive recursive. Moreover, the authors prove that the addition of any other modalities from the HS repository immediately leads to undecidability, thus showing its maximality with respect to decidability. In addition, they show that the satisfiability problem for $AB\bar{B}\bar{A}$ (resp., $AE\bar{E}\bar{A}$) becomes undecidable as soon as it is interpreted over classes of linear orders that contain at least one linear order with an infinitely ascending (resp., descending) sequence, thus including the natural time flows $\mathbb{N}$ (resp., $\mathbb{Z} \setminus \mathbb{N}$), $\mathbb{Z}$, and $\mathbb{R}$ (in fact, $\bar{A}B$ and $\bar{A}\bar{B}$, resp., AE and $A\bar{E}$, are already undecidable over these classes of linear orders).

$\langle A \rangle$	$[a,b]R_A[c,d] \Leftrightarrow b = c$
$\langle \bar{L} \rangle$	$[a,b]R_{\bar{L}}[c,d] \Leftrightarrow d < a$
$\langle B \rangle$	$[a,b]R_B[c,d] \Leftrightarrow a = c, d < b$
$\langle \bar{B} \rangle$	$[a,b]R_{\bar{B}}[c,d] \Leftrightarrow a = c < b < d$

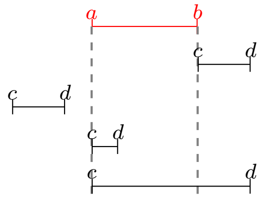

TABLE I
ALLEN'S RELATIONS AND CORRESPONDING HS MODALITIES.

In this paper, we focus our attention on the logic $AB\bar{B}\bar{L}$ (resp., $\bar{A}E\bar{E}L$). An ordered pair of intervals (I, I') satisfies the Allen's relation *later* if the ending point of I strictly precedes the starting point of I'. As we will show in Section II, the corresponding operator $\langle L \rangle$ can be easily defined in terms of $\langle A \rangle$ and $\langle B \rangle$ (or, equivalently, $\langle A \rangle$ and $\langle E \rangle$), $\langle B \rangle$ (resp., $\langle E \rangle$) being used to capture (non-)point intervals. The same holds for the transposed operators $\langle \overline{L} \rangle$ and $\langle \overline{A} \rangle$. Thus, decidability of $AB\bar{B}\bar{L}$ (resp., $\bar{A}E\bar{E}L$) over finite linear orders immediately follows from results in [13]. On the contrary, those results say nothing about the decidable/undecidable status of $AB\bar{B}\bar{L}$ when interpreted over infinite linear orders. In the following, we consider the satisfiability problem for $AB\bar{B}\bar{L}$ interpreted over the classes of *(i)* all linear orders, *(ii)* dense linear orders, and *(iii)* discrete linear orders. As for the latter, we distinguish between *strong* (there exists a finite number of points in between any given pair of points) and *weak* (if a point has a successor, resp., predecessor, then it has an immediate successor, resp., predecessor) discreteness. In [14], Bresolin et al. prove that $AB\bar{B}\bar{L}$ (resp., $\bar{A}\bar{E}EL$), interpreted over the integers (in fact, over the class of strongly discrete linear orders), is decidable. Unfortunately, the proof heavily relays on strong discreteness and it cannot be adapted to the other classes of linear orders. We prove that the satisfiability problem for $AB\bar{B}\bar{L}$ over all linear orders is EXPSPACE-complete, using a completely different proof technique based on a regular tree decomposition of (infinite) models. Decidability of $AB\bar{B}\bar{L}$ over the class of dense linear orders immediately follows, as density can be expressed in $AB\bar{B}\bar{L}$ by a constant formula. This is not the case with weakly discrete linear orders. However, we show that the proof can be adapted to cope with them. As a by-product, we solve some open problems about $AB\bar{B}$ (resp., $\bar{A}E\bar{E}$). More interestingly, pairing the results given in this paper with those in [13], it immediately follows that, over the considered classes of linear orders, $\langle \overline{A} \rangle$ cannot be defined in terms of $\langle \overline{L} \rangle$ and $\langle B \rangle$ (the operator $\langle \overline{B} \rangle$ does not help in this respect). The same holds for $\langle A \rangle$, $\langle L \rangle$, and $\langle E \rangle$. Furthermore, thanks to the undecidability results reported in [5], [6], [13], we can conclude that the addition to $AB\bar{B}\bar{L}$ of any other interval modality immediately leads to undecidability. Hence, $AB\bar{B}\bar{L}$ turns out to be maximal with respect to decidability over all interesting classes of linear orders, but that of finite ones.

The paper is organized as follows. In Section II, we give syntax and semantics of $AB\bar{B}\bar{L}$ and we provide a spatial interpretation of interval temporal structures that will prove itself extremely useful in decidability proofs. In Section III, we first prove the decidability of the fragment $AB\bar{B}$, interpreted over all linear orders, and then we show how to generalize the proof to full $AB\bar{B}\bar{L}$. We also show how to adapt the proof to the case of (weakly) discrete linear orders. Complexity issues are dealt with in Section IV. Conclusions provide an assessment of the work done.

II. THE INTERVAL TEMPORAL LOGIC $AB\bar{B}\bar{L}$

In this section, we briefly introduce syntax and semantics of $AB\bar{B}\bar{L}$, together with some examples of its application to the specification of temporal properties, and the basic notions of atom, type, and dependency. In addition, we provide an alternative interpretation of $AB\bar{B}\bar{L}$ over labeled grid-like structures.

A. Syntax and semantics

The logic $AB\bar{B}\bar{L}$ features the four modal operators $\langle A \rangle$, $\langle B \rangle$, $\langle \bar{B} \rangle$, and $\langle \bar{L} \rangle$, and it is interpreted in interval temporal structures over a linear order endowed with the four Allen's relations A ("meets"), B ("begins"), $\bar{B}$ ("begun by"), and $\bar{L}$ ("before"). A graphical account of Allen's relations $A, \bar{L}, B$, and $\bar{B}$ and of the corresponding HS modalities is given in Table I.

Given a set $\mathcal{P}rop$ of propositional variables, formulas of $AB\bar{B}\bar{L}$ are built up from $\mathcal{P}rop$ using the Boolean connectives $\neg$ and $\vee$, and the unary modal operators $\langle A \rangle$, $\langle B \rangle$, $\langle \bar{B} \rangle$, and $\langle \bar{L} \rangle$. As usual, we take advantage of the shorthands $\top$ (true) for $p \vee \neg p$, with $p \in \mathcal{P}rop$, $\varphi_1 \wedge \varphi_2$ for $\neg(\neg\varphi_1 \vee \neg\varphi_2)$, $[A]\varphi$ for $\neg\langle A \rangle\neg\varphi$, $[B]\varphi$ for $\neg\langle B \rangle\neg\varphi$, and so on. Moreover, we will use π as a shorthand for $\neg\langle B \rangle\top$, that is, π holds over all and only point intervals. Hereafter, we denote by $|\varphi|$ the size of φ. Given a linear order $\mathbb{O} = \langle O, < \rangle$, we define $\mathbb{I}_{\mathbb{O}}$ as the set of all closed intervals $[x, y]$, with $x, y \in O$ and $x \leq y$. For any pair of intervals $[x, y], [x', y'] \in \mathbb{I}_{\mathbb{O}}$, Allen's relations "meets" A, "begins" B, "begun by" $\bar{B}$, and "before" $\bar{L}$ are defined as follows:

- **"meets" relation:** $[x, y]\ A\ [x', y']$ iff $y = x'$;
- **"begins" relation:** $[x, y]\ B\ [x', y']$ iff $x = x'$ and $y' < y$;
- **"begun by" relation:** $[x, y]\ \bar{B}\ [x', y']$ iff $x = x'$ and $y < y'$;
- **"before" relation:** $[x, y]\ \bar{L}\ [x', y']$ iff $y' < x$.

We define an *interval structure* as a tuple $\mathcal{S} = (\mathbb{I}_{\mathbb{O}}, A, B, \bar{B}, \bar{L}, \sigma)$, where $\sigma : \mathbb{I}_{\mathbb{O}} \to \mathscr{P}(\mathcal{P}rop)$ is a labeling function that maps intervals in $\mathbb{I}_{\mathbb{O}}$ to sets of propositional variables. Given an interval structure $\mathcal{S}$ and an interval $I = [x, y]$, we define the semantics of an $AB\bar{B}\bar{L}$ formula as follows:

- $\mathcal{S}, I \vDash p$ iff $p \in \sigma(I)$, for any $p \in \mathcal{P}rop$;
- $\mathcal{S}, I \vDash \neg\varphi$ iff $\mathcal{S}, I \nvDash \varphi$;
- $\mathcal{S}, I \vDash \varphi_1 \vee \varphi_2$ iff $\mathcal{S}, I \vDash \varphi_1$ or $\mathcal{S}, I \vDash \varphi_2$;
- $\forall R \in \{A, B, \bar{B}, \bar{L}\}$, $\mathcal{S}, I \vDash \langle R \rangle\varphi$ iff there exists $J \in \mathbb{I}_{\mathbb{O}}$ such that $I\ R\ J$ and $\mathcal{S}, J \vDash \varphi$.

Given an interval structure $\mathcal{S}$ and a formula φ, we say that $\mathcal{S}$ *satisfies* φ (and hence φ is *satisfiable*) if $\mathcal{S}, I \vDash \varphi$ for some interval I in $\mathcal{S}$. We define the *satisfiability problem* for $AB\bar{B}\bar{L}$ as the problem of establishing whether a given $AB\bar{B}\bar{L}$-formula φ is satisfiable or not.

