Relational dual tableaux for interval temporal logics *

Davide Bresolin* — **Joanna Golińska-Pilarek**** — **Ewa Orłowska****

* *Department of Mathematics and Computer Science*
University of Udine (Italy)

bresolin@dimi.uniud.it

** *National Institute of Telecommunications*
Warsaw (Poland)

{J.Golinska-Pilarek,E.Orlowska}@itl.waw.pl

ABSTRACT. *Interval temporal logics provide both an insight into a nature of time and a framework for temporal reasoning in various areas of computer science. In this paper we present sound and complete relational proof systems in the style of dual tableaux for relational logics associated with modal logics of temporal intervals and we prove that the systems enable us to verify validity and entailment of these temporal logics. We show how to incorporate in the systems various relations between intervals and/or various time orderings.*

KEYWORDS: *interval temporal logics, relational logics, dual tableau systems, proof systems.*

1. Introduction

Interval temporal logics provide both an insight into a nature of time and a framework for temporal reasoning in the area of artificial intelligence (reasoning about action and change, qualitative reasoning, planning, and natural language processing), theoretical computer science (specification and automatic verification of programs and reactive systems) and databases (temporal and spatio-temporal databases). In the literature various propositional and first-order interval temporal logics have been proposed (a comprehensive survey can be found in [GOR 04]). Among others, the most significant propositional ones are Halpern and Shoham's HS [HAL 91, VEN 90], Venema's CDT logic [GOR 03a, GOR 06, VEN 91], Moszkowski's Propositional Interval Temporal Logic (PITL) [MOS 83], and Goranko, Montanari, and Sciavicco's Propositional Neighborhood Logic (PNL) [BRE 05b, BRE 06, GOR 03b].

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Propositional interval temporal logics are very expressive (it can be shown that both HS and CDT are strictly more expressive than every point-based temporal logic on linear orders): they make it possible to express properties of *pairs* of time points (think of intervals as constructed out of points), rather than *single* time points. In linear orders 13 different binary relations between intervals are possible [ALL 83]: *equals* ($1'$), *ends* (E), *during* (D), *begins* (B), *overlaps* (O), *meets* (M), *precedes* (P) together with their converses.

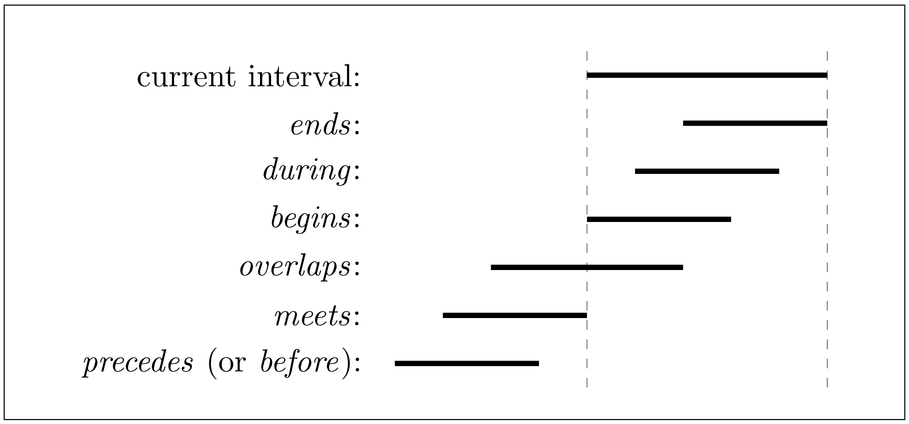

Figure 1. *caption of the figure*

These relations are usually called *Allen's relations*, and lead to a rich interval algebra, called Allen's Interval Algebra. Propositional interval temporal logics are usually characterized by modalities of the form $\langle R\rangle$ and $\langle\overline{R}\rangle$, where R is any of these relations and $\overline{R}$ denotes the converse of R. To the best of our knowledge, there are no interval logics where the modalities corresponding to *overlaps* and its converse are chosen as primitive.

In this paper we present relational proof systems in the style of dual tableaux for relational logics associated with modal logics of temporal intervals and we prove that the systems enable us to verify validity and entailment of these temporal logics. In constructing the systems we apply the method known for various non-classical logics, in particular for standard modal and temporal logics [ORŁ 95, ORŁ 96]. The key steps of the method are:

– Development of a relational logic RL_L appropriate for a given interval temporal logic L.

– Development of a validity preserving translation from the language of logic L into the language of logic RL_L.

– Construction of a proof system for RL_L such that for every formula φ of L, φ is valid in L iff its translation $\tau(\varphi)$ is provable in RL_L.

Each logic RL_L is based on the classical relational logic of binary relations, $\mathsf{RL}(1, 1')$, which provides a means for proving the identities valid in the class of representable relation algebras (see e.g., [GOL 06a, ORŁ 96]). RL_L is capable of expressing both binary relations holding between points of time and binary relations holding between time intervals. The proof systems developed in this paper are extensions of the proof system for $\mathsf{RL}(1, 1')$ originated in [ORŁ 88], see also [GOL 06a, ORŁ 96]. The systems are founded on the Rasiowa-Sikorski system for the first order logic [RAS 63] which is extended with the rules for equality predicate in [GOL 06b]. In constructing deduction rules for our systems we follow the general principles of defining relational deduction rules presented in [MAC 02]. In sections 2, 3, 4, 5, and 6 we develop a relational proof system for the Halpern and Shoham's logic HS [HAL 91] in accordance with the three steps mentioned above. Next, in section 7 we show how this system can be extended or modified in order to incorporate the remaining interval relations of Allen [ALL 83, LAD 87] and/or other time orderings.

A recent implementation of the proof system for $\mathsf{RL}(1, 1')$ is described in [DAL 05]. The system is available at `http://www.logic.stfx.ca/reldt/`. In [FOR 05] an implementation of translation procedures from non-classical logics to relational logic $\mathsf{RL}(1, 1')$ is presented. The system can be downloaded from `http://www.di.univaq.it/TARSKI/transIt/`.

2. Syntax and semantics of HS

Halpern and Shoham's logic [HAL 91, VEN 90] is a propositional interval logic characterized by four temporal modalities, that correspond to Allen's relations *begins*, *ends*, and their converses. These four modalities suffice to define all unary modalities corresponding to Allen's relations. Hence, HS is the most expressive interval temporal logic featuring only *unary* modalities. Formally, HS-formulas are generated by the following abstract syntax:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle B\rangle\varphi \mid \langle E\rangle\varphi \mid \langle\overline{B}\rangle\varphi \mid \langle\overline{E}\rangle\varphi.$$

The other propositional connectives, such as $\wedge$, $\rightarrow$ and the propositional constants $\top$ (true) and $\bot$ (false), as well as the necessity modalities $[B]$, $[E]$, $[\overline{B}]$ and $[\overline{E}]$ can be defined as usual.

Given a strict linear ordering $\langle D, <\rangle$, a *non-strict interval* on D is a pair $[c, d]$ such that $c \leq d$. We denote the set of all non-strict intervals on D as $\mathbb{I}(D)^+$. A *model* for an HS-formula is a tuple $\mathbf{M}^+ = \langle D, \mathbb{I}(D)^+, \mathcal{V}\rangle$, where $\langle D, <\rangle$ is a strict linear ordering and $\mathcal{V}\colon AP \rightarrow 2^{\mathbb{I}(D)^+}$ is a *valuation function* assigning to every propositional letter $p \in AP$ the set of intervals where p holds. The semantics of HS is defined recursively by the *satisfiability relation* $\models$ as follows. Let $\mathbf{M}^+ = \langle D, \mathbb{I}(D)^+, \mathcal{V}\rangle$ be some given HS-model, and let $[c, d] \in \mathbb{I}(D)^+$:

– for every propositional letter $p \in AP$, $\mathbf{M}^+, [c, d] \models p$ iff $[c, d] \in \mathcal{V}(p)$;
– $\mathbf{M}^+, [c, d] \models \neg\psi$ iff $\mathbf{M}^+, [c, d] \not\models \psi$;
– $\mathbf{M}^+, [c, d] \models \psi_1 \vee \psi_2$ iff $\mathbf{M}^+, [c, d] \models \psi_1$, or $\mathbf{M}^+, [c, d] \models \psi_2$;

– $\mathbf{M}^+, [c, d] \models \langle B \rangle \psi$ iff $\exists c' \in D$ such that $c' < d$, and $\mathbf{M}^+, [c, c'] \models \psi$;

– $\mathbf{M}^+, [c, d] \models \langle E \rangle \psi$ iff $\exists c' \in D$ such that $c < c'$, and $\mathbf{M}^+, [c', d] \models \psi$;

– $\mathbf{M}^+, [c, d] \models \langle \overline{B} \rangle \psi$ iff $\exists c' \in D$ such that $d < c'$, and $\mathbf{M}^+, [c, c'] \models \psi$;

– $\mathbf{M}^+, [c, d] \models \langle E \rangle \psi$ iff $\exists c' \in D$ such that $c' < c$, and $\mathbf{M}^+, [c', d] \models \psi$.

Note that $\mathbb{I}(D)^+$ includes also intervals of the form $[c, c]$, that are called *point intervals*. Since point intervals have no intervals that begin and/or end them, they can be distinguished by the formulas $[B]\bot$ and $[E]\bot$. This allow us to define two derived operators, $[\![BP]\!]$ and $[\![EP]\!]$, that express properties that hold on the begin point and on the end point of the current interval, respectively:

$$[\![BP]\!]\varphi ::= ([B]\bot \wedge \varphi) \vee \langle B \rangle([B]\bot \wedge \varphi)$$

$$[\![EP]\!]\varphi ::= ([E]\bot \wedge \varphi) \vee \langle E \rangle([E]\bot \wedge \varphi)$$

In the presence of point intervals, it is possible to define in HS the modalities corresponding to the other Allen's relations as follows:

$$\begin{array}{ll}
\langle D \rangle\varphi ::= \langle B \rangle\langle E \rangle\varphi & \langle \overline{D} \rangle\varphi ::= \langle \overline{B} \rangle\langle \overline{E} \rangle\varphi \\
\langle O \rangle\varphi ::= \langle B \rangle\langle \overline{E} \rangle\varphi & \langle \overline{O} \rangle\varphi ::= \langle \overline{B} \rangle\langle E \rangle\varphi \\
\langle M \rangle\varphi ::= [\![BP]\!]\langle \overline{E} \rangle\varphi & \langle \overline{M} \rangle\varphi ::= [\![EP]\!]\langle \overline{B} \rangle\varphi \\
\langle P \rangle\varphi ::= \langle M \rangle\langle M \rangle\varphi & \langle \overline{P} \rangle\varphi ::= \langle \overline{M} \rangle\langle \overline{M} \rangle\varphi
\end{array}$$

It is worth noticing that in [HAL 91], as well as in most of the interval logic literature [GOR 03b, GOR 04], the modalities $\langle M \rangle$ and $\langle \overline{M} \rangle$ are denoted as $\langle \overline{A} \rangle$ and $\langle A \rangle$ (*after*), respectively. In this paper we choose to change the usual notation, in order to be coherent with Allen's terminology.