We conclude the section with a little discussion of the relationships between the modalities for Allen's relations *before* and *met-by* . Let $\langle\bar{A}\rangle$ be the modal operator corresponding to the *met-by* relation. It can be easily checked that $\mathcal{S}, I \models \langle\bar{L}\rangle\varphi$ if and only if $\mathcal{S}, I \models \langle\bar{A}\rangle(\neg\pi \wedge \langle\bar{A}\rangle\varphi)$, and thus $\langle\bar{L}\rangle$ turns out to be definable in the language of $AB\bar{B}\bar{A}$ [1]. Given the undecidability results in [13], the undefinability of $\langle\bar{A}\rangle$ in $AB\bar{B}\bar{L}$, over the considered classes of linear orders, is an immediate consequence of the decidability results we will prove in Section III. As it is apparent that $\langle\bar{L}\rangle$ cannot be defined in $AB\bar{B}$ ($\langle A\rangle$, $\langle B\rangle$, and $\langle\bar{B}\rangle$ only allow one to refer to intervals whose left endpoint is greater than or equal to that of the initial interval), this allows us to properly locate $AB\bar{B}\bar{L}$ in between $AB\bar{B}$ and $AB\bar{B}\bar{A}$: it is strictly more expressive than the former and strictly less expressive than the latter. The very same conclusion can be drawn about the relationships among $\bar{A}\bar{E}EL$, $\bar{A}E\bar{E}$, and $AE\bar{E}\bar{A}$.

B. $AB\bar{B}\bar{L}$ at work

Finding an optimal balance between expressiveness and (computational) complexity is a challenge for every temporal logic. In the following, we provide some examples showing that $AB\bar{B}\bar{L}$ is expressive enough to specify a number of meaningful temporal requirements.

First, it allows one to model telic statements, that is, statements (verbs) that denote goals. As an example, the formula $\langle A\rangle\big(\varphi \wedge [B](\neg\varphi \wedge [A]\neg\varphi) \wedge [\bar{B}]\neg\varphi\big)$ can be used to express accomplishments (think of formula φ as the assertion: "The airplane flew from Venice to Toronto"). Moreover, formulas of point-based temporal logics of the form $\psi U \varphi$, using the standard until operator, can be encoded in $AB\bar{B}\bar{L}$ as follows: $\langle A\rangle\big(\pi \wedge \varphi\big) \vee \langle A\rangle\big(\langle A\rangle(\pi \wedge \varphi) \wedge [B](\langle A\rangle(\pi \wedge \psi))\big)$.

If we restrict ourselves to discrete linear orders, we can use the operator $[B]$ to constrain the length of intervals. As an example, the safety constraint imposing that the gas burner will not leak uninterruptedly for (at least) k time units after the last leakage, borrowed from the well-known *gas-burner* example [15], can be formalized as follows: $[G](Leak \rightarrow \neg\langle A\rangle(\underbrace{[B]\ldots[B]}_{k+1}\perp \wedge \langle A\rangle Leak))$, where $[G]\psi$ is a shorthand for the formula $\psi \wedge [\bar{L}](\psi \wedge [A]\psi) \wedge [B](\psi \wedge [A]\psi) \wedge [\bar{B}](\psi \wedge [A]\psi) \wedge [A]\psi$ (the derived operator $[G]$ thus being the *global* modality).

When interpreted over dense linear orderings, $AB\bar{B}\bar{L}$ can be used to specify properties of continuous and hybrid systems (systems mixing continuous and discrete behaviors). Consider the case of an autonomous vehicle that moves following some reference trajectory. A typical problem is to check whether the tracking error, measured as the root-mean-square of the real trajectory with respect to the reference one, is under a certain threshold. First, to evaluate the root-mean-square error, one must take into account the entire time interval of the trajectory rather than its single points (an interval temporal logic is needed). Now, suppose that the propositional variable $Near$ holds only over the intervals where the tracking error is below the threshold. The monotonicity restriction constraining the tracking error not to increase when we extend a trajectory to the right can be expressed by the $AB\bar{B}\bar{L}$-formula $[G](Near \rightarrow [\bar{B}]Near)$. The fact that eventually the vehicle reaches a target position, following a trajectory where the tracking error is below the given threshold, can be expressed by the $AB\bar{B}\bar{L}$-formula $Source \rightarrow \langle A\rangle(Near \wedge [A]Target)$, where $Source$ and $Target$ hold only at the initial and the target position of the vehicle, respectively.

During its movement, the vehicle can hit some obstacles. Hitting an obstacle is an example of a *discrete event* in an hybrid system: it is an instantaneous event that causes a discontinuous change in the behavior of the vehicle. This can be expressed by means of the $AB\bar{B}\bar{L}$-formula $[G](Hit \rightarrow \pi)$, that forces the propositional variable Hit to hold over point intervals only. After hitting an obstacle, the vehicle is allowed to deviate from the reference trajectory to avoid it, but not too much: the tracking error should eventually decrease below the threshold again, as imposed by the the $AB\bar{B}\bar{L}$-formula $[G](Hit \rightarrow \langle A\rangle Near)$. Finally, we may constrain any deviation from the reference trajectory to be caused by a past hitting event by means of the $AB\bar{B}\bar{L}$-formula $[G](\neg Near \rightarrow \langle\bar{L}\rangle Hit)$.

C. Atoms, types, and dependencies

In this section we introduce some basic notions that will be used in the rest of the paper. Let φ be an $AB\bar{B}\bar{L}$-formula. The basic notion of *φ-atom* is defined as follows. Let the *closure* $\mathcal{C}l(\varphi)$ of φ be the set of all sub-formulas of φ and of their negations (we identify $\neg\neg\alpha$ with α, $\neg\langle A\rangle\alpha$ with $[A]\neg\alpha$, $\neg[A]\alpha$ with $\langle A\rangle\neg\alpha$, and the same for the other modal operators). For technical reasons, we define the *extended closure* $\mathcal{C}l^+(\varphi)$ as the set of all formulas in $\mathcal{C}l(\varphi)$ plus all formulas of the forms $\langle R\rangle\alpha$ and $[R]\neg\alpha$, with $R \in \{A, B, \bar{B}, \bar{L}\}$ and $\alpha \in \mathcal{C}l(\varphi)$, and the two formulas π and $\neg\pi$. A *φ-atom* is a non-empty set $F \subseteq \mathcal{C}l^+(\varphi)$ such that (i) for every $\alpha \in \mathcal{C}l^+(\varphi)$, $\alpha \in F$ iff $\neg\alpha \notin F$ and (ii) for every $\gamma = \alpha \vee \beta \in \mathcal{C}l^+(\varphi)$, $\gamma \in F$ iff $\alpha \in F$ or $\beta \in F$ (intuitively, a *φ-atom* is a maximal, *locally consistent* set of formulas chosen from $\mathcal{C}l^+(\varphi)$). Let $\mathcal{A}_\varphi$ be the set of all possible atoms that can be built over $\mathcal{C}l^+(\varphi)$. Cardinality of both $\mathcal{C}l(\varphi)$ and $\mathcal{C}l^+(\varphi)$ is *linear* in $|\varphi|$, while cardinality of $\mathcal{A}_\varphi$ may be *exponential* in $|\varphi|$ (more precisely, we have $|\mathcal{C}l(\varphi)| \leq 2|\varphi|$, $|\mathcal{C}l^+(\varphi)| \leq 18|\varphi| + 2$, and $|\mathcal{A}_\varphi| \leq 2^{9|\varphi|+1}$).

Now, let $\mathcal{S} = (\mathbb{I}_{\mathbb{O}}, A, B, \bar{B}, \bar{L}, \sigma)$ be an interval structure that satisfies φ. In order to relate intervals in $\mathcal{S}$ to the set of sub-formulas of φ they satisfy, we introduce the notion *φ-type*. For every $I \in \mathbb{I}_{\mathbb{O}}$, we define the *$\varphi$-type* of I, denoted by $\mathcal{T}ype_{\mathcal{S}}(I)$, as the set of all formulas $\alpha \in \mathcal{C}l^+(\varphi)$ such that $\mathcal{S}, I \models \alpha$. It can be easily shown that every φ-type is a φ-atom, but not vice versa. Hereafter, we shall omit the argument φ, thus calling a φ-atom (resp., a φ-type) simply an atom (resp., a type).

Given an atom F, we denote by $\mathcal{O}bs(F)$ the set of all *observables* of F, namely, the set of formulas $\alpha \in \mathcal{C}l(\varphi)$

such that $\alpha \in F$. Similarly, given an atom F and a relation $R \in \{A, B, \bar{B}, \bar{L}\}$, we denote by $\mathcal{R}eq_R(F)$ the set of all R-*requests* of F, namely, the formulas $\alpha \in \mathcal{C}l(\varphi)$ such that $\langle R \rangle \alpha \in F$. Taking advantage of the above sets, we define the relation $\xrightarrow{A}$ (resp., $\xrightarrow{B}$, $\xrightarrow{\bar{L}}$) $\subseteq \mathcal{A}_\varphi \times \mathcal{A}_\varphi$ as follows:

$$F \xrightarrow{A} G \quad \text{iff} \quad \mathcal{R}eq_A(F) = \mathcal{O}bs(G) \cup \mathcal{R}eq_B(G) \cup \cup \mathcal{R}eq_{\bar{B}}(G)$$

$$F \xrightarrow{B} G \quad \text{iff} \quad \begin{cases} \mathcal{O}bs(F) \cup \mathcal{R}eq_{\bar{B}}(F) \subseteq \mathcal{R}eq_{\bar{B}}(G) \subseteq \\ \subseteq \mathcal{O}bs(F) \cup \mathcal{R}eq_{\bar{B}}(F) \cup \mathcal{R}eq_B(F) \\ \mathcal{O}bs(G) \cup \mathcal{R}eq_B(G) \subseteq \mathcal{R}eq_B(F) \subseteq \\ \subseteq \mathcal{O}bs(G) \cup \mathcal{R}eq_B(G) \cup \mathcal{R}eq_{\bar{B}}(G) \\ \mathcal{R}eq_{\bar{L}}(F) = \mathcal{R}eq_{\bar{L}}(G) \end{cases}$$

$$F \xrightarrow{\bar{L}} G \quad \text{iff} \quad \mathcal{O}bs(G) \cup \mathcal{R}eq_{\bar{L}}(G) \subseteq \mathcal{R}eq_{\bar{L}}(F)$$

Notice that the relation $\xrightarrow{\bar{B}}$ can be defined in the very same way. However, we did not introduce it as its addition is useless: the relation $\bar{B}$ is the inverse of the relation B and thus it holds that $F \xrightarrow{\bar{B}} G$ if and only if $G \xrightarrow{B} F$.