3. Relational logic for HS

The vocabulary of the language RL_{HS} consists of the pairwise disjoint sets listed below:

– a countable infinite set $\mathbb{IV} = \{i, j, k, \ldots\}$ of *interval variables*;

– since intervals are meant to be certain pairs of points, to every interval variable i we associate two point variables denoted i_1, i_2, with the intuition that $i = [i_1, i_2]$. We define the countable infinite set of *point variables* as $\mathbb{PV} = \{i_1, i_2 : i \in \mathbb{IV}\}$;

– a countable infinite set $\mathbb{IRV}$ of *interval relational variables*;

– a set $\mathbb{PRC} = \{1', <\}$ of *point relational constants*;

– a set $\mathbb{IRC} = \{1, B, E\}$ of *interval relational constants*;

– a set $\mathbb{OP} = \{-, \cup, \cap, ;, {}^{-1}\}$ of *relational operation symbols*.

The constants $1'$ and $<$ are intended to represent the identity relation and the ordering on the set of time points, respectively. We use a traditional relation-algebraic notation for constants $1'$ and 1 (Boolean unit). The unary operators $-$ and $^{-1}$ bind stronger than the binary $\cup, \cap$ and ;.

The specific relational operations of converse ($^{-1}$) and composition (;) are defined as usual. For binary relations A, B on a set U:

$$A^{-1} = \{(x, y) \in U \times U : (y, x) \in A\}$$

$$A;B = \{(x, y) \in U \times U : \exists z \in U\, [(x, z) \in A \wedge (z, y) \in B]\}$$

Relational terms and formulas:

– The set of *point relational terms* $\mathbb{PRT}$ is the smallest set of expressions that includes $\mathbb{PRC}$ and is closed with respect to the operation symbols from $\mathbb{OP}$.

– The set of *interval relational terms* $\mathbb{IRT}$ is the smallest set of expressions that includes $\mathbb{IRA} = \mathbb{IRV} \cup \mathbb{IRC}$ and is closed with respect to the operation symbols from $\mathbb{OP}$.

– The set of *point relational formulas* $\mathbb{PRF}$ consists of expressions of the form $x\ R\ y$ where $x, y \in \mathbb{PV}$ and $R \in \mathbb{PRT}$.

– The set of *interval relational formulas* $\mathbb{IFR}$ consists of expressions of the form $i\ R\ j$ where $i, j \in \mathbb{IV}$ and $R \in \mathbb{IRT}$.

– The set $\mathbb{RF}$ of RL_{HS}*-formulas* (or, simply formulas if it is clear from the context), consists of expressions from $\mathbb{PRF} \cup \mathbb{IRF}$.

– R is said to be an *atomic relational term* whenever $R \in \mathbb{PRC} \cup \mathbb{IRA}$. $x\ R\ y$ is said to be an *atomic formula* whenever R is an atomic relational term.

Semantics:

An RL_{HS}-model is a tuple $\mathcal{M} = (U, \mathbb{I}(U)^+, m)$, where U and $\mathbb{I}(U)^+$ are non-empty sets and $m\colon \mathbb{PRT} \cup \mathbb{IRT} \to 2^{U \times U} \cup 2^{\mathbb{I}(U)^+ \times \mathbb{I}(U)^+}$ is a meaning function which assigns binary relations on $U \times U$ to point relational terms and binary relations on $\mathbb{I}(U)^+ \times \mathbb{I}(U^+)$ to interval relational terms as follows:

(1) $m(1') = \mathrm{Id}_U$;

(2) $m(<)$ is a strict linear ordering on U, that is for every $c, d, e \in U$ the following holds:

(Irref)	$(c, c) \notin m(<)$;
(Trans)	if $(c, d) \in m(<)$ and $(d, e) \in m(<)$, then $(c, e) \in m(<)$;
(Lin)	$(c, d) \in m(<)$ or $(d, c) \in m(<)$ or $(c, d) \in m(1')$;

(3) m extends to all compound relational terms $R \in \mathbb{PRT}$ as follows:

- $m(-R) = (U \times U) \setminus m(R)$;
- $m(R \cup S) = (m(R) \cup m(S))$;
- $m(R \cap S) = (m(R) \cap m(S))$;
- $m(R^{-1}) = m(R)^{-1}$;

- $m(R;S) = (m(R);m(S))$;

(4) $\mathbb{I}(U)^+ = \{[c,d] \in U \times U : (c,d) \in m(< \cup 1')\}$;

(5) $m(1) = \mathbb{I}(U)^+ \times \mathbb{I}(U)^+$;

(6) $m(B) = \{([c,d],[c',d']) \in m(1) : (c,c') \in m(1') \wedge (d',d) \in m(<)\}$;

(7) $m(E) = \{([c,d],[c',d']) \in m(1) : (c,c') \in m(<) \wedge (d,d') \in m(1')\}$;

(8) m extends to all compound relational terms $R \in \mathbb{IRT}$ as in (3) except for the clause for $-R$: $m(-R) = m(1) \setminus m(R)$.

An RL_{HS}-*valuation* in a model $\mathcal{M} = (U, \mathbb{I}(U)^+, m)$ is any function $v\colon \mathbb{PV} \cup \mathbb{IV} \to U \cup \mathbb{I}(U)^+$ such that:

– if $x \in \mathbb{PV}$ then $v(x) \in U$;

– if $i \in \mathbb{IV}$ then $v(i) = [v(i_1), v(i_2)] \in \mathbb{I}(U)^+$.

We say that v *satisfies a formula* $x\ R\ y$ ($\mathcal{M}, v \models x\ R\ y$ for short) iff $(v(x), v(y)) \in m(R)$. A formula is *true* in $\mathcal{M}$ whenever it is satisfied in $\mathcal{M}$ by every valuation v. A formula is RL_{HS}-*valid* whenever it is true in every RL_{HS}-model.

4. Translation

In this section we present a translation of the formulas of logic HS into relational terms of RL_{HS}. We follow a general principle of translation of modal formulas presented in [ORŁ 88]: modal formulas should be mapped into terms which represent right ideal relations, that is the relations satisfying the condition $R;1 = R$. It is known that the Boolean operations preserve the property of being a right ideal relation, and the composition of any relation with a right ideal relation results in a right ideal relation. So our definition of translation enforces the property of having a right ideal translation for propositional variables. It follows that the property is guaranteed for the formulas built with the classical propositional connectives. Moreover, since the translation of the formulas built with the possibility operator is defined as a composition of the constant denoting an accessibility relation with the translation of the formula to which the possibility operator is applied, the translation results in a term representing a right ideal relation.

We consider the following translation function τ, that maps HS-formulas φ to RL_{HS}-formulas of the form $x\ R\ y$ as follows:

– for every propositional letter $p \in AP$, $\tau(p) = P;1$, where $P \in \mathbb{IRV}$ is a relational variable;

– $\tau(\neg\psi) = -\tau(\psi)$;

– $\tau(\psi_1 \vee \psi_2) = \tau(\psi_1) \cup \tau(\psi_2)$;

– $\tau(\langle B\rangle\psi) = B;\tau(\psi)$;

– $\tau(\langle E\rangle\psi) = E;\tau(\psi)$;

– $\tau(\langle \overline{B}\rangle\psi) = B^{-1};\tau(\psi)$;

– $\tau(\langle\overline{E}\rangle\psi) = E^{-1}; \tau(\psi)$.

PROPOSITION 1. — *For every HS-model $\mathbf{M}^+$ and for every HS-formula ψ there is an RL_{HS}-model $\mathcal{M}$ such that ψ is true in $\mathbf{M}^+$ iff $i\ \tau(\psi)\ j$ is true in $\mathcal{M}$, where $i, j \in \mathbb{IV}$ and $i \neq j$.*

PROOF. — Let ψ be an HS-formula, and let $\mathbf{M}^+ = \langle D, \mathbb{I}(D)^+, \mathcal{V}\rangle$ be an HS-model. We define the corresponding RL_{HS}-model $\mathcal{M} = (U, \mathbb{I}(U)^+, m)$ as follows:

– $U = D$;

– $\mathbb{I}(U)^+ = \mathbb{I}(D)^+$;

– $m(<) = \{(c, d) \in U : c < d\}$;

– $m(1) = \mathbb{I}(U)^+ \times \mathbb{I}(U)^+$ and $m(1') = \mathrm{Id}_U$;

– for every $p \in AP$, $m(P) = \{([c, d], [c', d']) \in m(1) : [c, d] \in \mathcal{V}(p)\}$;

– $m(B) = \{([c, d], [c', d']) \in m(1) : c = c', d' < d\}$;

– $m(E) = \{([c, d], [c', d']) \in m(1) : c < c', d' = d\}$.

Given a valuation v we show by induction on the structure of ψ that the following property holds:

$$\mathbf{M}^+, v(i) \models \psi \text{ iff } \mathcal{M}, v \models i\ \tau(\psi)\ j.$$

From that, we can conclude that ψ is true in $\mathbf{M}^+$ iff $i\ \tau(\psi)\ j$ is true in $\mathcal{M}$. By way of example we prove the required condition for the formulas of the form: $\psi_1 \vee \psi_2$, $\langle B\rangle\psi_1$ and $\langle\overline{E}\rangle\psi_1$.

– If $\psi = \psi_1 \vee \psi_2$ then $\mathbf{M}^+, v(i) \models \psi_1 \vee \psi_2$ iff $\mathbf{M}^+, v(i) \models \psi_1$ or $\mathbf{M}^+, v(i) \models \psi_2$, iff, by inductive hypothesis, $\mathcal{M}, v \models i\ \tau(\psi_1)\ j$ or $\mathcal{M}, v \models i\ \tau(\psi_2)\ j$, iff $\mathcal{M}, v \models i\ (\tau(\psi_1) \cup \tau(\psi_2))\ j$ iff $\mathcal{M}, v \models i\ \tau(\psi_1 \vee \psi_2)\ j$.