Relations $\xrightarrow{B}$ and $\xrightarrow{\bar{L}}$ are transitive, while $\xrightarrow{A}$ is not. Moreover, the three relations satisfy a *view-to-type dependency*, namely, for every pair of intervals $I, J \in \mathbb{I}_\mathbb{O}$, and for every relation $R \in \{A, B, \bar{L}\}$, we have that $I \; R \; J$ implies $\mathcal{T}ype_\mathcal{S}(I) \xrightarrow{R} \mathcal{T}ype_\mathcal{S}(J)$.

The following proposition states two simple, but quite useful, properties of the relations $\xrightarrow{A}$, $\xrightarrow{B}$, and $\xrightarrow{\bar{L}}$ that will be exploited in the proofs of Lemma 3 and Theorem 4 below.

Proposition 1. *Let F, G, and H be three atoms in $\mathcal{A}_\varphi$. We have that:*

1) *if $F \xrightarrow{A} H$ and $G \xrightarrow{B} H$, then $F \xrightarrow{A} G$;*
2) *if $F \xrightarrow{B} G$ and $G \xrightarrow{\bar{L}} H$, then $F \xrightarrow{\bar{L}} H$.*

The proof is straightforward and thus omitted.

D. Compass structures

The logic $AB\bar{B}\bar{L}$ can be equivalently interpreted over grid-like structures, called *compass structures* (such structures have been originally proposed by Venema in [16]), by exploiting the existence of a natural bijection between the intervals $I = [x, y]$ in $\mathbb{I}_\mathbb{O}$ and the points $p = (x, y)$ of an $O \times O$ grid such that $x \leq y$. As an example, in Figure 1 we show five intervals $I_0, \ldots, I_4$, such that $I_0 \; B \; I_1$, $I_0 \; \bar{B} \; I_2$, $I_0 \; A \; I_3$, and $I_0 \; \bar{L} \; I_4$, together with the corresponding points $p_0, \ldots, p_4$ of the grid (the four Allen's relations $A, B, \bar{B}$, and $\bar{L}$ between pairs of intervals are mapped into corresponding spatial relations between points; for the sake of readability, we name the latter ones as the former ones).

Definition 1. *Given an $AB\bar{B}\bar{L}$ formula φ, a (consistent and fulfilling)* compass *(φ)*-structure *is a pair $\mathcal{G} = (\mathbb{P}_\mathbb{O}, \mathcal{L})$, where $\mathbb{P}_\mathbb{O}$ is the set of points of the form $p = (x, y)$, with $x, y \in O$ and $x \leq y$, and $\mathcal{L}$ is a function that maps every point $p \in \mathbb{P}_\mathbb{O}$ into a (φ-)atom $\mathcal{L}(p)$ in such a way that:*

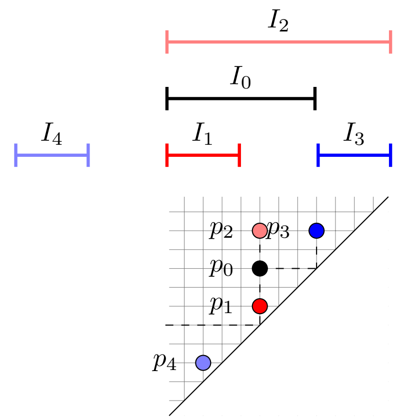

Fig. 1. Correspondence between intervals and the points of a grid.

- *for every pair of points $p, q \in \mathbb{P}_\mathbb{O}$ and every relation $R \in \{A, B, \bar{L}\}$, if $p \; R \; q$, then $\mathcal{L}(p) \xrightarrow{R} \mathcal{L}(q)$* (**consistency**)*;*
- *for every point $p \in \mathbb{P}_\mathbb{O}$, every relation $R \in \{A, B, \bar{B}, \bar{L}\}$, and every formula $\alpha \in \mathcal{R}eq_R(\mathcal{L}(p))$, there is a point $q \in \mathbb{P}_\mathbb{O}$ such that $p \; R \; q$ and $\alpha \in \mathcal{O}bs(\mathcal{L}(q))$* (**fulfillment**)*.*

We say that a compass (φ-)structure $\mathcal{G} = (\mathbb{P}_\mathbb{O}, \mathcal{L})$ *features* a formula α if there is a point $p \in \mathbb{P}_\mathbb{O}$ such that $\alpha \in \mathcal{L}(p)$. It is easy to see that an $AB\bar{B}\bar{L}$-formula φ is satisfied by some interval structure if and only if it is featured by some compass structure, and thus that the satisfiability problem for $AB\bar{B}\bar{L}$ is reducible to the problem of deciding, for any given formula φ, whether there exists a compass structure featuring φ or not.

III. Decidability of $AB\bar{B}\bar{L}$

We are now ready to prove the main result of the paper, namely, the decidability of the satisfiability problem for $AB\bar{B}\bar{L}$ when interpreted over the class of all linear orders. The decidability of $AB\bar{B}\bar{L}$ over the class of dense linear orders immediately follows, as density can be defined in $AB\bar{B}\bar{L}$ by a constant formula. Formally, given a $AB\bar{B}\bar{L}$-formula φ, we have that φ is satisfiable over the class of dense linear orders if and only if the (constant) formula $\varphi \wedge [G](\neg\pi \rightarrow \langle B \rangle \neg\pi)$ is satisfiable over the class of all linear orders. Unfortunately, we cannot apply a similar argument to weakly discrete linear orders (if a point has a successor, resp., predecessor, then it has an immediate successor, resp., predecessor), as we do not have a constant formula defining weak discreteness in $AB\bar{B}\bar{L}$. At the end of the section, we will show how to tailor the decidability proof for the general case to weakly discrete linear orders.

Let φ be an $AB\bar{B}\bar{L}$-formula. We say that a compass φ-structure $\mathcal{G} = (\mathbb{P}_\mathbb{O}, \mathcal{L})$ is *bounded* if $\mathbb{O}$ is a bounded linear order, that is, it has both a *minimum* element $\min(\mathbb{O})$ and a *maximum* element $\max(\mathbb{O})$. It is possible to show that satisfiability over arbitrary interval structures can be reduced to satisfiability over bounded compass structures. Without loss of generality, we assume $\min(\mathbb{O}) = 0$ and $\max(\mathbb{O}) = 1$, and we restrict our attention to formulas φ featured by point $(0, 0)$, that is, to formulas $\varphi \in \mathcal{L}(0, 0)$. For any given $AB\bar{B}\bar{L}$-formula

φ, let $\bar{\varphi}$ be the formula $\varphi \vee \langle \bar{B} \rangle \varphi \vee \langle \bar{B} \rangle \langle A \rangle \varphi$. Moreover, let $\#$ be a fresh propositional letter. We define the *immersion* $f_\#(\psi)$ of ψ in (the space of) $\#$ as $f_\#(\psi) = \# \wedge f'_\#(\psi)$, where:

- if $\psi = p$, then $f'_\#(\psi) = p$;
- if $\psi = \neg\gamma$, then $f'_\#(\psi) = \neg f'_\#(\gamma)$;
- if $\psi = \gamma \vee \delta$, then $f'_\#(\psi) = f'_\#(\gamma) \vee f'_\#(\delta)$;
- if $\psi = \langle R \rangle \gamma$, then $f'_\#(\psi) = \langle R \rangle f_\#(\gamma)$, for every $R \in \{A, B, \bar{B}, \bar{L}\}$.

Finally, to deal with bounded (resp., past unbounded, future unbounded, unbounded) satisfiability, we define the formula $\psi_{bounded}$ (resp., ψ_{past}, ψ_{future}, $\psi_{unbounded}$) as follows:

- $\psi_{bounded} = \bar{\psi}$;
- $\psi_{past} = \neg\# \wedge [\bar{B}]\neg\# \wedge [\bar{B}][A]\# \wedge \overline{f_\#(\psi)} \wedge [\bar{B}]\langle B \rangle \neg\pi$;
- $\psi_{future} = [G](\langle A \rangle \neg\pi \leftrightarrow \#) \wedge \overline{f_\#(\psi)} \wedge [\bar{B}](\# \rightarrow \langle \bar{B} \rangle \top)$;
- $\psi_{unbounded} = \neg\# \wedge [\bar{B}]\neg\# \wedge [\bar{B}][A](\langle A \rangle \neg\pi \leftrightarrow \#) \wedge \overline{f_\#(\psi)} \wedge [\bar{B}](\# \rightarrow \langle \bar{B} \rangle \top) \wedge [\bar{B}]\langle B \rangle \neg\pi$.

Consider, for instance, the formula ψ_{future}. It is possible to prove that ψ is satisfied by a future unbounded compass structure if and only if ψ_{future} is satisfied by a bounded compass structure.