– If $\psi = \langle B\rangle\psi_1$ then $\mathbf{M}^+, v(i) \models \langle B\rangle\psi_1$ iff there exists $c' < v(i_2)$ such that $\mathbf{M}^+, [v(i_1), c'] \models \psi_1$, iff, by inductive hypothesis and by definition of $\mathcal{M}$, $(v(i), [v(x_1), c']) \in m(B)$ and $([v(i_1), c'], [v(j_1), v(j_2)]) \in m(\tau(\psi_1))$, iff $\mathcal{M}, v \models i\ (B; \tau(\psi_1))\ j$ iff $\mathcal{M}, v \models i\ \tau(\langle B\rangle\psi_1)\ j$.

– Finally, if $\psi = \langle\overline{E}\rangle\psi_1$ then $\mathbf{M}^+, v(i) \models \langle\overline{E}\rangle\psi_1$ iff there exists $c' < v(i_1)$ such that $\mathbf{M}^+, [c', v(i_2)] \models \psi_1$, iff, by inductive hypothesis and by definition of $\mathcal{M}$, $(v(i), [c', v(i_2)]) \in m(E^{-1})$ and $([c', v(i_2)], [v(j_1), v(j_2)]) \in m(\tau(\psi_1))$, iff $\mathcal{M}, v \models i\ (E^{-1}; \tau(\psi_1))\ j$ iff $\mathcal{M}, v \models i\ \tau(\langle\overline{E}\rangle\psi_1)\ j$. ■

PROPOSITION 2. — *For every RL_{HS}-model $\mathcal{M}$ and for every HS-formula ψ there is an HS-model $\mathbf{M}^+$ such that ψ is true in $\mathbf{M}^+$ iff $i\ \tau(\psi)\ j$ is true in $\mathcal{M}$, where $i, j \in \mathbb{IV}$ and $i \neq j$.*

PROOF. — Let ψ be an HS-formula, and let $\mathcal{M} = (U, \mathbb{I}(U)^+, m)$ be an RL_{HS}-model. We define the corresponding HS-model $\mathbf{M}^+ = \langle D, \mathbb{I}(D)^+, \mathcal{V}\rangle$ as follows:

– $D = U$;

– for all $c, d \in U$, $c < d$ iff $(c, d) \in m(<)$;

– for all $p \in AP$, $[c,d] \in \mathcal{V}(p)$ iff $([c,d],[c',d']) \in m(P;1)$, for all $[c',d'] \in \mathbb{I}(U)^+$.

Since $m(<)$ is a strict linear ordering on U, then $\langle D, < \rangle$ is a strict linear ordering, and thus $\mathbf{M}^+$ is correctly defined.

Given a valuation v we show by induction on the structure of ψ that the following property holds:

$$\mathcal{M}, v \models i\ \tau(\psi)\ j \text{ iff } \mathbf{M}^+, v(i) \models \psi.$$

From that, we can conclude that $i\ \tau(\psi)\ j$ is true in $\mathcal{M}$ iff ψ is true in $\mathbf{M}^+$. By way of example we prove the required condition for the formulas of the form: $\neg\psi_1$, $\langle E\rangle\psi_1$ and $\langle \overline{B}\rangle\psi_1$.

– If $\psi = \neg\psi_1$ then $\mathcal{M}, v \models i\ \tau(\neg\psi_1)\ j$ iff $\mathcal{M}, v \models i\ (-\tau(\psi_1))\ j$ iff $\mathcal{M}, v \not\models i\ \tau(\psi_1)\ j$ iff, by inductive hypothesis, $\mathbf{M}^+, v(i) \not\models \psi_1$ iff, $\mathbf{M}^+, v(i) \models \neg\psi_1$.

– If $\psi = \langle E\rangle\psi_1$ then $\mathcal{M}, v \models i\ \tau(\langle E\rangle\psi_1)\ j$ iff $\mathcal{M}, v \models i\ (E;\tau(\psi_1))\ j$ iff, there exists $[c',d'] \in \mathbb{I}(D)^+$ such that $(v(i),[c',d']) \in m(E)$ and $([c',d'],v(j)) \in m(\tau(\psi_1))$, iff, by the definition of $m(E)$ and by inductive hypothesis, $v(i_2) = d'$, $v(i_1) < c'$, and $\mathbf{M}^+, [c',d'] \models \psi_1$, iff $\mathbf{M}^+, v(i) \models \langle E\rangle\psi_1$.

– If $\psi = \langle \overline{B}\rangle\psi_1$ then $\mathcal{M}, v \models i\ \tau(\langle \overline{B}\rangle\psi_1)\ j$ iff $\mathcal{M}, v \models i\ (B^{-1};\tau(\psi_1))\ j$ iff, there exists $[c',d'] \in \mathbb{I}(D)^+$ such that $(v(i),[c',d']) \in m(B^{-1})$ and $([c',d'],v(j)) \in m(\tau(\psi_1))$, iff, by the definition of $m(B^{-1})$ and by inductive hypothesis, $v(i_1) = c'$, $d' > v(i_2)$, and $\mathbf{M}^+, [c',d'] \models \psi_1$, iff $\mathbf{M}^+, v(i) \models \langle \overline{B}\rangle\psi_1$. ■

From the above propositions we obtain:

THEOREM 3. — *For every HS-formula ψ, ψ is HS-valid iff $i\ \tau(\psi)\ j$ is RL_{HS}-valid.*

5. The proof system for logic RL_{HS}

The proof system for logic RL_{HS} presented in this section belongs to the family of dual tableau systems, as mentioned in Section 1. It consists of axiomatic sets of formulas and rules which apply to finite sets of formulas. The axiomatic sets take the place of axioms. There are three groups of rules: the rules which reflect definitions of the standard relational operations; the rules which enable us to decompose interval relations into point relations according to the definitions recalled in Section 1; the rules which reflect the properties of the temporal ordering assumed in the models of the HS logic. The rules have the following general form:

$$(*) \qquad \frac{\Phi}{\Phi_1 \mid \ldots \mid \Phi_n}$$

where $\Phi_1, \ldots, \Phi_n$ are finite non-empty sets of formulas, $n \geq 1$, and Φ is a finite (possibly empty) set of formulas. Φ is called the *premise* of the rule, and $\Phi_1, \ldots, \Phi_n$ are called its *conclusions*. A rule of the form $(*)$ is said to be *applicable* to a set X of formulas whenever $\Phi \subseteq X$. As a result of application of a rule of the form $(*)$ to a set X, we obtain the sets $(X \setminus \Phi) \cup \Phi_i$, $i = 1, \ldots, n$. As usual, any concrete rule will always be presented in a short form without set brackets.

In dual tableau systems proofs have the form of finitely branching trees. Branching is interpreted as conjunction and the sets of formulas in the nodes of the trees are interpreted as disjunctions of their members. A formula is provable whenever there exists a closed proof tree for it. We close a branch of the proof tree whenever it contains a node with an axiomatic set of formulas. The tree is closed if all of its branches are closed.

The completeness theorem states that any valid formula has a closed proof tree. This theorem is usually proved by contradiction. Assuming that a valid formula does not have a closed proof tree, we consider any of those trees. It necessarily has an infinite branch (it is guaranteed by König's lemma). We make this tree complete: whenever a rule is applicable to a node of the tree, then it has been applied. The principles of construction of a complete proof tree are stated in the form of what is called completion conditions (Section 5.5). Next, from the syntactic resources of an infinite branch we construct a branch structure (Section 5.6) and we prove that it is a model of logic RL_{HS} in which the original formula is not true.

We say that a variable in a rule is *new* whenever it appears in a conclusion of the rule and does not appear in its premise.

5.1. *Decomposition rules*

Standard decomposition rules

Let $x, y, z \in \mathbb{PV}$ and $R, S \in \mathbb{PRT}$ or $x, y, z \in \mathbb{IV}$ and $R, S \in \mathbb{IRT}$.

$(\cup)$ $\dfrac{x\ (R \cup S)\ y}{x\ R\ y, x\ S\ y}$ $\qquad$ $(-\cup)$ $\dfrac{x\ -(R \cup S)\ y}{x\ -R\ y \mid x\ -S\ y}$

$(\cap)$ $\dfrac{x\ (R \cap S)\ y}{x\ R\ y \mid x\ S\ y}$ $\qquad$ $(-\cap)$ $\dfrac{x\ -(R \cap S)\ y}{x\ -R\ y, x\ -S\ y}$

$(-)$ $\dfrac{x\ - -R\ y}{x\ R\ y}$

$(^{-1})$ $\dfrac{x\ R^{-1}\ y}{y\ R\ x}$ $\qquad$ $(-^{-1})$ $\dfrac{x\ -R^{-1}\ y}{y\ -R\ x}$

$(;)$ $\dfrac{x\ (R;S)\ y}{x\ R\ z, x\ (R;S)\ y \mid z\ S\ y, x\ (R;S)\ y}$ $\qquad$ z is any variable

$(-;)$ $\dfrac{x\ -(R;S)\ y}{x\ -R\ z, z\ -S\ y}$ $\qquad$ z is a new variable

Decomposition rules from interval relations to point relations

For $i, j \in \mathbb{IV}$ and $R \in \mathbb{IRA}$:

$(R_1)\quad \dfrac{i\ R\ j}{i_1\ 1'\ k_1, i\ R\ j \mid i_2\ 1'\ k_2, i\ R\ j \mid k\ R\ j, i\ R\ j}$ with k any interval variable.

$(R_2)\quad \dfrac{i\ R\ j}{j_1\ 1'\ k_1, i\ R\ j \mid j_2\ 1'\ k_2, i\ R\ j \mid i\ R\ k, i\ R\ j}$ with k any interval variable.

For $i, j \in \mathbb{IV}$:

$(B)\quad \dfrac{i\ B\ j}{i_1\ 1'\ j_1, i\ B\ j \mid j_2 < i_2, i\ B\ j}$ $\qquad (-B)\quad \dfrac{i\ -B\ j}{i_1\ -1'\ j_1, j_2 -< i_2, i\ -B\ j}$

$(E)\quad \dfrac{i\ E\ j}{i_2\ 1'\ j_2, i\ E\ j \mid i_1 < j_1, i\ E\ j}$ $\qquad (-E)\quad \dfrac{i\ -E\ j}{i_2\ -1'\ j_2, i_1 -< j_1, i\ -E\ j}$

5.2. *Specific rules*

Rules for $1'$

For $x, y \in \mathbb{PV}$ and $R \in \mathbb{PRC}$:

$(1'1)\quad \dfrac{x\ R\ y}{xRz, x\ R\ y \,|\, y\ 1'\ z, x\ R\ y}$ $\qquad (1'2)\quad \dfrac{x\ R\ y}{x\ 1'\ z, x\ R\ y \,|\, z\ R\ y, x\ R\ y}$

with z any point variable.

Rules for $<$

For $x, y \in \mathbb{PV}$:

(Irref$<$) $\quad \dfrac{}{x < x}$

(Tran$<$) $\quad \dfrac{x < y}{x < y, x < z \,|\, x < y, z < y}$ $\qquad z$ is any point variable

5.3. *Axiomatic sets*

An axiomatic set is a set including a subset of any of the following forms:

(a1) $x\ R\ y, x\ -R\ y$, for either $x, y \in \mathbb{PV}$ and $R \in \mathbb{PRT}$ or $x, y \in \mathbb{IV}$ and $R \in \mathbb{IRT}$;
(a2) $x\ 1'\ x$ for $x \in \mathbb{PV}$;
(a3) $x < y, x\ 1'\ y, y < x$ for $x, y \in \mathbb{PV}$;
(a4) $i\ 1\ j$ for $i, j \in \mathbb{IV}$;
(a5) $i_1 < i_2, i_1\ 1'\ i_2$ for $i \in \mathbb{IV}$.

5.4. *Proof trees and soundness of the proof system*

A finite set of formulas $\{x_1\ R_1\ y_1, \ldots, x_n\ R_n\ y_n\}$ is said to be an RL_{HS}-*set* whenever for every RL_{HS}-model $\mathcal{M}$ and every valuation v in $\mathcal{M}$ there exists $i \in \{1, \ldots, n\}$ such that $x_i\ R_i\ y_i$ is satisfied by v in $\mathcal{M}$.

Let Φ be a non-empty set of RL_{HS}-formulas. A rule $\dfrac{\Phi}{\Phi_1 \mid \ldots \mid \Phi_n}$ is RL_{HS}-correct whenever the following property holds: Φ is an RL_{HS}-set if and only if Φ_i is an RL_{HS}-set, for every $i \in \{1, \ldots, n\}$. When Φ is empty, RL_{HS}-correctness can be expressed as follows: rule $\dfrac{}{\Phi_1 \mid \ldots \mid \Phi_n}$ is RL_{HS}-correct if and only if there exists $i \in \{1, \ldots, n\}$ such that Φ_i is not an RL_{HS}-set.

DEFINITION 4. — *Let $x\ R\ y$ be an RL_{HS}-formula. An RL_{HS}*-proof tree for *$x\ R\ y$ is a tree with the following properties:*

– the formula $x\ R\ y$ is at the root of the tree;

– each node except the root is obtained by application of an RL_{HS}-rule to its predecessor node;

– a node does not have successors whenever it is an RL_{HS}-axiomatic set.

Due to the forms of the rules we obtain the following:

REMARK 5. — If a node of an RL_{HS}-proof tree does not contain an axiomatic subset and contains an RL_{HS}-formula $x\ R\ y$ or $x\ {-R}\ y$, for atomic R, then all of its successors contain this formula as well. □

A branch of an RL_{HS}-proof tree is said to be RL_{HS}-*closed* whenever it contains a node with an axiomatic set of formulas. A proof tree is RL_{HS}-closed if and only if all of its branches are closed.

A formula is *provable* whenever there is a closed RL_{HS}-proof tree for it.

PROPOSITION 6. —

1) All RL_{HS}-rules are correct.

2) All RL_{HS}-axiomatic sets are RL_{HS}-sets.

PROOF. —

Proof of 1) We show the correctness of rules (B) and $(-E)$. Proving correctness of the other rules is similar. Let $\mathcal{M} = (U, \mathbb{I}(U)^+, m)$ be an RL_{HS}-model and let v be an RL_{HS}-valuation.

It is easy to see that if $\{i\ B\ j\}$ is an RL_{HS}-set, then $\{i_1\ 1'\ j_1, i\ B\ j\}$ and $\{j_2 < i_2, i\ B\ j\}$ are RL_{HS}-sets. Assume $\mathcal{M}, v \models i_1\ 1'\ j_1$ and $\mathcal{M}, v \models j_2 < i_2$, that is $v(i), v(j) \in \mathbb{I}(U)^+$, $(v(i_1), v(j_1)) \in m(1')$ and $(v(j_2), v(i_2)) \in m(<)$. By the definition of $m(B)$, we obtain $(v(i), v(j)) \in m(B)$. In the remaining cases the proofs are similar.

The proof of correctness of the rule $(-E)$ is analogous. Assume $\mathcal{M}, v \models i_2 -1' \ j_2$ or $\mathcal{M}, v \models i_1 -< j_1$, that is $v(i), v(j) \in \mathbb{I}(U)^+$ and $(v(i_2), v(j_2)) \notin m(1')$ or $(v(i_1), v(j_1)) \notin m(<)$. By the definition of $m(E)$, we obtain $(v(i), v(j)) \notin m(E)$, hence $(v(i), v(j)) \in m(-E)$. The remaining parts of the proof are obvious.

Proof of 2) It suffices to show that all sets of the forms (a1)-(a5) are RL_{HS}-sets. We prove it for sets (a4) and (a5). In the remaining cases the proofs are similar.

By the definition of an RL_{HS}-model, for every RL_{HS}-valuation v and for every $i, j \in \mathbb{IV}$, $(v(i), v(j)) \in \mathbb{I}(U)^+ \times \mathbb{I}(U)^+$, hence $(v(i), v(j)) \in m(1)$. Therefore $\{i \ 1 \ j\}$ is an RL_{HS}-set.

By the definition, for every RL_{HS}-valuation v and for every $i \in \mathbb{IV}$, $v(i) \in \mathbb{I}(U)^+$, that is $(v(i_1), v(i_2)) \in m(< \cup 1')$. Therefore in every RL_{HS}-model $\mathcal{M}$, $\mathcal{M}, v \models i_1 < i_2$ or $\mathcal{M}, v \models i_1 \ 1' \ i_2$. Hence $\{i_1 < i_2, i_1 \ 1' \ i_2\}$ is an RL_{HS}-set. ■

Due to Proposition 6, we obtain the following theorem.

THEOREM 7. — *Let $x \ R \ y$ be an RL_{HS}-formula. If $x \ R \ y$ is provable, then it is RL_{HS}-valid.*

5.5. *Completion conditions*

Given a proof tree and a branch b in it, we write, by abusing the notation, $x \ R \ y \in b$ if $x \ R \ y$ belongs to a set of formulas of a node of branch b. A non-closed branch b is said to be RL_{HS}-complete whenever it satisfies the following *completion conditions*.

For all variables x, y, z and relational terms R, S such that either $x, y, z \in \mathbb{PV}$ and $R, S \in \mathbb{PRT}$ or $x, y, z \in \mathbb{IV}$ and $R, S \in \mathbb{IRT}$:

– Cpl($\cup$) (Cpl($-\cap$)) If $x \ (R \cup S) \ y \in b$ (resp. $x \ -(R \cap S) \ y \in b$), then both $x \ R \ y \in b$ (resp. $x \ -R \ y \in b$) and $x \ S \ y \in b$ (resp. $x \ -S \ y \in b$).

– Cpl($\cap$) (Cpl($-\cup$)) If $x \ (R \cap S) \ y \in b$ (resp. $x \ -(R \cup S) \ y \in b$), then either $x \ R \ y \in b$ (resp. $x \ -R \ y \in b$) or $x \ S \ y \in b$ (resp. $x \ -S \ y \in b$).

– Cpl($-$) If $x \ (- - R) \ y \in b$, then $x \ R \ y \in b$.

– Cpl($^{-1}$) If $x \ R^{-1} \ y \in b$, then $y \ R \ x \in b$.

– Cpl($-^{-1}$) If $x \ -R^{-1} \ y \in b$, then $y \ -R \ x \in b$.

– Cpl(;) If $x \ (R; S) \ y \in b$, then for every z either $x \ R \ z \in b$ or $z \ S \ y \in b$.

– Cpl($-$;) If $x \ -(R; S) \ y \in b$, then for some z both $x \ -R \ z \in b$ and $z \ -S \ y \in b$.

For all $x, y \in \mathbb{PV}$ and $R \in \mathbb{PRC}$:

– Cpl($1'1$) If $x \ R \ y \in b$ then, for every $z \in \mathbb{PV}$, $x \ R \ z \in b$ or $y \ 1' \ z \in b$.

– Cpl($1'2$) If $x \ R \ y \in b$ then, for every $z \in \mathbb{PV}$, $z \ 1' \ x \in b$ or $z \ R \ y \in b$.

For all $x, y \in \mathbb{PV}$:

– Cpl(Irref<) $x < x \in b$.

– Cpl(Tran<) If $x < y \in b$ then, for every $z \in \mathbb{PV}$, $x < z \in b$ or $z < y \in b$.

For all $i, j \in \mathbb{IV}$:

– Cpl(R_1) If $i\ R\ j \in b$, then for every $k \in \mathbb{IV}$ either $i_1\ 1'\ k_1 \in b$, $i_2\ 1'\ k_2 \in b$, or $k\ R\ j \in b$.

– Cpl(R_2) If $i\ R\ j \in b$, then for every $k \in \mathbb{IV}$ either $j_1\ 1'\ k_1 \in b$, $j_2\ 1'\ k_2 \in b$, or $i\ R\ k \in b$.

– Cpl(B) If $i\ B\ j \in b$ then either $i_1\ 1'\ j_1 \in b$ or $j_2 < i_2 \in b$.

– Cpl($-B$) If $i\ -B\ j \in b$ then $i_1\ -1'\ j_1, j_2\ -< i_2 \in b$.

– Cpl(E) If $i\ E\ j \in b$ then either $i_2\ 1'\ j_2 \in b$ or $i_1 < j_1 \in b$.

– Cpl($-E$) If $i\ -E\ j \in b$ then $i_2\ -1'\ j_2, i_1\ -< j_1 \in b$.

An RL_{HS}-proof tree is said to be RL_{HS}-complete if and only if all of its non-closed branches are RL_{HS}-complete. An RL_{HS}-complete non-closed branch is said to be RL_{HS}-open.

By Remark 5 and since the set containing a subset $\{x\ R\ y, x\ -R\ y\}$ is axiomatic, the following fact can be easily proved by induction:

FACT 8. — Let b be an open branch of an RL_{HS}-proof tree. Then there is no RL_{HS}-formula $x\ R\ y$ such that $x\ R\ y \in b$ and $x\ -R\ y \in b$.