Theorem 2. *Let φ be an $AB\bar{B}\bar{L}$ formula. It holds that φ is satisfied by some interval structure if and only if there exists a bounded compass structure $\mathcal{G} = (\mathbb{P}_\mathbb{O}, \mathcal{L})$ such that $\varphi_{bounded} \vee \varphi_{past} \vee \varphi_{future} \vee \varphi_{unbounded} \in \mathcal{L}(0,0)$.*

Hereafter, we will call the bounded compass structure of Theorem 2 a *bounded compass structure for* φ. For the sake of readability, we prove our decidability result in two steps. First, we prove that the satisfiability problem for the fragment $AB\bar{B}$ over all linear orders is decidable; then, we show how to generalize the proof to $AB\bar{B}\bar{L}$.

A. A preliminary step: decidability of $AB\bar{B}$

In the following, we first define a suitable notion of pseudo-model for a satisfiable formula of $AB\bar{B}$, and then we prove that the problem of establishing whether or not such a pseudo-model exists is decidable.

As a preliminary step, we introduce the key notion of shading. Let $\mathcal{G} = (\mathbb{P}_\mathbb{O}, \mathcal{L})$ be a compass structure. The *shading of the row y of $\mathcal{G}$* is the set $\mathcal{S}hading_\mathcal{G}(y) = \{\mathcal{L}(x, y) : x \leq y\}$, that is, the set of the atoms of all points in the row y of $\mathcal{G}$. The following lemma easily follows from the definition of shading and from the semantics of $AB\bar{B}$.

Lemma 1. *Let $\mathcal{G} = (\mathbb{P}_\mathbb{O}, \mathcal{L})$ be a compass structure and let $y \in O$. We have that $\mathcal{S}hading_\mathcal{G}(y)$ satisfies the following properties:*

(S1) *for every pair of atoms F and F' in $\mathcal{S}hading_\mathcal{G}(y)$, $\mathcal{R}eq_A(F) = \mathcal{R}eq_A(F')$;*

(S2) *there exists one and only one atom $F \in \mathcal{S}hading_\mathcal{G}(y)$ such that $\pi \in F$;*

(S3) *$\mathcal{R}eq_A(\pi(\mathcal{S}hading_\mathcal{G}(y))) = \mathcal{R}eq_{\bar{B}}(\pi(\mathcal{S}hading_\mathcal{G}(y)))$, where $\pi(\mathcal{S}hading_\mathcal{G}(y))$ is the atom whose existence and uniqueness is guaranteed by* **(S2)**.

With a little abuse of notation, we will call *shading* any set S of atoms that satisfies properties **(S1)-(S3)** of Lemma 1. Moreover, we will denote with S^- the set $S \setminus \{\pi(S)\}$.

Let φ be a satisfiable $AB\bar{B}$ formula. By Theorem 2, there exists a bounded compass structure $\mathcal{G} = (\mathbb{P}_\mathbb{O}, \mathcal{L})$ such that $\varphi \in \mathcal{L}(0,0)$.

Definition 2. *Given two shadings S_1 and S_2, we say that $M_B(S_1, S_2) \subseteq S_1 \times S_2^-$ is a* matching set *if it satisfies the following properties:*

(M1) *for every $(F, G) \in M_B(S_1, S_2)$, $G \xrightarrow{B} F$;*

(M2) *for every $F \in S_1$, there exists $G \in S_2^-$ such that $(F, G) \in M_B(S_1, S_2)$;*

(M3) *there exists one and only one element $(F, G) \in M_B(S_1, S_2)$ such that $F = \pi(S_1)$.*

Consider now the following, more restrictive, variant of the relation $\xrightarrow{B}$:

$$F \xmapsto{B} G \qquad \text{iff} \qquad \begin{cases} \mathcal{R}eq_B(F) = \mathcal{O}bs(G) \cup \mathcal{R}eq_B(G) \\ \mathcal{R}eq_{\bar{B}}(G) = \mathcal{O}bs(F) \cup \mathcal{R}eq_{\bar{B}}(F). \end{cases}$$

Note that $F \xmapsto{B} G$ implies $F \xrightarrow{B} G$, but the converse implication is not true in general.

Definition 3. *A matching set $M_B(S_1, S_2)$ is said to be* strong *if the following two additional properties hold:*

(M4) *for every $(F, G) \in M_B(S_1, S_2)$, $G \xmapsto{B} F$;*

(M5) *for every $G \in S_2^-$, there exists $F \in S_1$ such that $(F, G) \in M_B(S_1, S_2)$.*

Intuitively, a matching set connects two shadings S_1 and S_2 such that the row corresponding to S_1 is below the row corresponding to S_2. It is indeed easy to prove that, given two rows y, y' of a compass structure $\mathcal{G}$ such that $y < y'$, $S_1 = \mathcal{S}hading_\mathcal{G}(y)$, and $S_2 = \mathcal{S}hading_\mathcal{G}(y')$, the set $\{(F, G) :$ there exists $x \leq y$ such that $F = \mathcal{L}(x, y)$ and $G = \mathcal{L}(x, y')\}$ is a matching set. Moreover, if y' is the immediate successor of y in $\mathcal{G}$, that is, there are no points between y and y' in O, then it is a strong matching set (obviously, the vice versa does not hold).

To compose sequences of matching sets, we introduce the notion of *matching graph*.

Definition 4. *Given k shadings $S_1, \ldots, S_k$ and $k-1$ matching sets $M_B(S_1, S_2), \ldots, M_B(S_{k-1}, S_k)$, we define the* matching graph *$M_B(S_1, S_2) \circ \ldots \circ M_B(S_{k-1}, S_k)$ as the k-level graph such that:*

(G1) *the nodes are all pairs (F, j) such that $F \in S_j$, for $j = 1, \ldots, k$;*

(G2) *the edges are all pairs $((F, j), (G, j+1))$ such that $(F, G) \in M_B(S_j, S_{j+1})$, for $j = 1, \ldots, k-1$.*

Given a matching set $M_B(S, T)$ and a matching graph $\mathcal{M} = M_B(S_1, S_2) \circ \ldots \circ M_B(S_{k-1}, S_k)$, we say that $\mathcal{M}$ *covers* $M_B(S, T)$ if *(i)* $S_1 = S$, *(ii)* $S_k = T$, and *(iii)* for every $(F, G) \in M_B(S, T)$ there exists a path $p = (F_1, 1) \ldots (F_k, k) \in \mathcal{M}$ such that $F_1 = F$ and $F_k = G$. Given a path $p = (F_1, 1) \ldots (F_k, k)$ in a matching graph $\mathcal{M}$, we say that p is *fulfilling* if *(i)* for every $\psi \in \mathcal{R}eq_{\bar{B}}(F_1) \setminus \mathcal{R}eq_{\bar{B}}(F_k)$, there exists $2 \leq j \leq k$ such that $\psi \in \mathcal{O}bs(F_j)$, and *(ii)* for every $\psi \in \mathcal{R}eq_B(F_k) \setminus \mathcal{R}eq_B(F_1)$, there exists $1 \leq j \leq k-1$ such that $\psi \in \mathcal{O}bs(F_j)$. We say that a matching graph $\mathcal{M} = M_B(S_1, S_2) \circ \ldots \circ M_B(S_{k-1}, S_k)$ covering $M_B(S,T)$ is *fulfilling for* $M_B(S,T)$ if and only if for every $(F,G) \in M_B(S,T)$, there exists a fulfilling path $p = (F,1)\ldots(G,k) \in \mathcal{M}$.

The concepts of matching set and matching graph allow us to define the key notion of *decomposition tree* (part of a decomposition tree is graphically depicted in Figure 2).

Definition 5. *Let φ be an $AB\bar{B}$-formula. A* decomposition tree *for φ is a labeled tree $T_\varphi = \langle \mathcal{T}, \nu \rangle$ that satisfies the following properties:*

(T1) *$\mathcal{T} = \{\mathcal{N}, \downarrow_1, ..., \downarrow_m\}$ is a ranked tree of rank m, for some $m \in \mathbb{N}$ (for every node n, there exists $i \leq m$ such that n has i labeled successors $\downarrow_1(n), \ldots, \downarrow_i(n)$);*

(T2) *ν is a labeling function mapping every node $n \in \mathcal{N}$ into a tuple (S_n, T_n, M_n), where S_n and T_n are shadings, and M_n is a matching set between S_n and T_n;*

(T3) *the label of the root n_0 is a triple (S_0, T_0, M_0), such that $S_0 = \{F_0\}$, $\varphi \in F_0$, $\mathcal{R}eq_B(F_0) = \mathcal{R}eq_{\bar{L}}(F_0) = \emptyset$ and $\mathcal{R}eq_{\bar{B}}(G) = \emptyset$ for every $G \in T_0$;*

(T4) *for every node $n \in \mathcal{N}$, with $\nu(n) = (S_n, T_n, M_n)$, if M_n is a strong matching set, then n has no successors in $\mathcal{T}$;*

(T5) *for every node $n \in \mathcal{N}$, with $\nu(n) = (S_n, T_n, M_n)$, if M_n is not a strong matching set, then n has $k \leq m$ successors $n_1, \ldots, n_k$ such that a) $\nu(n_1) = (S_1, T_1, M_1)$, with $S_1 = S_n$, b) $\nu(n_k) = (S_k, T_k, M_k)$, with $T_k = T_n$, c) for every $1 \leq j \leq k-1$, $T_j = S_{j+1}$, and d) the matching graph $\mathcal{M} = M_1 \circ \ldots \circ M_k$ is fulfilling for M_n.*

A decomposition tree for a formula φ can be viewed as the unfolding of a finite graph, which provides a finite representation of a (possibly infinite) bounded compass structure.