5.6. *Branch model*

Let b be an open branch of a proof tree. The branch structure $\mathcal{M}^b = (U^b, \mathbb{I}(U^b)^+, m^b)$ is defined as follows:

– $U^b = \mathbb{PV}$;

– $m^b(R) = \{(x, y) \in U^b \times U^b : x\ R\ y \notin b\}$ for $R \in \mathbb{PRC}$;

– m^b extends to all compound relational terms $R \in \mathbb{PRT}$ as in RL_{HS}-models;

– $\mathbb{I}(U^b)^+ = \{[c, d] : c, d \in U^b, (c, d) \in m^b(< \cup 1')\}$;

– $m^b(R) = \{(i, j) \in \mathbb{I}(U^b)^+ \times \mathbb{I}(U^b)^+ : i\ R\ j \notin b\}$ for $R \in \mathbb{IRV}$;

– $m^b(1) = \mathbb{I}(U^b)^+ \times \mathbb{I}(U^b)^+$;

– $m^b(B) = \{([c, d], [c', d']) \in \mathbb{I}(U^b)^+ \times \mathbb{I}(U^b)^+ : (c, c') \in m^b(1') \wedge (d', d) \in m^b(<)\}$;

– $m^b(E) = \{([c, d], [c', d']) \in \mathbb{I}(U^b)^+ \times \mathbb{I}(U^b)^+ : (c, c') \in m^b(<) \wedge (d, d') \in m^b(1')\}$;

– m^b extends to all compound relational terms $R \in \mathbb{IRT}$ as in RL_{HS}-models.

PROPOSITION 9. — *$m^b(1')$ is an equivalence relation on U^b.*

PROOF. — $x\ 1'\ x \notin b$ for every $x \in U^b$, because $\{x\ 1'\ x\}$ is axiomatic. Thus, $(x, x) \in m^b(1')$ for every $x \in U^b$. Therefore $m^b(1')$ is reflexive. Assume $(x, y) \in$

$m^b(1')$, that is $x\ 1'\ y \notin b$. Suppose $(y, x) \notin m^b(1')$. Then $y\ 1'\ x \in b$. By the completion condition Cpl($1'1$) we get $y\ 1'\ y \in b$ or $x\ 1'\ y \in b$, a contradiction. Therefore $m^b(1')$ is symmetric. Assume $(x, y) \in m^b(1')$ and $(y, z) \in m^b(1')$ that is $x\ 1'\ y \notin b$ and $y\ 1'\ z \notin b$. Suppose $(x, z) \notin m^b(1')$. Then $x\ 1'\ z \in b$. By the completion condition Cpl($1'1$) we obtain $x\ 1'\ y \in b$ or $z\ 1'\ y \in b$. In the first case we get a contradiction. In the second case, by the application of the completion condition Cpl($1'1$) to $z\ 1'\ y \in b$ we obtain $z\ 1'\ z \in b$ or $y\ 1'\ z \in b$, and in both cases we get a contradiction. Therefore $m^b(1')$ is transitive, and hence $m^b(1')$ is an equivalence relation. ■

PROPOSITION 10. — *Let b be an open branch. A structure $\mathcal{M}^b$ satisfies the conditions (2)-(8) from the definition of RL_{HS}-models.*

PROOF. — The conditions (3)-(8) are satisfied by the definition of a branch structure. Therefore it suffices to show that $m^b(<)$ satisfies the conditions (Irref), (Trans) and (Lin).

By the completion condition Cpl(Irref<), for every $x \in U^b$, we have $x < x \in b$, but it means that $(x, x) \notin m^b(<)$ for every $x \in U^b$, therefore $m^b(<)$ is irreflexive.

To prove transitivity, assume $(x, y) \in m^b(<)$ and $(y, z) \in m^b(<)$, that is $x < y \notin b$ and $y < z \notin b$. Suppose $(x, z) \notin m^b(<)$. Then $x < z \in b$. By the completion condition Cpl(Tran<) $x < y \in b$ or $y < z \in b$, a contradiction. Therefore $m^b(<)$ satisfies the condition (Trans).

Since b is open, for all $x, y \in U^b$, $x < y \notin b$ or $y < x \notin b$ or $x\ 1'\ y \notin b$. It means that $(x, y) \in m^b(<)$ or $(y, x) \in m^b(<)$ or $(x, y) \in m^b(1')$, therefore $m^b(<)$ satisfies the condition (Lin). ■

Given a structure $\mathcal{M}^b = (U^b, \mathbb{I}(U^b)^+, m^b)$, let $v^b \colon \mathbb{PV} \cup \mathbb{IV} \to U^b \cup \mathbb{I}(U^b)^+$ be such that $v^b(x) = x$ for every $x \in \mathbb{PV}$ and $v(i) = [i_1, i_2]$ for every $i \in \mathbb{IV}$.

PROPOSITION 11. — *Let b be an open branch, and let $\mathcal{M}^b$ be the corresponding branch structure. The function v^b satisfies the definition of RL_{HS}-valuation.*

PROOF. — By the definition of v^b, if $x \in \mathbb{PV}$ then $v^b(x) \in U^b$ and, if $i \in \mathbb{IV}$ then $v^b(i) = [v^b(i_1), v^b(i_2)]$. It remains to show that for every $i \in \mathbb{IV}$, $(v^b(i_1), v^b(i_2)) \in m^b(< \cup 1')$. Suppose that there exists $i \in \mathbb{IV}$ such that $(v^b(i_1), v^b(i_2)) \notin m^b(< \cup 1')$. This implies that $(v^b(i_1), v^b(i_2)) \notin m^b(<)$ and $(v^b(i_1), v^b(i_2)) \notin m^b(1')$. By the definition of m^b, this implies that $i_1 < i_2 \in b$ and $i_1\ 1'\ i_2 \in b$, which means that b is closed, a contradiction. ■

Let satisfiability of formulas in $\mathcal{M}^b$ be defined as for RL_{HS}-models.

PROPOSITION 12. — *Let b be an open branch and let $x\ R\ y$ be an RL_{HS}-formula. Then the following holds:*

$$(*) \text{ if } \mathcal{M}^b, v^b \models x\ R\ y, \text{ then } x\ R\ y \notin b$$

PROOF. — The proof is by induction on the complexity of formulas. For $R \in \mathbb{PRC} \cup \mathbb{IRV}$ and its complement, $(*)$ holds by the definition.

– For $R = 1$, $(*)$ holds trivially, since $i\ 1\ j$ is axiomatic.

– Let $R = B$. Assume $(i, j) \in m^b(B)$, that is $(i_1, j_1) \in m^b(1')$ and $(j_2, i_2) \in m^b(<)$. Then $i_1\ 1'\ j_1 \notin b$ and $j_2 < i_2 \notin b$. Suppose $i\ B\ j \in b$. By the completion condition Cpl(B), either $i_1\ 1'\ j_1 \in b$ or $j_2 < i_2 \in b$, a contradiction.

– Let $R = -B$. Assume $(i, j) \notin m^b(B)$, that is $(i_1, j_1) \notin m^b(1')$ or $(j_2, i_2) \notin m^b(<)$. Then $i_1\ 1'\ j_1 \in b$ or $j_2 < i_2 \in b$. Suppose $i\ -B\ j \in b$. By the completion condition Cpl($-B$), both $i_1\ -1'\ j_1 \in b$ and $j_2\ -<\ i_2 \in b$, a contradiction.

– Let $R = E$. Assume $(i, j) \in m^b(E)$, that is $(i_1, j_1) \in m^b(<)$ and $(j_2, i_2) \in m^b(1')$. Then $i_1 < j_1 \notin b$ and $j_2\ 1'\ i_2 \notin b$. Suppose $i\ E\ j \in b$. By the completion condition Cpl(E), either $i_1 < j_1 \in b$ or $j_2\ 1'\ i_2 \in b$, a contradiction.

– Let $R = -E$. Assume $(i, j) \notin m^b(E)$, that is $(i_1, j_1) \notin m^b(<)$ or $(j_2, i_2) \notin m^b(1')$. Then $i_1 < j_1 \in b$ or $j_2\ 1'\ i_2 \in b$. Suppose $i\ -E\ j \in b$. By the completion condition Cpl($-E$), both $i_1\ -<\ j_1 \in b$ and $j_2\ -1'\ i_2 \in b$, a contradiction.

Therefore $(*)$ holds for all atomic formulas and its complements. The remaining cases can be proved in a standard way using the completion conditions and the property of Fact 8. See also [GOL 06a]. ■

It is easy to check that the branch structure satisfies the extensionality property.

PROPOSITION 13. — *Let $\mathcal{M}^b$ be a branch structure determined by an open branch b. Then the following hold:*

– For every $R \in \mathbb{PRC}$ and for all $x, y, z, t \in \mathbb{PV}$: if $(x, y) \in m^b(R)$ and $(x, z), (y, t) \in m^b(1')$, then $(z, t) \in m^b(R)$.

– For every $R \in \mathbb{IRA}$ and for all $i, j, k, l \in \mathbb{IV}$ such that $i = [i_1, i_2], j = [j_1, j_2], k = [k_1, k_2], l = [l_1, l_2]$: if $(i, j) \in m^b(R)$ and $(i_1, k_1), (i_2, k_2), (j_1, l_1), (j_2, l_2) \in m^b(1')$, then $(k, l) \in m^b(R)$.

Since $m^b(1')$ is an equivalence relation on U^b, given a branch structure $\mathcal{M}^b$, we may define the quotient model $\mathcal{M}_q^b = (U_q^b, \mathbb{I}(U_q^b)^+, m_q^b)$ as follows:

– $U_q^b = \{\|x\| : x \in U^b\}$, where $\|x\|$ is the equivalence class of $m^b(1')$ generated by x;

– $\mathbb{I}(U_q^b)^+ = \{[\|c\|, \|d\|] : [c, d] \in \mathbb{I}(U^b)^+\}$;

– $m_q^b(R) = \{(\|x\|, \|y\|)) \in U_q^b \times U_q^b : (x, y) \in m^b(R)\}$, for every $R \in \mathbb{PRC}$;

– m_q^b extends to all compound relational terms $R \in \mathbb{PRT}$ as in RL_{HS}-models;

– $m_q^b(R) = \{([\|c\|, \|d\|], [\|c'\|, \|d'\|]) \in \mathbb{I}(U_q^b)^+ \times \mathbb{I}(U_q^b)^+ : ([c, d], [c', d']) \in m^b(R)\}$, for every $R \in \mathbb{IRA}$;

– m_q^b extends to all compound relational terms $R \in \mathbb{IRT}$ as in RL_{HS}-models.

Due to Proposition 13, the quotient model $\mathcal{M}_q^b$ is well defined, that is the definitions of $m_q^b(R)$ and $\mathbb{I}(U_q^b)^+$ do not depend on the choice of the representatives of the equivalence classes.

PROPOSITION 14. — *The structure $\mathcal{M}_q^b$ is an RL_{HS}-model.*

PROOF. — We have to show that $m_q^b(1')$ is the identity on U_q^b. Indeed, for every $x, y \in \mathbb{PV}$ we have:

$$(\|x\|, \|y\|) \in m_q^b(1') \text{ iff } (x, y) \in m^b(1') \text{ iff } \|x\| = \|y\|$$

■

Let v_q^b be such that $v_q^b(x) = \|x\|$, for every $x \in \mathbb{PV}$, and $v_q^b(i) = [\|i_1\|, \|i_2\|]$, for every $i \in \mathbb{IV}$. It is easy to see that v_q^b is an RL_{HS}-valuation in $\mathcal{M}_q^b$, since v^b satisfies the definition of RL_{HS}-valuation.

By an easy induction we can prove the following:

PROPOSITION 15. — *For every RL_{HS}-formula $x\ R\ y$:*

$$(*) \quad \mathcal{M}^b, v^b \models x\ R\ y \quad \text{iff} \quad \mathcal{M}_q^b, v_q^b \models x\ R\ y$$

The above propositions enable us to prove the completeness of the proof system.

THEOREM 16 (COMPLETENESS OF RL_{HS}-SYSTEM). — *Let $x\ R\ y$ be an RL_{HS}-formula. If $x\ R\ y$ is RL_{HS}-valid, then $x\ R\ y$ is RL_{HS}-provable.*

PROOF. — Assume $x\ R\ y$ is RL_{HS}-valid. Suppose there is no closed RL_{HS}-proof tree for $x\ R\ y$. Consider a non-closed RL_{HS}-proof tree for $x\ R\ y$. We may assume that this tree is complete. Let b be an open branch of the complete RL_{HS}-proof tree for $x\ R\ y$. Since $x\ R\ y \in b$, by Proposition 12, the branch structure $\mathcal{M}^b$ does not satisfy $x\ R\ y$. By Proposition 15 also the quotient model $\mathcal{M}_q^b$ does not satisfy $x\ R\ y$. Since $\mathcal{M}_q^b$ is an RL_{HS}-model, $x\ R\ y$ is not RL_{HS}-valid, a contradiction. ■

6. HS-validity and RL_{HS}-provability

In this section we conclude the discussion of Sections 4 and 5 and we show how the proof system of logic RL_{HS} can be used to verify the validity and entailment of formulas of logic HS. We also present examples of derivations.

The following theorem follows from Theorems 3 and 16.

THEOREM 17. — *For every HS-formula φ, φ is HS-valid if and only if $i\ \tau(\varphi)\ j$ is RL_{HS}-provable.*

As an example of validity checking, consider the HS-formula $\varphi = \langle B\rangle\langle B\rangle p \rightarrow \langle B\rangle p$, which express the fact that $\langle B\rangle$ is a transitive modality. By the semantics of

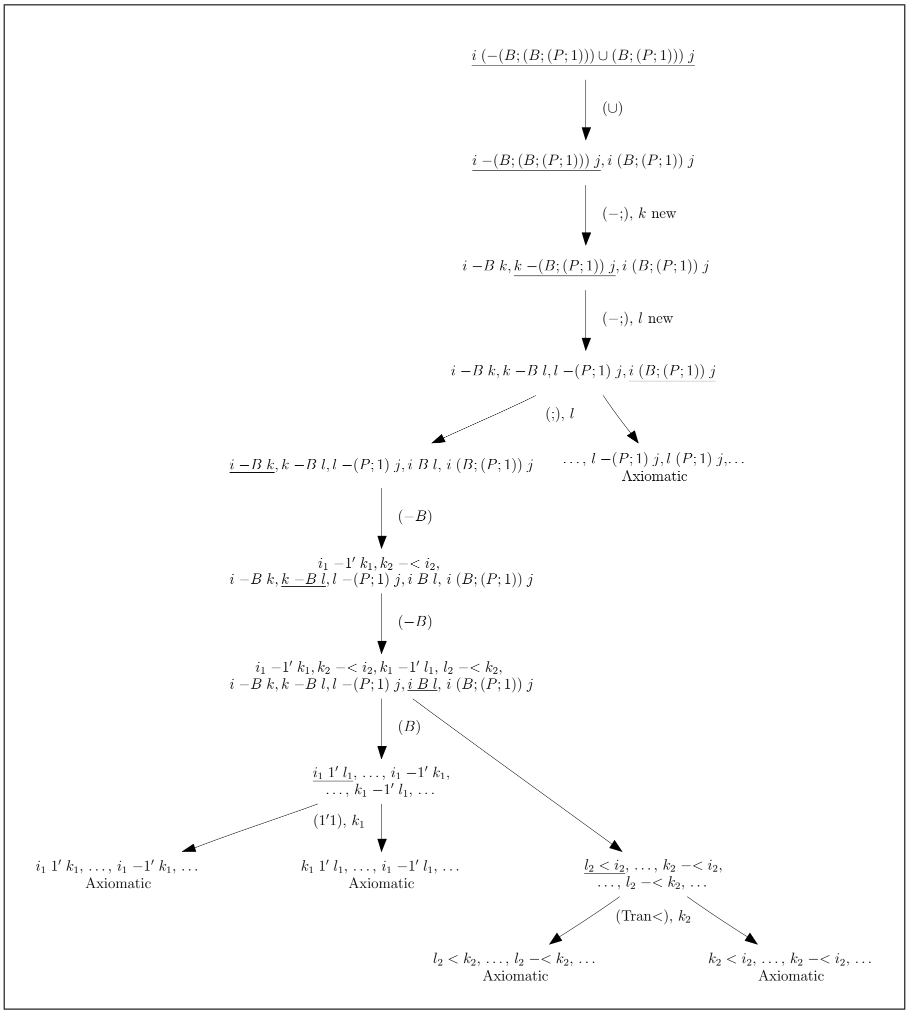

Figure 2. *Proof tree for* $\langle B\rangle\langle B\rangle p \to \langle B\rangle p$

HS, it is easy to see that φ is valid. The translation $\tau(\varphi)$ of the above formula into a relational term of RL_{HS} is $-(B;(B;(P;1))) \cup (B;(P;1))$. Figure 2 depicts an RL_{HS}-proof tree that shows that the relational formula $i\ \tau(\varphi)\ j$ is RL_{HS}-valid, and thus that φ is HS-valid. In each node of the proof tree we underline the formula to which a rule has been applied during the construction of the proof tree.

Let $R_1, \ldots, R_n, R$ be binary relations on $\mathbb{I}(U)^+$ and let $1 = \mathbb{I}(U)^+ \times \mathbb{I}(U)^+$. It is known that $R_1 = 1, \ldots, R_n = 1$ imply $R = 1$ iff $(1; -(R_1 \cap \ldots \cap R_n); 1) \cup R = 1$. Therefore, for every RL_{HS}-model $\mathcal{M}$, $\mathcal{M} \models i\ R_1\ j$, …, $\mathcal{M} \models i\ R_n\ j$ imply $\mathcal{M} \models$

$i\ R\ j$ iff $\mathcal{M} \models i\ (1; -(R_1 \cap \ldots \cap R_n); 1) \cup R)\ j$ which means that entailment in RL_{HS} can be expressed in its language.

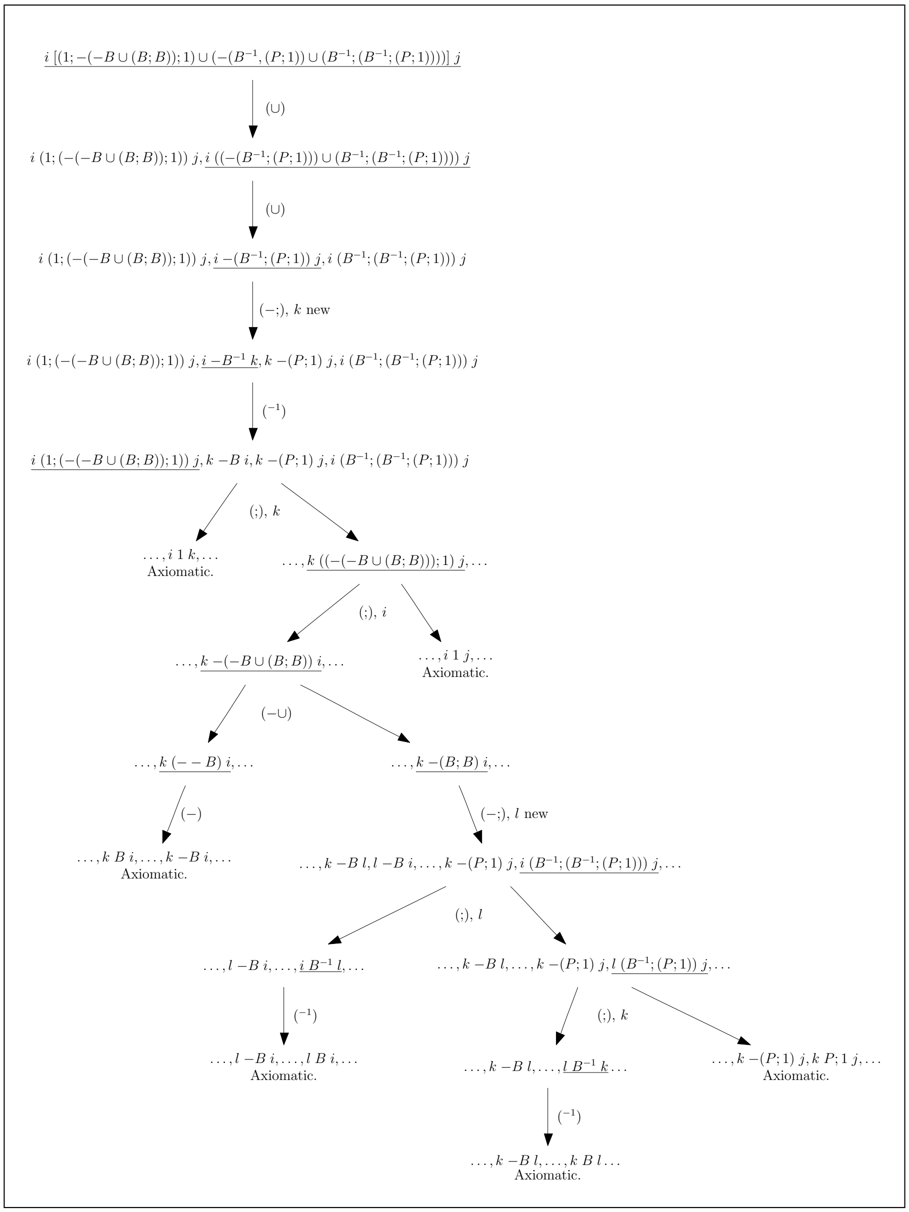

Figure 3. *Proof tree showing that* $\mathrm{Dense}_{\mathsf{RL}_{HS}}$ *entails* Dense_{HS}