Lemma 2 (Completeness). *Let φ be an $AB\bar{B}$-formula and $\mathcal{G} = \langle \mathbb{P}_{\mathbb{O}}, \mathcal{L} \rangle$ be a bounded compass structure for φ. Then, there exists a decomposition tree $T_\varphi = \langle \mathcal{T}, \nu \rangle$ for φ with rank $\leq 4 \cdot |\varphi| \cdot 2^{18|\varphi|+2} + 2^{9|\varphi|+1} + 1$.*

Proof: Let $\mathcal{G} = \langle \mathbb{P}_{\mathbb{O}}, \mathcal{L} \rangle$ be a bounded compass structure for φ. We show how to build step-by-step a decomposition tree $T_\varphi = \langle \mathcal{T}, \nu \rangle$ for φ by using information in $\mathcal{G}$. Given $y, y' \in O$, with $y < y'$, we define the matching set $M_B(y, y')$ between the shadings of y and y' as the set: $\{(F,G) :$ there exists $x \leq y$ such that $F = \mathcal{L}(x,y)$ and $G = \mathcal{L}(x,y')\}$.

By hypothesis, $\mathbb{O}$ is a linear order with minimum element $\min(\mathbb{O}) = 0$ and maximum element $\max(\mathbb{O}) = 1$. We start the building procedure with the one-node labeled tree $T_0 = (\{n_0\}, \nu_0)$, with $\nu_0(n_0) = (\mathcal{S}hading_{\mathcal{G}}(0), \mathcal{S}hading_{\mathcal{G}}(1), M_B(0,1))$. It can be easily checked that n_0 satisfies property **(T3)** of Definition 5. Now, let $T_i = (\mathcal{T}_i, \nu_i)$ be the labeled tree obtained at the i-th step. The labeled tree $T_{i+1} = (\mathcal{T}_{i+1}, \nu_{i+1})$ can be built as follows. For every leaf n of $\mathcal{T}_i$, with $\nu_i(n) = (\mathcal{S}hading_{\mathcal{G}}(y), \mathcal{S}hading_{\mathcal{G}}(y'), M_B(y,y'))$, such that $M_B(y,y')$ is not a strong matching set, we define the set $WitSet(y,y')$ of "witness rows" as follows:

- for every $(F,G) \in M_B(y,y')$, $WitSet(y,y')$ contains a set of rows $\{y_1^{F,G}, \ldots, y_h^{F,G}\}$ such that there exists a point $x \leq y$ and a fulfilling path $(\mathcal{L}(x,y), \mathcal{L}(x, y_1^{F,G}), \ldots, \mathcal{L}(x, y_h^{F,G}), \mathcal{L}(x,y'))$, with $F = \mathcal{L}(x,y)$ and $G = \mathcal{L}(x,y')$;
- for every $G \in \mathcal{S}hading_{\mathcal{G}}(y')^-$ such that there exists no pair $(F,G) \in M_B(y,y')$, for some atom F, $WitSet(y,y')$ contains a row y^G such that $\mathcal{L}(y^G, y') = G$.

Let $WitSet(y,y') = \{y_1 < y_2 < \ldots < y_k\}$. We add $k+1$ successors $n_1, \ldots, n_{k+1}$ to n such that $\nu_{k+1}(n_1) = (\mathcal{S}hading_{\mathcal{G}}(y), \mathcal{S}hading_{\mathcal{G}}(y_1), M_B(y,y_1))$, $\nu_{k+1}(n_{k+1}) = (\mathcal{S}hading_{\mathcal{G}}(y_k), \mathcal{S}hading_{\mathcal{G}}(y'), M_B(y_k,y'))$, and $\nu_{k+1}(n_j) = (\mathcal{S}hading_{\mathcal{G}}(y_{j-1}), \mathcal{S}hading_{\mathcal{G}}(y_j), M_B(y_{j-1},y_k))$, for every $2 \leq j \leq k$. The number of successors is bounded by $4 \cdot |\varphi| \cdot 2^{18|\varphi|+2} + 2^{9|\varphi|+1} + 1$, as (i) for every atom F, the number of requests in $\mathcal{R}eq_B(F)$ (resp., $\mathcal{R}eq_{\bar{B}}(F)$) is bounded by $2 \cdot |\varphi| + 1$, (ii) the number of distinct pairs (F,G) in a matching set is bounded by $2^{9|\varphi|+1} \cdot 2^{9|\varphi|+1} = 2^{18|\varphi|+2}$, (iii) every pair (F,G) in a matching set needs at most $4 \cdot |\varphi|$ distinct points to fulfill all $\bar{B}$-requests in F and all B-requests in G (the formula π does not force the addition of a new point), and (iv) the number of distinct atoms in the shading $\mathcal{S}hading_{\mathcal{G}}(y')^-$ is bounded by $2^{9|\varphi|+1}$ (as a matter of fact, this is a bound to the rank of the decomposition tree, as it does not depend on the considered node, and thus property **(T1)** of Definition 5 immediately follows). Moreover, by definition of $WitSet(y,y')$, the successors of n satisfy property **(T5)** of Definition 5. Finally, it can be easily checked that properties **(T2)** and **(T4)** of Definition 5 are satisfied as well.

The decomposition tree for φ is $T_\omega = \bigcup_{i=0}^{\infty} T_i$. ■

Lemma 3 (Soundness). *Let φ be an $AB\bar{B}$-formula and $T_\varphi = \langle \mathcal{T}, \nu \rangle$ be a decomposition tree for φ. Then, there exists a bounded compass structure $\mathcal{G} = \langle \mathbb{P}_{\mathbb{O}}, \mathcal{L} \rangle$ for φ.*

Proof: Let $T_\varphi = \langle \mathcal{T}, \nu \rangle$ be a decomposition tree for φ. We exploit information provided by T_φ to build a (possibly infinite) sequence of finite compass structures $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \ldots$, whose (possibly infinite) union $\mathcal{G}_\omega = \bigcup_{i=0}^{\infty} \mathcal{G}_i$ is a bounded compass structure for φ.

Let us introduce the notation we are going to use. Let $\mathcal{G}_i = \langle \mathbb{P}_{\mathbb{O}_i}, \mathcal{L}_i \rangle$ be the compass structure generated at the i-th step and let $\mathbb{O}_i = \{y_0 < \ldots < y_k\}$. We define a function $f_i : \{y_0 < \ldots < y_{k-1}\} \mapsto \mathcal{T}$ that maps every row y, but the maximum one, to a node of the decomposition tree. At every step, we guarantee that the following invariant holds:

(INV) for every row $y_j < y_k$, if $\nu(f_i(y_j)) = (S_j, S_{j+1}, M_j)$, then $\mathcal{S}hading_{\mathcal{G}_i}(y_j) \subseteq S_j$, $\mathcal{S}hading_{\mathcal{G}_i}(y_{j+1}) \subseteq S_{j+1}$, and, for every $x \leq y_j$, $(\mathcal{L}_i(x,y_j), \mathcal{L}_i(x,y_{j+1})) \in M_j$.

We start with the root n_0 of T_φ, with $\nu(n_0) = (\{F_0\}, T_0, M_0)$, and we build the initial two-rows compass structure $\mathcal{G}_0 = \langle \mathbb{P}_{\mathbb{O}_0}, \mathcal{L}_0 \rangle$, where $\mathbb{O}_0 = \{0 < 1\}$, $\mathcal{L}_0(0,0) = F_0$, $\mathcal{L}_0(0,1) = G_0$, $(F_0, G_0) \in M_0$, and $\mathcal{L}_0(1,1) = \pi(T_0)$. The function f_0 such that $f_0(0) = n_0$ respects the invariant. Now, let $\mathcal{G}_i$ and f_i respectively be the compass structure and the mapping function generated at the i-th step. We extend $\mathcal{G}_i$

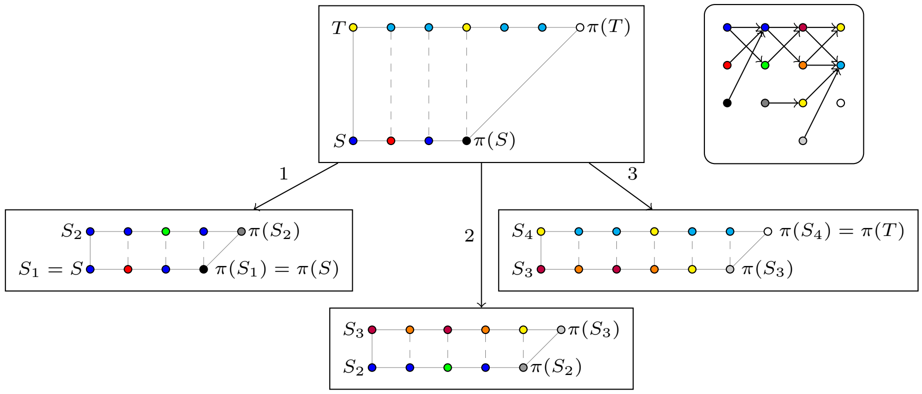

Fig. 2. A node of a decomposition tree, its successors, and the matching graph.

to $\mathcal{G}_{i+1}$ and we define the new mapping function f_{i+1} as follows. Let $\mathbb{O}_i = \{y_0 < \ldots < y_k\}$. For every $0 \le j < k$, let $f_i(y_j) = n_j$ and $\nu(n_j) = (S_j, S_{j+1}, M_j)$. For every node n_j, we execute the following two steps.

P1. If n_j is a leaf of $\mathcal{T}$, then we do not add any new point, we put $\mathcal{L}_{i+1}(x, y_j) = \mathcal{L}_i(x, y_j)$, for all $x \le y_j$, and we let $f_{i+1}(y_j) = f_i(y_j) = n_j$.

P2. If n_j has h successors $m_1, \ldots, m_h$, then add $h-1$ new points $z_1 < \ldots < z_{h-1}$ between y_j and y_{j+1} to $\mathbb{O}_{i+1}$. For $1 \le l \le h$, let $\nu(m_l) = (S_l, T_l, R_l)$, and let $\mathcal{M} = R_1 \circ \ldots \circ R_h$ be the corresponding fulfilling matching graph. We define the labeling $\mathcal{L}_{i+1}$ as follows:

- for every $x, y \in \mathbb{O}_i$, with $x \le y$, $\mathcal{L}_{i+1}(x,y) = \mathcal{L}_i(x,y)$;
- for every $x \in \mathbb{O}_i$ such that $x \le y_j$, let $p = (F_1, 1) \ldots (F_{h+1}, h+1)$ be a fulfilling path in $\mathcal{M}$ such that $F_1 = \mathcal{L}_i(x, y_j)$ and $F_{h+1} = \mathcal{L}_i(x, y_{j+1})$ (the existence of such a p is guaranteed by **(INV)** and **(T5)**). For all $1 \le l \le h-1$, we put $\mathcal{L}_{i+1}(x, z_l) = F_{l+1}$;
- for every $1 < l \le h$, let $p = (F_l, l) \ldots (F_{h+1}, h+1)$ be a path such that $F_l = \pi(S_l)$ and $F_{h+1} \in S^-_{j+1}$ (the existence of such a p is guaranteed by **(T5)**). We put $\mathcal{L}_{i+1}(z_{l-1}, y_{j+1}) = F_{h+1}$, and, for every $l \le q \le h$, we put $\mathcal{L}_{i+1}(z_{l-1}, z_{q-1}) = F_l$;
- for every $1 \le l \le h-1$, let $F_{j+2}, \ldots, F_k$ be a sequence of atoms such that $(\mathcal{L}_{i+1}(z_l, y_{j+1}), F_{j+2}) \in M_{j+1}$ and, for every $j+2 \le q \le k-1$, $(F_q, F_{q+1}) \in M_q$ (the existence of such a sequence is guaranteed by **(T2)** and **(T5)**). For every $j+1 < q \le k$, we put $\mathcal{L}_{i+1}(z_l, y_q) = F_q$.

Finally, we put $f_{i+1}(y_j) = m_1$, and $f_{i+1}(z_l) = m_{l+1}$, for every $1 \le l \le h-1$.

It is easy to see that, by the above construction, $\mathcal{G}_{i+1}$ and f_{i+1} preserve the invariant.

Let $\mathcal{G}_\omega = \bigcup_{i=0}^{\infty} \mathcal{G}_i$. We prove that $\mathcal{G}_\omega$ is a consistent and fulfilling compass structure that features φ. First, we show that $\mathcal{G}_\omega$ satisfies the consistency conditions for the relations B and A; then that it satisfies the fulfillment conditions for the $\bar{B}$-, B-, and A-requests; finally, that it features φ.

CONSISTENCY WITH RELATION B. Consider two points $p = (x, y)$ and $p' = (x', y')$ in $\mathcal{G}_\omega$ such that $p \; B \; p'$, that is, $0 \le x = x' \le y' < y \le 1$. Let i be the earliest stage of the above construction at which both points are present in the compass structure. Since $\mathcal{G}_i$ is a finite compass structure, let $y' = y_1 < y_2 < \ldots < y_h = y$ be the sequence of all points between y' and y in $\mathbb{O}_i$. By the above construction, we have that $\mathcal{L}_\omega(x,y) = \mathcal{L}_i(x, y_h) \xrightarrow{B} \mathcal{L}_i(x, y_{h-1}) \xrightarrow{B} \ldots \xrightarrow{B} \mathcal{L}(x, y_1) = \mathcal{L}_\omega(x', y')$, and thus, by the transitivity of $\xrightarrow{B}$, $\mathcal{L}_\omega(x,y) \xrightarrow{B} \mathcal{L}_\omega(x', y')$.

CONSISTENCY WITH RELATION A. Consider two points $p = (x, y)$ and $p' = (x', y')$ in $\mathcal{G}_\omega$ such that $p \; A \; p'$, that is, $0 \le x \le y = x' \le y' \le 1$. Let i be the earliest stage of the above construction at which both points are present in the compass structure, and consider the point $(y, x') = (y, y) = (x', x')$. Let $f_i(y) = n_y$, and let $\nu(n_y) = (S_y, T_y, M_y)$. By **(INV)**, we have that $\mathcal{S}hading(y) \subseteq S_y$ and, by **(S1)**, **(S2)** and **(S3)**, $\mathcal{R}eq_A(\mathcal{L}_i(x,y)) = \mathcal{R}eq_A(\mathcal{L}_i(y,y)) = \mathcal{R}eq_{\bar{B}}(\mathcal{L}(y,y))$. This implies that $\mathcal{L}_i(x,y) \xrightarrow{A} \mathcal{L}_i(y,y)$. Since $(x', y') \; B \; (x', x') = (y, y)$, by the consistency with relation B, we have that $\mathcal{L}_i(x', y') \xrightarrow{B} \mathcal{L}_i(y,y)$. By Proposition 1, we can conclude that $\mathcal{L}_\omega(x,y) = \mathcal{L}_i(x,y) \xrightarrow{A} \mathcal{L}_i(x', y') = \mathcal{L}_\omega(x', y')$.

FULFILLMENT OF B-REQUESTS. Let $p = (x, y)$ be a point in $\mathcal{G}_\omega$ such that there exists $\psi \in \mathcal{R}eq_B(\mathcal{L}_\omega(x,y))$ for some formula ψ, and suppose, by contradiction, that ψ is not fulfilled in $\mathcal{G}_\omega$. Now, let i be the earliest stage of the above construction at which the point (x, y) is present in the compass structure. Let y' be the smallest row in $\mathcal{G}_i$ such that $\psi \in \mathcal{R}eq_B(\mathcal{L}_i(x, y'))$ and let y'' be the immediate predecessor of y' in $\mathbb{O}_i$ (whose existence is guaranteed by **T3**). Let $f_i(y'') = n''$, and suppose that $\nu(n'') = (S'', T'', M'')$. By hypothesis, we have that $\psi \notin \mathcal{O}bs(\mathcal{L}_i(x, y''))$ and $\psi \notin \mathcal{R}eq_B(\mathcal{L}_i(x, y''))$. This implies that M'' cannot be a strong match, and thus that n'' cannot be a leaf of $\mathcal{T}$. Let h be the rank of n''. According to the above construction, at step $i+1$, $h-1$ points $z_1, \ldots, z_{h-1}$, with $y'' < z_1 < \ldots < z_{h-1} < y'$, have been added to $\mathbb{O}_i$. Moreover, by construction, $\mathcal{L}_{i+1}(x, y''), \mathcal{L}_{i+1}(x, z_1), \ldots, \mathcal{L}_{i+1}(x, z_{h-1}), \mathcal{L}_{i+1}(x, y')$ is a fulfilling path. This implies that there exists an index $1 \le l \le h-1$ such that $\psi \in \mathcal{L}_{i+1}(x, z_l)$, against the hypothesis that the B-request ψ is not fulfilled for (x, y) in $\mathcal{G}_\omega$ (contradiction).

FULFILLMENT OF $\bar{B}$-REQUESTS. The proof that $\mathcal{G}_\omega$ fulfills all $\bar{B}$-requests of its atoms is symmetric to the one for B-requests, and thus it is omitted.

FULFILLMENT OF A-REQUESTS. Let $p = (x, y)$ be a point in $\mathcal{G}_\omega$ such that there exists $\psi \in \mathcal{R}eq_A(\mathcal{L}_\omega(x, y))$ for some formula ψ. By the definition of shading, we have that $\mathcal{R}eq_A(\mathcal{L}_\omega(x, y)) = \mathcal{R}eq_A(\mathcal{L}_\omega(y, y)) = \mathcal{R}eq_{\bar{B}}(\mathcal{L}_\omega(y, y))$. This implies that fulfillment of A-requests directly follows from the fulfillment of $\bar{B}$-requests.

FEATURED FORMULAS. By the definition of decomposition tree, we have that the root n_0 of $\mathcal{T}$ is labeled with (S_0, T_0, M_0), with $S_0 = \{F_0\}$ and $\varphi \in F_0$. By the above construction, we have that $\mathcal{L}_\omega(0, 0) = F_0$, and thus $\mathcal{G}_\omega$ is a bounded compass structure for φ. ∎

Theorem 3. *Let φ an $AB\bar{B}$-formula. Then, φ is satisfiable in the class of all linear orders if and only if there exists a* decomposition tree *$T_\varphi = \langle \mathcal{T}, \nu \rangle$ for φ with rank $\leq 4 \cdot |\varphi| \cdot 2^{18|\varphi|+2} + 2^{9|\varphi|+1} + 1$.*

It easily follows from Theorem 2, Lemma 2, and Lemma 3.

B. The main result: decidability of $AB\bar{B}\bar{L}$

To deal with $AB\bar{B}\bar{L}$, the notion of decomposition tree must be suitably generalized. As a preliminary step, we show that, when the $\langle \bar{L} \rangle / [\bar{L}]$ modalities are considered, the shading of a row y satisfies the following additional property.