As an example of entailment in RL_{HS}, suppose that $<$ is a dense linear ordering. It can be shown that density can be expressed in terms of the relation B by the following axiom:

$$\text{Dense}_{\mathsf{RL}_{HS}} := B \subseteq (B; B),$$

that is equivalent to $-B \cup (B; B) = 1$. In [VEN 90], the following HS-axiom is proposed to express density:

$$\text{Dense}_{HS} := \langle \overline{B} \rangle p \to \langle \overline{B} \rangle \langle \overline{B} \rangle p.$$

Its RL_{HS}-translation is $\tau(\text{Dense}_{HS}) = -(B^{-1}; (P; 1)) \cup (B^{-1}; (B^{-1}; (P; 1)))$. To prove that $\text{Dense}_{\mathsf{RL}_{HS}}$ entails Dense_{HS} it is sufficient to show that the relational formula

$$i\ [(1; -(-B \cup (B; B)); 1) \cup (-(B^{-1}; (P; 1)) \cup (B^{-1}; (B^{-1}; (P; 1))))]\ j$$

is RL_{HS}-valid. Figure 3 depicts a closed proof tree for this formula, thus proving that Dense_{HS} is valid for every dense ordering. As in the previous example, in each node of the proof tree we underline the formula to which a rule has been applied during the construction of the proof tree.

It is known that the formula Dense_{HS} is satisfiable in a non-dense model, so Dense_{HS} does not entail $\text{Dense}_{\mathsf{RL}_{HS}}$.

7. Extensions of the relational system

In the previous sections we have provided a relational proof system for the interval temporal logic HS, interpreted over linear temporal domains. In this section we exploit the modularity of the relational approach, and we show how to adapt it to cope with other interval relations and other meaningful temporal domains.

7.1. *Considerations on the nature of intervals*

In Section 2, we considered the *non-strict* semantics of HS, where, given a strict ordering $\langle D, < \rangle$, the set of non-strict intervals $\mathbb{I}(D)^+$ is defined as the set of all $[c, d]$ such that $c \leq d$. This choice includes in the set of intervals also *point intervals*, that are intervals of the form $[c, c]$. In the literature another natural semantics for interval logics is considered, namely, the *strict* one, where point intervals are excluded. Given a strict ordering $\langle D, < \rangle$, a *strict interval* is a pair $[c, d]$ where $c < d$. The set of all strict intervals on D will be denoted by $\mathbb{I}(D)^-$, and the semantics of formulas is defined over *strict interval structures* $\langle D, \mathbb{I}(D)^- \rangle$ in a way analogous to the non-strict case.

In this section we show how to modify the relational proof system for RL_{HS} in the case of the strict semantics. To this end, we define the relational logic RL_{HS}^- (strict

RL_{HS}), characterized by the same syntax as non-strict RL_{HS}, but with a different semantics. An RL^-_{HS}-model is a tuple $\mathcal{M}^- = (U, \mathbb{I}(U)^-, m)$ where U and m are defined as in RL_{HS}-models, and $\mathbb{I}(U)^- = \{[c,d] \in U \times U : (c,d) \in m(<)\}$. An RL^-_{HS}-valuation is any function $v\colon \mathbb{PV} \cup \mathbb{IV} \to 2^U \cup 2^{\mathbb{I}(U)^- \times \mathbb{I}(U)^-}$ such that:

– if $x \in \mathbb{PV}$ then $v(x) \in U$;

– if $i \in \mathbb{IV}$ then $v(i) = [v(i_1), v(i_2)] \in \mathbb{I}(U)^-$.

The notions of satisfiability and validity of a formula are defined as in RL_{HS}.

A proof system for RL^-_{HS}

A proof system for RL^-_{HS} can be obtained from the proof system for RL_{HS} by substituting the axiomatic set (a5) with a new one:

(a5$^-$) $\quad i_1 < i_2$ for $i \in \mathbb{IV}$.

In the case of the strict semantics, for every valuation v and every interval variable i, we have $v(i) = [v(i_1), v(i_2)]$ with $v(i_1) < v(i_2)$. Hence, (a5$^-$) is an RL^-_{HS}-set. Correctness of the other rules of the proof system follows directly from the correctness of the rules for RL_{HS}. Thus, soundness of the RL^-_{HS}-proof system is straightforward.

Completeness of the proof system can be proved as in the case of RL_{HS}, with the only difference that, given an open branch b, the branch structure $\mathcal{M}^b = (U^b, \mathbb{I}(U^b)^-, m^b)$ is defined such that $\mathbb{I}(U^b)^- = \{[c,d] : c, d \in U^b, (c,d) \in m^b(<)\}$.

7.2. *Incorporating the other interval relations*

In this section we show how to modify the relational logic RL_{HS} and its proof system to obtain a relational logic RL_L (and a corresponding proof system) that is appropriate to any interval logic L that is based on unary modalities corresponding to Allen's relations. Generally speaking, any interval logic L is defined by the following abstract syntax:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \{\langle R_i \rangle\varphi, \langle \overline{R_i} \rangle\varphi : i \in I\}.$$

with I determining any choice of basic interval relations from the 13 Allen's relations. Given an interval logic L, the corresponding relational logic RL_L differs from RL_{HS} only in the choice of the set of interval relational constants, that is defined as $\mathbb{IRC} = \{1\} \cup \{R_i : i \in I\}$. Models of RL_L are defined as in the case of RL_{HS} while the semantics of the relational constants R_i has to be defined in accordance with the semantics of the chosen primitive interval relations. Any L-formula φ can be translated to an RL_L-formula $i\ R\ j$ by means of the following validity preserving translation τ:

– for propositional letters and for propositional connectives, τ is defined as in the case of RL_{HS};

– for every basic modality $\langle R \rangle$, $\tau(\langle R \rangle\psi) = R; \tau(\psi)$;

– for every converse modality $\langle\overline{R}\rangle$, $\tau(\langle\overline{R}\rangle\psi) = R^{-1};\tau(\psi)$.

A proof system for RL_L can be obtained from the proof system for RL_{HS} (in the case of the non-strict semantics for intervals) or for RL^{-}_{HS} (in the case of the strict semantics), by substituting rules (B), (E), $(-B)$, and $(-E)$ with rules that are appropriate for the choice of basic Allen's relations. Rules for *begins* and *ends* are presented in Section 5, while the rules for the remaining relations are the following.

For $i, j \in \mathbb{IV}$:

(D) $\quad \dfrac{i\ D\ j}{i_1 < j_1, i\ D\ j \mid j_2 < i_2, i\ D\ j}$ $\qquad (-D)$ $\quad \dfrac{i\ -D\ j}{i_1 -\!\!< j_1, j_2 -\!\!< i_2, i\ -D\ j}$

(M) $\quad \dfrac{i\ M\ j}{j_2\ 1'\ i_1, i\ M\ j}$ $\qquad (-M)$ $\quad \dfrac{i\ -M\ j}{j_2\ -1'\ i_1, i\ -M\ j}$

(P) $\quad \dfrac{i\ P\ j}{j_2 < i_1, i\ P\ j}$ $\qquad (-P)$ $\quad \dfrac{i\ -P\ j}{j_2 -\!\!< i_1, i\ -P\ j}$

(O) $\quad \dfrac{i\ O\ j}{j_1 < i_1, i\ O\ j \mid i_1 < j_2, i\ O\ j \mid j_2 < i_2, i\ O\ j}$

$(-O)$ $\quad \dfrac{i\ -O\ j}{j_1 -\!\!< i_1, i_1 < j_2, j_2 < i_2, i\ -O\ j}$

It is easy to check that the rules correspond to the semantics of Allen's relations, as depicted in Section 1. Hence, soundness of the rules is straightforward. To prove completeness we need to appropriately expand the completion conditions and the notion of branch structure. For instance, rules (M) and $(-M)$ require the following completion conditions:

Cpl(M) If $i\ M\ j \in b$ then $j_2\ 1'\ i_1 \in b$.

Cpl$(-M)$ If $i\ -M\ j \in b$ then $j_2\ -1'\ i_1 \in b$.

Consider now the branch structure $\mathcal{M}^b = (U^b, \mathbb{I}(U^b), m^b)$ (where $\mathbb{I}(U^b)$ can be either $\mathbb{I}(U^b)^-$ or $\mathbb{I}(U^b)^+$). The meaning of M in $\mathcal{M}^b$ is defined as follows:

$$m^b(M) = \{([c, d], [c', d']) \in \mathbb{I}(U^b) \times \mathbb{I}(U^b) : (d', c) \in m^b(1')\}.$$

The valuation v^b and the notion of satisfiability in $\mathcal{M}^b$ are defined as in RL_{HS}. To prove completeness, we have to show that $\mathcal{M}^b, v^b \models i\ R\ j$ if and only if $i\ R\ j \notin b$, where R can be either M or $-M$.

– Let $R := M$. Assume $(i, j) \in m^b(M)$, that is $(j_2, i_1) \in m^b(1')$. Then $j_2\ 1'\ i_1 \notin b$. Suppose $i\ M\ j \in b$. By the completion condition Cpl(M), $j_2\ 1'\ i_1 \in b$, a contradiction.

– Let $R := -M$. Assume $(i, j) \notin m^b(M)$, that is $(j_2, i_1) \notin m^b(1')$. Then $j_2\ 1'\ i_1 \in b$. Suppose $i\ -M\ j \in b$. By the completion condition Cpl$(-M)$, $j_2\ -1'\ i_1 \in b$, a contradiction.

The rest of the completeness proof is as in RL_{HS}.

Relational systems for other interval temporal logics.

The rules presented above allow us to easily adapt the proof system for RL_{HS} to any propositional interval temporal logic that is a proper fragment of HS. Here we show two examples of such a modification.

The logic BE.