Lemma 4. *Let $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ be a compass structure for an $AB\bar{B}\bar{L}$ formula φ and let $y \in O$. Then, $\mathcal{S}hading_{\mathcal{G}}(y)$ satisfies the following property:*

(S4) $\psi \in \mathcal{R}eq_{\bar{L}}(\pi(\mathcal{S}hading_{\mathcal{G}}(y))$ *if and only if there exists $F \in \mathcal{S}hading_{\mathcal{G}}(y)$, with $F \neq \pi(\mathcal{S}hading_{\mathcal{G}}(y))$, such that $\psi \in \mathcal{R}eq_B(F)$.*

Proof: It is an easy consequence of the semantics of the $\langle \bar{L} \rangle$ and $\langle B \rangle$ modalities: $\psi \in \mathcal{R}eq_{\bar{L}}(\pi(\mathcal{S}hading_{\mathcal{G}}(y)))$ iff $\psi \in \mathcal{R}eq_{\bar{L}}(\mathcal{L}(y, y))$ iff $\exists(x', y') : y' < y \wedge \psi \in \mathcal{L}(x', y')$ iff $\psi \in \mathcal{R}eq_B(\mathcal{L}(x', y))$. As $\mathcal{L}(x', y) \in \mathcal{S}hading_{\mathcal{G}}(y)$, the thesis immediately follows. ∎

Lemma 4 shows that fulfillment of $\bar{L}$-requests can be reduced to fulfillment of B-requests of an appropriate set of points. However, condition **(S4)** alone is not sufficient to guarantee that from a decomposition tree for a formula φ we can build a fulfilling bounded compass structure for it: it may happen that, in the final bounded compass structure, there exist a point (y, y) and a formula $\psi \in \mathcal{R}eq_{\bar{L}}(\mathcal{L}(y, y))$, but no points in the row y have ψ in their set of B-requests, and thus the $\bar{L}$-request ψ is not fulfilled for (y, y).

Now, given a consistent and fulfilling bounded compass structure $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ and a formula $\psi \in \mathcal{C}l(\varphi)$ that occurs in $\mathcal{G}$, we distinguish among the following three cases:

(Type1) there exists a point (x_ψ, y_ψ) such that $\psi \in \mathcal{L}(x_\psi, y_\psi)$ and $\psi \notin \mathcal{R}eq_{\bar{L}}(y_\psi, y_\psi)$;

(Type2) there exists an horizontal coordinate x_ψ and an infinite descending sequence of rows $y_1 > y_2 > \ldots$ such that, for every $i \in \mathbb{N}$, *(i)* $\psi \in \mathcal{L}(x_\psi, y_i)$, *(ii)* $\psi \in \mathcal{R}eq_B(x_\psi, y_i)$, and *(iii)* if $y' < y_j$ for every $j \geq 1$, then $\psi \notin \mathcal{R}eq_{\bar{L}}(y', y')$;

(Type3) there exists an infinite descending sequence of rows $y_1 > y_2 > \ldots$ such that, for every $i \in \mathbb{N}$, *(i)* there exists x_i such that $\psi \in \mathcal{L}(x_i, y_i)$, *(ii)* $\psi \notin \mathcal{R}eq_B(x_i, y_i)$, and *(iii)* if $y' < y_j$ for every $j \geq 1$, then $\psi \notin \mathcal{R}eq_{\bar{L}}(y', y')$.

These different types of formula describe the different ways of fulfilling a $\bar{L}$-request in a bounded compass structure. Given a point (x, y) and a formula $\psi \in \mathcal{R}eq_{\bar{L}}(\mathcal{L}(x, y))$, one of the following situations may arise:

- if ψ is a Type1-formula, then x must be strictly greater than y_ψ (otherwise, ψ must belong to $\mathcal{R}eq_{\bar{L}}(\mathcal{L}(y_\psi, y_\psi))$, in contradiction with the definition) and thus the point (x_ψ, y_ψ) fulfills the request for (x, y);
- if ψ is a Type2-formula, then there must be a row y_i in the infinite descending sequence such that $y_i < x$, and thus the point (x_ψ, y_i) fulfills the request for (x, y);
- if ψ is a Type3-formula, then there must be a row y_i in the infinite descending sequence such that $y_i < x$, and thus the point (x_i, y_i) fulfills the request for (x, y).

It is worth noticing that while for Type1-formulas one single point (x_ψ, y_ψ) suffices for fulfilling all the occurrences of $\langle \bar{L} \rangle \psi$ in the compass structure, for Type2-formulas and Type3-formulas an infinite number of points is needed.

To cope with all the three possible types of formula, we extend the definitions and the construction given in Section III-A as follows. First of all, we call *extended shading* any shading S that also satisfies condition **(S4)**. The definitions of matching set (Definition 2), strong matching set (Definition 3), matching graph (Definition 4), and fulfilling matching graph remain unchanged. An extended decomposition tree for a formula φ of $AB\bar{B}\bar{L}$ is then defined as follows.

Definition 6. *Let φ be an $AB\bar{B}\bar{L}$-formula. An* extended decomposition tree *for φ is a labeled tree $T_\varphi = \langle \mathcal{T}, \nu, \tau_1, \tau_2, \tau_3 \rangle$ such that the following conditions hold:*

(ET1) *$\langle \mathcal{T}, \nu \rangle$ is a decomposition tree for φ;*

(ET2) *for every node n of $\mathcal{T}$, with $\nu(n) = (S_n, T_n, M_n)$, S_n and T_n are extended shadings;*

(ET3) *let n_0 be the root of $\mathcal{T}$, with $\nu(n_0) = (S_0, T_0, M_0)$. Then, τ_1, τ_2, and τ_3 form a partition of $\mathcal{R}eq_{\bar{L}}(\pi(T_0))$;*

(ET4) *for every formula $\psi \in \tau_1$, there exists an immediate successor n_ψ of the root, with $\nu(n_\psi) = (S_\psi, T_\psi, M_\psi)$, such that a) $\psi \notin \mathcal{R}eq_{\bar{L}}(\pi(S_\psi))$, and b) there exists an atom $F \in S_\psi$ such that $\psi \in F$;*

(ET5) *let $\mathcal{M} = M_1 \circ \ldots \circ M_k$ be the matching graph defined by the successors of the root and let Θ be a partition of τ_2. For every $\theta \in \Theta$, there exists an immediate successor of the root $n_{loop} = \downarrow_l (n_0)$, with $\nu(n_{loop}) = (S_{n_{loop}}, T_{n_{loop}}, M_{n_{loop}})$, such that $\theta = \tau_2 \cap (\mathcal{R}eq_{\bar{L}}(\pi(T_{n_{loop}})) \setminus \mathcal{R}eq_{\bar{L}}(\pi(S_{n_{loop}})))$ and there exist $(F_1, G_1), \ldots, (F_o, G_o)$ in $M_{n_{loop}}$, with $o \leq |\theta|$, such that for every $\psi \in \theta$ there exists $1 \leq i \leq o$ with $\psi \in \mathcal{R}eq_B(G_i)$. Moreover, there exist o distinct successors of the root $\downarrow_{i_1} (n_0), \ldots, \downarrow_{i_o} (n_0)$, with $i_1 < l, \ldots, i_o < l$,* *such that for every $j = 1, \ldots, o$, there exists a path $p = (H_{i_j}, i_j) \ldots (H_l, l)$ in $\mathcal{M}$ with $\pi \in H_{i_j}$ and $H_l = F_j$. Finally, let n' be any node with $\nu(n') = \nu(n_{loop})$, $n_1, \ldots, n_h$ be its h immediate successors, and $\mathcal{M}'$ be the corresponding matching graph. Then, there exists $1 \leq j \leq h$ such that $\nu(n_j) = \nu(n')$ and for every (F_i, G_i), with $1 \leq i \leq o$, there exists a fulfilling path $p = (H_1, 1) \ldots (H_h, h)$ in $\mathcal{M}'$ with $H_1 = H_j = F_i$ and $H_h = H_{j+1} = G_i$*

(ET6) *let $\mathcal{M} = M_1 \circ \ldots \circ M_k$ be the matching graph defined by the successors of the root. For every $\psi \in \tau_3$, there exist two successors of the root $m_\psi =\downarrow_i (n_0)$ and $n_\psi =\downarrow_j (n_0)$, with $i < j$, such that a) $\nu(n_\psi) = (S_\psi, T_\psi, M_\psi)$, b) $\nu(m_\psi) = (U_\psi, U_\psi, N_\psi)$, c) $\psi \in \mathcal{R}eq_{\bar{L}}(\pi(S_\psi))$, d) there exists $G_\psi \in S_\psi$ such that $\psi \in G_\psi$, $\psi \notin \mathcal{R}eq_B(G_\psi)$, and there exists a path $p = (H_i, i) \ldots (H_j, j)$ in $\mathcal{M}$ with $\pi \in H_i$ and $H_j = G_\psi$, e) every node n, with $\nu(n) = (S_n, T_n, M_n)$, such that $\psi \in \mathcal{R}eq_{\bar{L}}(\pi(T_n) \backslash \pi(S_n))$, has a successor with the same labeling as n_ψ, and f) every node n', with $\nu(n') = \nu(m_\psi)$ has a successor with the same labeling as m_ψ.*

The three sets τ_1, τ_2, and τ_3 partition all $\bar{L}$-requests that occur in the extended decomposition tree in Type1, Type2, and Type3 formulas, respectively (condition **(ET3)**). Condition **(ET4)** guarantees fulfilling of Type1-formulas. Conditions **(ET5)** and **(ET6)** guarantee sufficient conditions for the existence of the infinite descending chains of rows needed for Type2 and Type3 formulas, respectively. By using techniques similar to the ones of Lemma 2 and Lemma 3, we can prove an analogous of Theorem 3 for $AB\bar{B}\bar{L}$.

Theorem 4. *Let φ be an $AB\bar{B}\bar{L}$-formula. Then, φ is satisfiable in the class of all linear orders if and only if there exists an* extended decomposition tree *$T_\varphi = \langle \mathcal{T}, \nu, \tau_1, \tau_2, \tau_3 \rangle$ for φ with rank $m \leq 4 \cdot |\varphi| \cdot 2^{18|\varphi|+2} + 2^{9|\varphi|+1} + |\varphi| + 1$.*

Proof: (sketch) We have to show that (i) the existence of a consistent and fulfilling bounded compass structure for a formula φ implies the existence of an extended decomposition tree for it (completeness), (ii) the existence of an extended decomposition tree implies the existence of a consistent and fulfilling bounded compass structure (soundness), and (iii) the number of successors of a node of an extended decomposition tree is bounded by $4 \cdot |\varphi| \cdot 2^{18|\varphi|+2} + 2^{9|\varphi|+1} + |\varphi| + 1$.

The proof of completeness follows that of Lemma 2: we start from a bounded compass structure $\mathcal{G} = \langle \mathbb{P}_{\mathbb{O}}, \mathcal{L} \rangle$ and we iteratively define the labeling of the corresponding extended decomposition tree by appropriately selecting, at each step, a set of "witness rows" satisfying all conditions of Definition 6.

The proof of soundness is a bit more involved. First, given an extended decomposition tree $T_\varphi = \langle \mathcal{T}, \nu, \tau_1, \tau_2, \tau_3 \rangle$, we consider its root n_0 and its immediate successors $n_1, \ldots, n_k$ and we build an *initial bounded compass structure* $\mathcal{G}_0 = \langle \mathbb{P}_{\mathbb{O}_0}, \mathcal{L}_0 \rangle$ that satisfies the following conditions (the possibility to build such a structure is guaranteed by properties **(ET4)**, **(ET5)**, and **(ET6)** of Definition 6):

(IS1) for every $\psi \in \tau_1$, there exists a point (x_ψ, y_ψ) such that $\psi \in \mathcal{L}(x_\psi, y_\psi)$ and $\psi \notin \mathcal{R}eq_{\bar{L}}(y_\psi, y_\psi)$;

(IS2) for every $\psi \in \tau_2$, there exists a point (x_ψ, y_ψ) such that $\psi \in \mathcal{L}(x_\psi, y_\psi)$ and $\psi \in \mathcal{R}eq_B(x_\psi, y_\psi)$;

(IS3) for every $\psi \in \tau_3$, there exists a point (x_ψ, y_ψ) such that $\psi \in \mathcal{L}(x_\psi, y_\psi)$, $\psi \notin \mathcal{R}eq_B(x_\psi, y_\psi)$, $\psi \in \mathcal{R}eq_{\bar{L}}(y_\psi, y_\psi)$, and $f_0(x_\psi) = m_\psi$.

Then, we proceed with the very same construction as in Lemma 3. Let $\mathcal{G}_i$ and f_i respectively be the compass structure and the mapping function generated at the i-th iteration. Moreover, let $u_i : \tau_3 \mapsto \mathcal{T}$ be an auxiliary function that maps every formula $\psi \in \tau_3$ to a node $u_i(\psi)$ in $\mathcal{T}$ such that $\nu(u_i(\psi)) = \nu(m_\psi)$. At step 0, we put $u_0(\psi) = m_\psi$, for every $\psi \in \tau_3$.

At the $(i+1)$-iteration, we extend $\mathcal{G}_i$ to $\mathcal{G}_{i+1}$ and we define functions f_{i+1} and u_{i+1} as follows. Let $\mathbb{O}_i = \{y_0 < \ldots < y_k\}$. For every $0 \leq j < k$, let $f_i(y_j) = n_j$ and $\nu(n_j) = (S_j, S_{j+1}, M_j)$. For every node n_j, we execute steps **P1**-**P4**. Steps **P1** and **P2** have been already described in the proof of Lemma 3. Steps **P3** and **P4** behave as follows:

P3. for every formula $\psi \in \tau_2$ such that $\psi \in \mathcal{R}eq_{\bar{L}}(\mathcal{L}(y_{j+1}, y_{j+1})) \setminus \mathcal{R}eq_{\bar{L}}(\mathcal{L}(y_j, y_j))$ and ψ is not fulfilled, proceed as follows:
- let (x_ψ, y_ψ) be the point whose existence is guaranteed by condition **(IS2)**, and let m_l be the successor of n_j, with $\nu(m_l) = \nu(n_{loop})$, whose existence is guaranteed by property **(ET5)**. We put $\mathcal{L}_{i+1}(x_\psi, z_{l-1}) = \mathcal{L}(x_\psi, y_\psi)$;

P4. for every formula $\psi \in \tau_3$ such that $\psi \in \mathcal{R}eq_{\bar{L}}(\mathcal{L}(y_{j+1}, y_{j+1})) \setminus \mathcal{R}eq_{\bar{L}}(\mathcal{L}(y_j, y_j))$ and ψ is not fulfilled, proceed as follows:
- let (x_ψ, y_ψ) be the point whose existence is guaranteed by condition **(IS3)**, and let m_l and m'_ψ be the successors of n_j and $u_i(\psi)$, respectively, with $\nu(m_l) = \nu(n_\psi)$ and $\nu(m'_\psi) = \nu(m_\psi)$, whose existence is guaranteed by property **(ET6)**. We execute the following sequence of operations:
  - we apply steps **P1** and **P2** to node $u_i(\psi)$;
  - we put $\mathcal{L}_{i+1}(x'_\psi, z_{l-1}) = \mathcal{L}(x_\psi, y_\psi)$, where $x'_\psi = f_{i+1}(m'_\psi)$;
  - we put $u_{i+1}(\psi) = m'_\psi$;
  - we complete the labeling of all emerging points by using information from the labeling of m_l and m'_ψ.

In this way, we obtain a sequence of finite compass structures $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \ldots$ such that $\mathcal{G}_\omega = \bigcup_{i=0}^{\infty} \mathcal{G}_i$ is a fulfilling bounded compass structure for φ.

Finally, to prove that the rank of an extended decomposition tree is bounded by $4 \cdot |\varphi| \cdot 2^{18|\varphi|+2} + 2^{9|\varphi|+1} + |\varphi| + 1$ it is sufficient to observe that **(ET5)** and **(ET6)** force the existence of at most $|\varphi|$ additional successors of a node. ■

C. A decomposition tree for (weakly) discrete linear orders

As already pointed out, (weak) discreteness is not definable in $AB\bar{B}\bar{L}$ by a constant formula. However, to tailor Theorem 4 to the class of weakly discrete linear orders, it suffices to add the following condition to the definition of extended decomposition tree.

Definition 7. *Let φ be an $AB\bar{B}\bar{L}$-formula. A* discrete extended decomposition-tree *for φ is an extended decomposition tree $T_\varphi = \langle \mathcal{T}, \nu, \tau_1, \tau_2, \tau_3 \rangle$ such that the following additional property holds:*

(ETD) *if $n_1, \ldots, n_k$ are the k successors of a node n, with $k > 0$, then $\nu(n_1) = (E_1, H_1, M_1)$, $\nu(n_k) = (E_k, H_k, M_k)$, and M_1, M_k are strong matching sets.*

To prove completeness and soundness, it is sufficient to recall that a strong matching set between two rows y and y' corresponds to the case in which y' is the immediate successor of y. The following theorem follows directly from **(ETD)** and Theorem 4.

Theorem 5. *Let φ be an $AB\bar{B}\bar{L}$-formula. Then, φ is satisfiable over weakly discrete linear orders if and only if there exists a* discrete extended decomposition tree *$T_\varphi = \langle \mathcal{T}, \nu, \tau_1, \tau_2, \tau_3 \rangle$ for φ with rank $m \leq 4 \cdot |\varphi| \cdot 2^{18|\varphi|+2} + 2^{9|\varphi|+1} + |\varphi| + 1$.*

IV. Complexity bounds to the satisfiability problem for $AB\bar{B}\bar{L}$

In [8], Montanari et al. give an automaton-based algorithm to check satisfiability of formulas of a spatial modal logic based on an encoding of the problem into a suitable fragment of CTL. The very same technique can be used to check the satisfiability of an $AB\bar{B}\bar{L}$-formula φ. The effectiveness of such an approach stems from the fact that the properties that characterize an extended decomposition tree for φ can be expressed by a CTL formula $\vec{\varphi}$, with $|\vec{\varphi}|$ exponential in $|\varphi|$, that is, extended decomposition trees for φ are all and only those ones that satisfy $|\vec{\varphi}|$. Next, satisfiability of $\vec{\varphi}$ over extended decomposition trees can be reduced to the universality problem for a suitable Büchi tree automaton $\mathcal{A}_{\vec{\varphi}}$, which can be obtained from $\vec{\varphi}$ in polynomial time with respect to $|\vec{\varphi}|$. Since the universality problem for regular ω-languages is in PSPACE [17] and $|\mathcal{A}_{\vec{\varphi}}|$ is exponential in $|\varphi|$, the resulting decision procedure for $AB\bar{B}\bar{L}$ is in EXPSPACE. An EXPSPACE lower bound to the complexity of the satisfiability problem for $AB\bar{B}\bar{L}$ immediately follows from the reduction of the *exponential-corridor tiling problem* to the satisfiability problem for $AB\bar{B}$ given in [11].

Theorem 6. *The satisfiability problem for $AB\bar{B}\bar{L}$ over the class of all (resp., dense, weakly discrete) linear orders is EXPSPACE-complete.*

V. Conclusions

This paper aimed at contributing to the identification of the decidability/undecidability border in interval temporal logics by completing the picture given in [13]. In that paper, the authors prove the maximality of $AB\bar{B}\bar{A}$ with respect to decidability over finite linear orders. Here, we show that, to recover decidability in the case of infinite linear orders, the operator $\langle \overline{A} \rangle$ must be replaced by the weaker operator $\langle \overline{L} \rangle$ (the undefinability of $\langle \overline{A} \rangle$, resp., $\langle A \rangle$, in terms of $\langle \overline{L} \rangle, \langle B \rangle$, resp., $\langle L \rangle, \langle E \rangle$, is a by-product of this pair of results).

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