The logic BE features the two modalities $\langle B\rangle$ and $\langle E\rangle$, and has been first studied in [LOD 00], where its undecidability has been proved. Its formulas are generated by the following abstract syntax:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle B\rangle\varphi \mid \langle E\rangle\varphi.$$

Since BE does not have converse modalities, the relational logic RL_{BE} appropriate for BE is logic RL_{HS} without the converse operator $^{-1}$. A relational proof system for RL_{BE} can be obtained from the one for RL_{HS} by removing rules ($^{-1}$) and ($-^{-1}$).

Propositional neighborhood logics.

Interval logics based on the relation *meet* and its converse are usually called *neighborhood logics*. First-order neighborhood logics were first introduced and studied in [CHA 98]. Its propositional variant, called *Propositional Neighborhood Logic* (PNL, for short) has been proposed and investigated recently in [GOR 03b].

Accordingly with our notation for the interval modalities, the syntax of PNL is the following:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle M\rangle\varphi \mid \langle \overline{M}\rangle\varphi.$$

In [GOR 03b] the authors studied both the case of the non-strict and strict semantics for PNL over linear orderings (denoted PNL$^+$ and PNL$^-$, respectively). The relational logic RL_{PNL^+} (appropriate for PNL$^+$) is logic RL_{HS} where the interval relational constant M takes place of B and E. A proof system for RL_{PNL^+} can be obtained from the one for RL_{HS} by substituting rules (B), ($-B$), (E), and ($-E$) with rules (M) and ($-M$). In the case of the strict semantics, the relational logic RL_{PNL^-} (appropriate for PNL$^-$) can be obtained from RL^-_{HS} in the same way.

7.3. *Properties of the temporal ordering*

In all the relational systems RL_L presented above, the strict ordering $<$ is considered to be linear, without any further assumption. In this section we propose some possible extensions and modifications of our systems in case of other temporal ordering.

Unbounded orderings

An ordering is said to be *unboundend below* (resp. *above*) if for every x there exists z such that $z < x$ (resp. $x < z$). Such a condition can be expressed in a relational system RL_L by means of the following rules.

For $x \in \mathbb{PV}$:

(No-min<) $\dfrac{}{z -\!< x}$ (No-max<) $\dfrac{}{x -\!< z}$ with z new point variable.

Soundness of the rules can be easily proved. Suppose that $<$ is unbounded below (the case where $<$ is unbounded above is similar). Then, for every x, there exists z such that $z < x$. Thus, $z -\!< x$ cannot be an RL_L-set and rule (No-min<) is correct.

To prove completeness of the system, we need to add the following completion conditions.

Cpl(No-min<) For all $x \in \mathbb{PV}$, there exists $z \in \mathbb{PV}$ such that $z -\!< x \in b$.

Cpl(No-max<) For all $x \in \mathbb{PV}$, there exists $z \in \mathbb{PV}$ such that $x -\!< z \in b$.

Consider now the branch structure $\mathcal{M}^b = (U^b, \mathbb{I}(U^b), m^b)$ of RL_L. To prove that $m^b(<)$ is unbounded below, suppose by contradiction that there exists $x \in \mathbb{PV}$ such that, for all $z \in \mathbb{PV}$, $(z, x) \notin m^b(<)$. This implies that $z < x \in b$ for all $z \in \mathbb{PV}$. By the completion condition Cpl(No-min<), there exists $z \in \mathbb{PV}$ such that $z -\!< x \in b$ and $z < x \in b$, a contradiction. Proving that the completion condition Cpl(No-max<) implies that $m^b(<)$ is unbounded above is similar.

Dense orderings

An ordering $<$ is *dense* if for every pair of different comparable points there exists another point in between, namely, if $\forall x, y(x < y \rightarrow \exists z(x < z \wedge z < y))$ holds. Density of the time domain can be expressed by the following rule.

For $x, y \in \mathbb{PV}$:

(Dense<) $\dfrac{}{x < y \mid x -\!< z, z -\!< y}$ with z new point variable.

Soundness is straightforward: the rule corresponds to the first-order formula $\exists x, y(x < y \wedge \forall z(x -\!< z \vee z -\!< y))$, that is exactly the negation of the density condition. As for the completeness, we add the following completion condition.

Cpl(Dense<) For all $x, y \in \mathbb{PV}$, either $x < y \in b$ or there exists $z \in \mathbb{PV}$ such that $x -\!< z \in b$ and $z -\!< y \in b$.

Consider now the branch structure, and suppose that $m^b(<)$ does not respect the density condition, that is, there exist $x, y \in \mathbb{PV}$ such that $(x, y) \in m^b(<)$ and, for all $z \in \mathbb{PV}$, $(x, z) \notin m^b(<)$ or $(z, y) \notin m^b(<)$. This implies that $x < y \notin b$ and, for all z, $x < z \in b$ or $z < y \in b$. By the completion condition Cpl(Dense<), we have that there exists z such that $x -\!< z \in b$ and $z -\!< y \in b$, a contradiction.

Discrete orderings

An ordering in *discrete* if every point with a successor/predecessor has an immediate successor/predecessor, that is:

$$(1) \quad \forall x, y(x < y \rightarrow \exists z(x < z \wedge \forall t(x -\!< t \vee t -\!< z))),$$

and

$$(2) \quad \forall x, y(y < x \rightarrow \exists z(z < x \wedge \forall t(z -\!\!< t \vee t -\!\!< x))).$$

Discreteness of the time domain is expressed by the following additional rules.

For $x, y, z, t \in \mathbb{PV}$:

$$(\text{Disc}<_1) \quad \frac{}{x < y \mid x -\!\!< z, x < t \mid x -\!\!< z, t < z}$$

$$(\text{Disc}<_2) \quad \frac{}{y < x \mid z -\!\!< x, z < t \mid z -\!\!< x, t < x}$$

with x, y, t any point variable, z new point variable.

The lower part of rule (Disc$<_1$) corresponds to the first-order formula $\exists x, y(x < y \wedge \forall z(x -\!\!< z \vee \exists t(x < t \wedge t < z)))$, that is exactly the negation of condition (1). Similarly, the lower part of rule (Disc$<_2$) corresponds to the negation of condition (2). Hence, soundness of the rules is straightforward.

To prove completeness, it is necessary to add the following completion conditions to the system:

Cpl(Disc$<_1$) For all $x, y \in \mathbb{PV}$, either $x < y \in b$, or there exists $z \in \mathbb{PV}$ such that $x -\!\!< z \in b$ and, for all $t \in \mathbb{PV}$, $x < t \in b$, or $t < z \in b$.

Cpl(Disc$<_2$) For all $x, y \in \mathbb{PV}$, either $y < x \in b$, or there exists $z \in \mathbb{PV}$ such that $z -\!\!< x \in b$ and, for all $t \in \mathbb{PV}$, $z < t \in b$, or $t < x \in b$.

Consider now the branch structure $\mathcal{M}^b$, and suppose that $m^b(<)$ does not respect condition (1). This implies that there exist $x, y \in \mathbb{PV}$ such that $(x, y) \in m^b(<)$ but, for all $z \in \mathbb{PV}$, either $(x, z) \notin m^b(<)$, or there exists $t \in \mathbb{PV}$ such that $(x, t) \in m^b(<)$ and $(t, z) \in m^b(<)$. By the definition of branch structure, this implies that $x < y \notin b$ and either $x < z \in b$ or $x < t \notin b$ and $t < z \notin b$. By the completion condition Cpl(Disc$<_1$), one of the following may arise:

– $x < y \in b$, a contradiction;

– $x -\!\!< z \in b$ and $x < t \in b$, a contradiction;

– $x -\!\!< z \in b$ and $t < z \in b$, a contradiction.

8. Conclusions

We presented a sound and complete relational proof system for HS interval temporal logic. Next we showed how to extend the system to the classes of interval temporal logics which may have some other interval relations as the accessibility relations in their models or to the logics where the interval relations may be based on orderings with various specific properties (e.g., unbounded, dense, discrete).

The rules presented in this paper provide also a means of a direct deduction in interval algebras considered in [LAD 87]. Let $<$ be a dense linear ordering on a non-empty set without endpoints. The 13 relations of Allen (the relations recalled in Section 1, their converses, and $1'$) based on such an ordering are the atoms of a proper relation algebra on $< \times <$, i.e., a subalgebra of the algebra $(2^{<\times<}, \cup, \cap, -, ;, -1, 1, 1')$, where $1 =< \times <$ and $1'$ is the identity on the field of $<$. The proof system for verification of validity of equations of the form $R = 1$ in this class of algebras consists of the standard decomposition rules (the rules of Section 5.1 with $x, y \in \mathbb{IV}$ and $R, S \in \mathbb{IRT}$), the decomposition rules for the six interval relations and their complements (the rules for B and E from Section 5.1 and the rules for D, O, M, and P from Section 7.2), the rules for $1'$ (the rules for $1'$ of Section 5.2 where the symbols of variables and relations range either over point variables and point relations or over interval variables and interval relations), and the rules that characterize the point ordering ((Irref<) and (Tran<) from Section 5.2, (No-min<), (No-max<), and (Dense<) from Section 7.3). The axiomatic sets of the system are (a1),...(a5), where in (a2) x can be either a point variable or an interval variable.

As indicated in the paper, relational dual tableaux are modular. The rules for an axiomatic and/or signature extension of a logic consist of the rules for this logic augmented with the new rules corresponding to the new concepts introduced in the extension. If an extension of a logic is defined in terms of new conditions on the models of the logic, then the new rules must be added which reflect these new semantic constraints. In defining the new rules the correspondence theory presented in [MAC 02] can be helpful. Due to their modularity, relational dual tableaux are well suited for providing deduction mechanisms for various classes of non-classical logics, in particular for interval temporal logics. If a logic from a class possess a Kripke-style semantics, then the construction of a relational logic adequate for embedding in it the given logic is usually straightforward and leads in a natural way to the appropriate translation. Since several interval temporal logics are known to be decidable (see [BOW 03, DIL 92, DIL 93, DIL 96b, DIL 96a, MON 02]), further work is needed on dual tableau decision procedures. In the literature there are some tableaux proof systems and decision procedures for interval temporal logics (see [BOW 03, BRE 05a, BRE 05b, BRE 06, GOR 03a]). Since tableaux and dual tableaux are known to be dual in a precisely defined sense ([GOL 06b]), it would be interesting to explore this relationship in order to extend the applicability of these systems. In particular, development of relational tableaux would be of interest. One could reasonably expect that they will have the modularity property similar to modularity of dual tableaux. Other proof systems for temporal logics can be found in [RAS 01b] and [RAS 01a].

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